

Wavefronts for degenerate diffusion-convection reaction equations with sign-changing diffusivity

Diego Berti* Andrea Corli[†] Luisa Malaguti*

July 23, 2021

Abstract

We consider in this paper a diffusion-convection reaction equation in one space dimension. The main assumptions are about the reaction term, which is monostable, and the diffusivity, which changes sign once or even more than once; then, we deal with a forward-backward parabolic equation. Our main results concern the existence of globally defined traveling waves, which connect two equilibria and cross both regions where the diffusivity is positive and regions where it is negative. We also investigate the monotony of the profiles and show the appearance of sharp behaviors at the points where the diffusivity degenerates. In particular, if such points are interior points, then the sharp behaviors are new and unusual.

AMS Subject Classification: 35K65; 35C07, 34B40, 35K57

Keywords: Sign-changing diffusivity, traveling-wave solutions, diffusion-convection reaction equations, sharp profiles.

1 Introduction

This paper deals with traveling-wave solutions to degenerate parabolic equations of forward-backward type. More precisely, we consider the equation

$$\rho_t + f(\rho)_x = (D(\rho)\rho_x)_x + g(\rho), \quad t \geq 0, x \in \mathbb{R}. \quad (1.1)$$

We denote with $\rho = \rho(t, x)$ the unknown function; also in view of applications we understand ρ as a (normalized) density or a concentration and then assume that ρ is valued in the interval $[0, 1]$. On the convective term f we only assume

$$(f) \quad f \in C^1[0, 1], f(0) = 0.$$

The condition $f(0) = 0$ just fixes a flux representative, since convection is only defined up to an additive constant. In the following, for brevity, we denote the derivative of f as $h(\rho) = \dot{f}(\rho)$. We assume that g satisfies

*Department of Sciences and Methods for Engineering, University of Modena and Reggio Emilia, Italy

[†]Department of Mathematics and Computer Science, University of Ferrara, Italy

(g) $g \in C^0[0, 1]$, $g > 0$ in $(0, 1)$, $g(0) = g(1) = 0$.

Condition (g) is natural in this framework and then g is said of *monostable* type. The diffusivity D is required to satisfy one of the following assumptions, for some $\alpha, \beta \in (0, 1)$:

(D_{pn}) $D \in C^1[0, 1]$, $D > 0$ in $(0, \alpha)$ and $D < 0$ in $(\alpha, 1)$;

(D_{np}) $D \in C^1[0, 1]$, $D < 0$ in $(0, \beta)$ and $D > 0$ in $(\beta, 1)$.

Each condition is labelled following the sign of D : “pn” in (D_{pn}) means that D is first positive and then negative, and the other so on. Above, α denotes a zero of D such that $\dot{D}(\alpha) \leq 0$ while β denotes a zero of D such that $\dot{D}(\beta) \geq 0$.

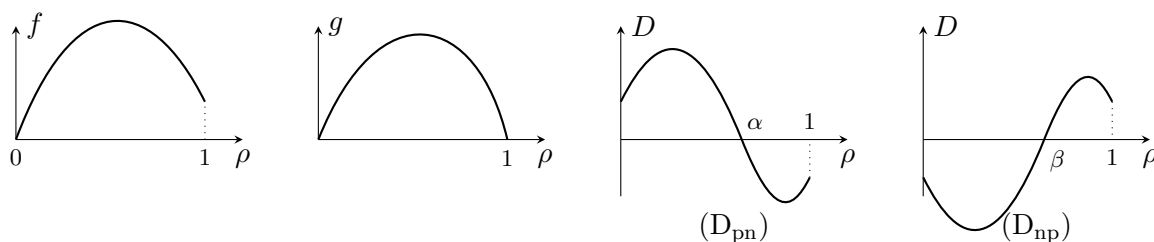


Figure 1: Typical plots of the functions f , g and D .

The above assumptions on D are the main issue of this paper, and make (1.1) a *forward* parabolic equation where $D > 0$ and a *backward* parabolic equation where $D < 0$. The above conditions leave also open the possibility that D vanishes at 0 or 1.

There are several motivations to study forward-backward parabolic equations as (1.1): for a short list of different modeling we quote [18, 21, 27] for biology, [13] for geophysics, [19] for thermodynamics. However, our main source of inspiration has been the recent modeling of collective movements, namely of vehicular flows and crowds dynamics. About this topic, we refer to [15, 16, 28] for general information, to [4, 5, 7] for the modeling using degenerate parabolic equations, to [9] for the study of star graphs, while for sign-changing diffusivities we refer to [11, 12, 26] and the references in the two former papers.

Our interest in this paper is about *traveling-wave* solutions (TWs for short) to (1.1); they are solutions to (1.1) of the form $\rho(t, x) = \varphi(x - ct)$. Here, the function $\varphi = \varphi(\xi)$ is the *profile* of the TW and the real number c is its *speed*. The equation for the profiles is then

$$(D(\varphi)\varphi')' + (c - h(\varphi))\varphi' + g(\varphi) = 0, \quad (1.2)$$

where $'$ denotes the derivative with respect to ξ . More precisely we focus on *wavefronts*, i.e., globally-defined, nonconstant and monotone profiles. To fix ideas we deal with *non-increasing* profiles and this leads us, because of (g), to impose the conditions

$$\varphi(-\infty) = 1, \quad \varphi(+\infty) = 0. \quad (1.3)$$

The study of non-decreasing profiles, in which the conditions in (1.3) are switched, is not explicitly treated in this paper. Nevertheless, all the results can be rephrased in that case,

once that (roughly speaking) the direction of speeds is reversed. Clearly, even under (1.3) the solutions to (1.2) are at most unique up to horizontal shifts. An interesting issue is whether the equilibria 0 and 1 can be reached by a wavefront φ at a finite value ξ_0 . This possibly occurs if D vanishes at those points, and in such cases φ is necessarily constant on either (ξ_0, ∞) or $(-\infty, \xi_0)$, with values 0 and 1, respectively. The profile is called *sharp* if it is not differentiable at ξ_0 . In this case φ is not a classical solution of (1.2) (see Definition 2.1). We refer to [17] for more information on traveling waves. The case where D changes sign has been considered by several authors *but only when $f = 0$* . About this case, we quote [2, 21] for D satisfying (D_{np}) and (D_{pn}) , respectively, and monostable g ; [22] for the case (D_{pn}) and bistable g (i.e., g changes sign once).

The main result of the current paper is that there still exist wavefronts joining 1 with 0, which travel across regions where D changes sign. In our approach the profiles are constructed by suitably pasting two *semi-wavefronts* and possibly a *traveling wave solution* in a bounded interval (see the next section for a definition), as in [11]. As a consequence of this procedure, one realizes that the assumptions on D can be somewhat relaxed, as in [2]. Indeed, for instance in case (D_{pn}) , it is sufficient to require $D \in C^0[0, 1]$ and $D \in C^1[0, \alpha] \cap C^1[\alpha, 1]$: the derivatives at α^- and α^+ may be different.

We prove that the wavefronts constructed in this way are unique (up to shifts) and provide results about the strict monotony of profiles; in particular, we characterize when they are sharp. At last, we give rather precise bounds on the critical thresholds by exploiting the estimates obtained in [6]; there, in turn, we used some related recent results proved in [25]. The main tool underlying our approach is a well-known reduction (in regions where D has constant sign) of Equation (1.2) to singular first-order systems [24] and its study by comparison-type techniques. More precisely, if we denote $z(\varphi) := D(\varphi)\varphi'$, where φ' is computed at $\varphi^{-1}(\varphi)$ (notice that the inverse function of φ exists because of the monotony of φ), then we are reduced to consider the problems, for instance in the case (D_{pn}) ,

$$\left\{ \begin{array}{l} \dot{z}(\varphi) = h(\varphi) - c - \frac{(Dg)(\varphi)}{z(\varphi)}, \\ z(\varphi) < 0, \\ z(0) = 0, \end{array} \right. \quad \varphi \in (0, \alpha), \quad \text{or} \quad \left\{ \begin{array}{l} \dot{z}(\varphi) = h(\varphi) - c - \frac{(Dg)(\varphi)}{z(\varphi)}, \\ z(\varphi) > 0, \\ z(1) = 0, \end{array} \right. \quad \varphi \in (\alpha, 1), \quad (1.4)$$

and similar problems in the other cases. We refer to [6] for a detailed study of $(1.4)_1$.

Problems (1.4) seems to suggest that the roles played by D and g are interchangeable; this is not true, in general. Consider for instance the bistable (or Allen-Cahn) equation, where $D > 0$ in $(0, 1)$ but g satisfies $g(0) = g(\alpha) = g(1) = 0$, $g < 0$ in $(0, \alpha)$ and $g > 0$ in $(\alpha, 1)$. In this case it is known [1], at least if $f = 0$, that equation (1.1) admits a *unique* admissible speed c corresponding to a wavefront from 1 to 0. In the current case (D_{np}) , on the contrary, where the role of D and g is commuted with respect to the bistable case, we shall find a whole half line of admissible speeds. The same result holds for the case (D_{pn}) .

Here follows an outline of the paper. Section 2 contains the statements of our main results. In Section 3, we first briefly recall some preliminary results in order to keep the paper self-contained; then we prove the main result under (D_{pn}) . Section 4 deals with the case (D_{np}) . In Section 5 we give some explicit examples; they aim at showing the role played by the convection term f , the qualitative difference of the thresholds in the cases (D_{pn}) and (D_{np}) , the occurrence of non-regular profiles. At last, Section 6 extends the previous results

to the case when D changes sign twice.

2 Main results

As we mentioned in the Introduction, traveling-waves can fail to be of class C^1 in the whole of their domain. The following definition makes precise what we mean by a TWs, see [17].

Definition 2.1. *Assume $f, D, g \in C^0[0, 1]$ and let $I \subset \mathbb{R}$ be an open interval. Let $\varphi \in C^0(I)$ be a function valued in $[0, 1]$, which is differentiable a.e. and such that $D(\varphi)\varphi' \in L^1_{\text{loc}}(I)$; at last, let c be a real constant.*

Then the function $\rho(x, t) := \varphi(x - ct)$, for (x, t) with $x - ct \in I$, is a traveling-wave solution (briefly, a TW) to equation (1.1) with wave speed c and wave profile φ if we have

$$\int_I \left(D(\varphi(\xi)) \varphi'(\xi) - f(\varphi(\xi)) + c\varphi(\xi) \right) \psi'(\xi) - g(\varphi(\xi)) \psi(\xi) d\xi = 0, \quad (2.1)$$

for every $\psi \in C_0^\infty(I)$.

We now give some definitions about TWs. A TW is: *global* if $I = \mathbb{R}$; *strict* if $I \neq \mathbb{R}$ and φ cannot be extended to \mathbb{R} ; *classical* if φ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (1.2) holds a.e.; *sharp at ℓ* if there exists $\xi_\ell \in I$, with $\varphi(\xi_\ell) = \ell$, such that φ is classical in $I \setminus \{\xi_\ell\}$ and not differentiable at ξ_ℓ . Analogously, a TW is *classical at ℓ* if it is classical in a neighborhood of ξ_ℓ .

Moreover, a TW is: *a wavefront* if it is global, with a monotonic, non-constant profile φ which satisfies either (1.3) or the converse condition; *a semi-wavefront to 1* (or *to 0*) if $I = (a, \infty)$ for $a \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \rightarrow 1$ (respectively, $\varphi(\xi) \rightarrow 0$) as $\xi \rightarrow \infty$; *a semi-wavefront from 1* (or *from 0*) if $I = (-\infty, b)$ for $b \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \rightarrow 1$ (respectively, $\varphi(\xi) \rightarrow 0$) as $\xi \rightarrow -\infty$. About semi-wavefronts, we say that φ connects $\varphi(a^+)$ (1 or 0) with 1 or 0 (resp., with $\varphi(b^-)$).

The problem of the loss of regularity of φ depends on whether the parabolic equation degenerates or not; more precisely, by arguing on the very equation (1.2), it is easy to see that if f, D are of class C^1 and g of class C^0 , then φ is classical in every interval $I_\pm \subseteq I$ where $\pm D(\varphi(\xi)) > 0$ for $\xi \in I_\pm$; moreover, $\varphi \in C^2(I_\pm)$ (see e.g. [17, Lemma 2.20]).

We recall that for a function $q : [0, 1] \rightarrow \mathbb{R}$, the notation $D_+q(\rho_0)$ and $D_-q(\rho_0)$, with $\rho_0 \in [0, 1]$, stands for the *right*, resp., *left lower Dini-derivative* of q at ρ_0 ; analogously, $D^\pm(q)$ represent the *right* and *left upper Dini-derivatives* of q . More explicitly,

$$D_\pm q(\rho_0) := \liminf_{\rho \rightarrow \rho_0^\pm} \frac{q(\rho) - q(\rho_0)}{\rho - \rho_0}, \quad D^\pm q(\rho_0) := \limsup_{\rho \rightarrow \rho_0^\pm} \frac{q(\rho) - q(\rho_0)}{\rho - \rho_0}.$$

In addition to the main assumptions (f), (D_{pn}) - (D_{np}) and (g) stated in the Introduction, we also need for some results two further regularity conditions on the product of Dg at the boundary of the interval $[0, 1]$, which are stated using the above notation:

$$D^+Dg(0) < +\infty \quad \text{and} \quad D^-Dg(1) < +\infty. \quad (2.2)$$

In general, $(2.2)_1$ implies that problem $(1.4)_1$ has a super-solution for sufficiently large c , see [23] in the case $f = 0$, and hence it is solvable for those c . Condition $(2.2)_1$ is always

satisfied if $D(0) = 0$, while, if $D(0) \neq 0$, it requires that g is sublinear close to 0; condition (2.2)₂ is commented analogously. At last, we denote the *difference quotient* of a function $F = F(\varphi)$ with respect to a point φ_0 as

$$\delta(F, \varphi_0)(\varphi) := \frac{F(\varphi) - F(\varphi_0)}{\varphi - \varphi_0}. \quad (2.3)$$

We first focus on the case (D_{pn}). The construction of a wavefront to (1.1) takes place by properly joining two semi-wavefronts, each of them with an intrinsic threshold. The existence of such semi-wavefronts in the region $(0, \alpha)$ has been done in [6, Theorem 2.2]; the main content of that result is the following. Under (f), (D_{pn}), (g) and (2.2)₁ there exists a real number, denoted by $c_{p,r}^*$, satisfying

$$\max \left\{ \sup_{(0,\alpha]} \delta(f, 0), h(0) + 2\sqrt{D_+ Dg(0)} \right\} \leq c_{p,r}^* \leq \sup_{(0,\alpha]} \delta(f, 0) + 2\sqrt{\sup_{(0,\alpha]} \delta(Dg, 0)}, \quad (2.4)$$

such that Equation (1.1) admits strict semi-wavefronts to 0, connecting α to 0, with speed c if and only if $c \geq c_{p,r}^*$. It is worth noting that the left- and the right-hand side of (2.4) describe a non-empty interval (possibly degenerating to a single point) of real numbers, as a direct inspection trivially shows. Moreover, in [25, Theorem 3.1] the authors proved that in case Dg differentiable at $\varphi = 0$ (e.g. in case $D(0) = 0$) the second addend of the right-hand side of (2.4) can be further enhanced by replacing $\delta(Dg, 0)(\varphi)$ with its mean value in $(0, \varphi)$, that is

$$c_{p,r}^* \leq \sup_{(0,\alpha]} \delta(f, 0) + 2\sqrt{\sup_{\varphi \in (0,\alpha]} \frac{1}{\varphi} \int_0^\varphi \frac{D(s)g(s)}{s} ds}. \quad (2.5)$$

We warn the reader that those semi-wavefronts are proved to be *intrinsically* nonunique, i.e., nonuniqueness holds even understanding profiles differing by a horizontal shift as a same profile (see Proposition 3.1). The subscripts p, r in $c_{p,r}^*$ mean that we are considering a case where D is *positive* in an interval I_+ and vanishes in the *right* extremum of I_+ , while g vanishes at the opposite extremum. Similarly, we shall use the notation $c_{n,l}^*$, $c_{p,l}^*$ and $c_{n,r}^*$.

An analogous result for semi-wavefronts from 1, connecting 1 to α , is first deduced in this paper. Assume (f), (D_{pn}), (g) and (2.2)₂. There exists $c_{n,l}^* \in \mathbb{R}$ satisfying

$$\max \left\{ \sup_{[\alpha,1)} \delta(f, 1), h(1) + 2\sqrt{D_- Dg(1)} \right\} \leq c_{n,l}^* \leq \sup_{[\alpha,1)} \delta(f, 1) + 2\sqrt{\sup_{[\alpha,1)} \delta(Dg, 1)}, \quad (2.6)$$

such that Equation (1.1) admits strict semi-wavefronts from 1, connecting 1 to α , with speed c , if and only if $c \geq c_{n,l}^*$; see Proposition 3.2. Also in this case profiles are intrinsically nonunique.

We now present our main results on *wavefronts*. Define, see Figure 2,

$$c_{pn}^* := \max \left\{ c_{p,r}^*, c_{n,l}^* \right\}. \quad (2.7)$$

For what concerns an estimate on c_{pn}^* , see Remark 3.3. We introduce the quantity $s_\pm(\gamma, c)$, defined formally by

$$s_\pm(\gamma, c) := \frac{1}{2} \left[h(\gamma) - c \pm \sqrt{(h(\gamma) - c)^2 - 4\dot{D}(\gamma)g(\gamma)} \right]. \quad (2.8)$$



Figure 2: The thresholds $c_{p,r}^*$, $c_{n,l}^*$ used in (2.7) and $c_{n,r}^*$, $c_{p,l}^*$, used below in (2.15).

In the next result, we make use of (2.8) with $\gamma = \alpha$. In this particular case, $s_-(\alpha, c)$ is well-defined since $\dot{D}(\alpha)g(\alpha) \leq 0$.

Theorem 2.1. *Assume (f), (g), (D_{pn}) and (2.2). Equation (1.1) admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying (1.3), if and only if $c \geq c_{pn}^*$.*

For such c , we have $\varphi'(\xi) < 0$ when $\varphi(\xi) \in (0, 1) \setminus \{\alpha\}$; there exists a unique $\xi_\alpha \in \mathbb{R}$ such that $\varphi(\xi_\alpha) = \alpha$ and

$$\varphi'(\xi_\alpha^\pm) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha, c)} < 0 & \text{if } \dot{D}(\alpha) < 0 \text{ or } c > h(\alpha), \\ -\infty & \text{if } \dot{D}(\alpha) = 0 \text{ and } c \leq h(\alpha). \end{cases} \quad (2.9)$$

At last, it holds that:

- (i) *if $D(0)D(1) < 0$, then φ is strictly decreasing and hence classical in $\mathbb{R} \setminus \{\xi_\alpha\}$;*
- (ii) *if $D(0)D(1) = 0$ and $c > c_{pn}^*$, then φ is classical in $\mathbb{R} \setminus \{\xi_\alpha\}$; φ is strictly decreasing if*

$$c > \max \left\{ h(0) + D^+g(0), h(1) + D^- \{-g\}(1) \right\};$$

- (iii) *if $D(0) = 0$ and $c = c_{pn}^* = c_{p,r}^* > h(0)$, then φ is sharp at 0 (reached at some $\xi_0 > \xi_\alpha$) and if $D(1) = 0$ and $c = c_{pn}^* = c_{n,l}^* > h(1)$, then φ is sharp at 1 (reached at some $\xi_1 < \xi_\alpha$). In these cases we have*

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0) - c_{p,r}^*}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) > 0, \\ -\infty & \text{if } \dot{D}(0) = 0, \end{cases} \quad \varphi'(\xi_1^+) = \begin{cases} \frac{h(1) - c_{n,l}^*}{\dot{D}(1)} < 0 & \text{if } \dot{D}(1) > 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

We refer to Figure 3 for a pictorial representation of Theorem 2.1. Theorem 2.1 extends [21, Theorem 1] to the case of a non-zero convection term f ; if $f = 0$ the estimates on c_{pn}^* deduced from (2.4) and (2.6) coincide with those in [21]. Estimates (2.4) and (2.6) imply $c_{pn}^* \geq \max \{h(0), h(1)\}$. However, if $h(\varphi) \geq h(0)$ for every φ in a right neighborhood of 0, then $c_{p,r}^* > h(0)$, see [6, Remark 6.4], and so $c_{pn}^* > h(0)$. An analogous remark holds for the point 1. This means that the lower estimate on c_{pn}^* is not sharp, in general. Therefore we also extend the corresponding result of [21], since they proved that the threshold c^* , which plays the role of c_{pn}^* here, satisfies the stricter estimate $c^* > 0$.

On the other hand, a new result that could not occur in [21] is that we can have $\varphi'(\xi_\alpha) = -\infty$, see (2.9)₂, while in the case $f = 0$ profiles are always C^1 at ξ_α . Explicit examples of fronts passing from a positive- to a negative-diffusivity region such that (2.9)₂ occurs, are given in Example 5.1. From a formal point of view, if $f = 0$ then this could occur if $c \leq 0$,

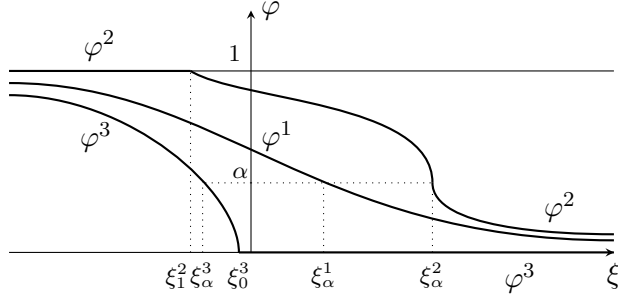


Figure 3: Some possible wavefronts joining 1 with 0 in case (D_{pn}) : a classical wavefront φ^1 (for $c > c_{pn}^*$ and either $\dot{D}(\alpha) < 0$ or $c > h(\alpha)$); a wavefront φ^2 which is sharp at 1, with finite right derivative at ξ_1^2 and $(\varphi^2)'(\xi_\alpha^2) = -\infty$ (for $c = c_{n,l}^* > h(1)$, $D(1) = 0$, $\dot{D}(1) > 0$, $\dot{D}(\alpha) = 0$ and $c \leq h(\alpha)$); a wavefront φ^3 which is sharp at 0 with $(\varphi^3)'(\xi_0^3) = -\infty$ (for $c = c_{p,r}^* > h(0)$, $D(0) = 0 = \dot{D}(0)$).

see $(2.9)_2$, while in that case $c > 0$ holds; more rigorously, see [21, (29)]. We emphasize that this phenomenon does not depend on the change of sign of D at α , but on the occurrence of the two conditions in $(2.9)_2$, see [6, Remark 9.3]. Moreover, just to get an insight on the problem, assume $\dot{D}(\alpha) = 0$; in order that $(2.9)_2$ takes place it is necessary that

$$\max\left\{\sup_{(0,\alpha]} \delta(f, 0), \sup_{[\alpha,1)} \delta(f, 1)\right\} \leq h(\alpha), \quad (2.10)$$

so that we have room for c . Condition (2.10) has a simple geometric interpretation: the slope of the tangent to the graph of f at α must be larger than the slope of any chord joining $(0, 0)$ with any other point of the graph of f , in the interval in $(0, \alpha]$, and analogously for the interval $[\alpha, 1)$. We easily see that $\sup_{(0,\alpha]} \delta(f, 0) \leq h(\alpha)$ fails if f is concave in $[0, \alpha]$, while it holds if f is convex; the converse result holds for the other condition. This means that $(2.9)_2$ may hold only if f changes its convexity. Moreover, if f changes convexity only once, for instance at α , then f must be convex in $[0, \alpha]$ and concave in $[\alpha, 1]$, and not conversely. In other words, the profile may become vertical if, at least in some subintervals, the behavior of f strongly contrast that of D .

We now comment on *(iii)*. The lack of results for the cases $c_{p,r}^* = h(0)$ or $c_{n,l}^* = h(1)$ is due to the fact that in these extremal cases the regularity depends on further properties of D and g . Indeed, under the mild assumptions of Theorem 2.1 the profile φ can be either sharp or classical when reaches the equilibria 0 or 1 (for explicit examples *at* 0, we refer to [6, Remark 10.1]; the discussion at 1 is analogous).

We focus now on (D_{np}) . Condition (D_{np}) is specular to (D_{pn}) , in an obvious sense: if D satisfies (D_{pn}) then $-D$ satisfies (D_{np}) . Despite this fact, the results in this case only partially mimic those of Theorem 2.1. In particular, if we focus on the interval $(\beta, 1)$, where $D > 0$, the contrast with [6, Theorem 2.2] emerges evident. Apart from a trivial horizontal translation, we are under the hypotheses considered in [10, Theorem 2.7] (compare (D_{np}) -*(g)* in the interval $(\beta, 1)$ with the corresponding assumptions of [10, Theorem 2.7]). In *contrast* with [6, Theorem 2.2] cited before, [10, Theorem 2.7] affirms that, for *each* $c \in \mathbb{R}$, Equation (1.1) admits (unique up to space shifts) strict semi-wavefronts from 1, connecting 1 to β .

Nevertheless, the system still admits a threshold speed, in the following sense. There exists $c_{p,l}^* \in \mathbb{R}$ such that the (unique up to space shifts) non-increasing wave profile φ , defined maximally in $(-\infty, \xi_\beta)$, satisfies [10, Theorem 2.6]

$$(D(\varphi)\varphi')(\xi_\beta^-) = \begin{cases} 0 & \text{if } c \geq c_{p,l}^*, \\ \ell < 0 & \text{if } c < c_{p,l}^*. \end{cases} \quad (2.11)$$

By improving the estimates of $c_{p,l}^*$ in [10], as for (2.4)-(2.5), it results that $c_{p,l}^*$ must satisfy

$$\max \left\{ \sup_{(\beta,1]} \delta(f, \beta), h(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)} \right\} \leq c_{p,l}^* \leq \sup_{(\beta,1]} \delta(f, \beta) + 2\sqrt{\sup_{\varphi \in (\beta,1]} \frac{1}{\varphi - \beta} \int_\beta^\varphi \frac{D(s)g(s)}{s - \beta} ds}. \quad (2.12)$$

See Proposition 4.1. Similarly, Equation (1.1) admits a (unique up to shifts) strict semi-wavefront to 0, connecting β to 0, for every $c \in \mathbb{R}$. Moreover, there exists $c_{n,r}^* \in \mathbb{R}$ satisfying

$$\max \left\{ \sup_{[0,\beta)} \delta(f, \beta), h(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)} \right\} \leq c_{n,r}^* \leq \sup_{[0,\beta)} \delta(f, \beta) + 2\sqrt{\sup_{\varphi \in [0,\beta)} \frac{1}{\beta - \varphi} \int_\varphi^\beta \frac{D(s)g(s)}{s - \beta} ds}, \quad (2.13)$$

such that the (unique up to shifts) non-increasing profile φ , defined maximally in $(\xi_\beta, +\infty)$, satisfies

$$(D(\varphi)\varphi')(\xi_\beta^+) = \begin{cases} 0 & \text{if } c \geq c_{n,r}^*, \\ s > 0 & \text{if } c < c_{n,r}^*, \end{cases} \quad (2.14)$$

see Proposition 4.2. We set (see Figure 2)

$$c_{np}^* := \max \{c_{n,r}^*, c_{p,l}^*\}. \quad (2.15)$$

In the next theorem, $s_\pm(\beta, c)$ is given by (2.8). Note that, in spite of $\dot{D}(\beta) \geq 0$, $s_\pm(\beta, c)$ is well-defined since in virtue of (2.15) and (2.12) (or (2.13)), we clearly have $(h(\beta) - c)^2 \geq (h(\beta) - c_{n,p}^*)^2 \geq 4\dot{D}(\beta)g(\beta)$.

Theorem 2.2. *Assume (f), (g), (D_{np}). Then, Equation (1.1) admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying (1.3), if and only if $c \geq c_{np}^*$.*

For such c , we have $\varphi' < 0$ if $\varphi \in (0, 1) \setminus \{\beta\}$. There exists a unique $\xi_\beta \in \mathbb{R}$ such that

$\varphi(\xi_\beta) = \beta$ and

$$\varphi'(\xi_\beta^-) = \begin{cases} \frac{g(\beta)}{s_-(\beta, c)} & \text{if } c > c_{p,l}^*, \\ \frac{g(\beta)}{s_+(\beta, c_{p,l}^*)} & \text{if } c = c_{p,l}^* \text{ and } \dot{D}(\beta) > 0, \\ -\infty & \text{if } c = c_{p,l}^* \text{ and } \dot{D}(\beta) = 0, \end{cases}$$

$$\varphi'(\xi_\beta^+) = \begin{cases} \frac{g(\beta)}{s_-(\beta, c)} & \text{if } c > c_{n,r}^*, \\ \frac{g(\beta)}{s_+(\beta, c_{n,r}^*)} & \text{if } c = c_{n,r}^* \text{ and } \dot{D}(\beta) > 0, \\ -\infty & \text{if } c = c_{n,r}^* \text{ and } \dot{D}(\beta) = 0. \end{cases} \quad (2.16)$$

At last, the following holds true:

- (i) if $D(0)D(1) < 0$ then φ is classical in $\mathbb{R} \setminus \{\xi_\beta\}$;
- (ii) if $D(0) = 0$, and either $c > h(0)$ or $c = h(0)$ and $\dot{D}(0) < 0$, then φ is classical at 0; if $D(1) = 0$, and either $c > h(1)$ or $c = h(1)$ and $\dot{D}(1) < 0$ then φ is classical at 1;
- (iii) if $D(0) = 0$ and $c < h(0)$, then φ is sharp at 0 (reached at some $\xi_0 > \xi_\beta$); if $D(1) = 0$ and $c < h(1)$, then φ is sharp at 1 (reached at some $\xi_1 < \xi_\beta$). In these cases we have

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0)-c}{\dot{D}(0)} & \text{if } \dot{D}(0) < 0, \\ -\infty & \text{if } \dot{D}(0) = 0, \end{cases} \quad \varphi'(\xi_1^+) = \begin{cases} \frac{h(1)-c}{\dot{D}(1)} & \text{if } \dot{D}(1) < 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

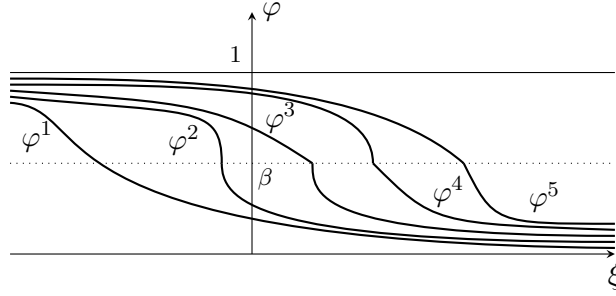


Figure 4: Some possible wavefronts joining 1 with 0 in case (D_{np}) ; Profiles are labelled according to the cases (1)–(4) of Remark 4.2; φ^5 occurs in both cases (3) and (4). For simplicity we only represented strictly monotone profiles.

About the regularity of φ at ξ_β , see also Remark 4.2. From (2.12) and (2.13), we have $c_{np}^* \geq h(\beta)$ and $c_{np}^* > h(\beta)$ if $\dot{D}(\beta) > 0$. Hence, all quantities represented as $s_\pm(\beta, \cdot)$ in (2.16) are negative and finite real values.

Theorem 2.2 above extends [2, Theorem 1] to the case of a non-zero convection term f ; if $f = 0$, the estimates on c_{np}^* deduced from (2.12) and (2.13) improve those in [2], not only because of a flaw in the upper estimate in c^* (the analog of c_{np}^*) in [2], see formula (14) there, but also because the more precise estimates from [6] and [25] are involved here (and

not in [2]). Moreover, by $c_{np}^* \geq h(\beta)$, with a reasoning as above Theorem 2.1 we can further show, as in [2], that $c_{np}^* > 0$ when $f = 0$. In Theorem 2.2 we also prove the existence of sharp profiles at either 0 or 1 even if $c > c_{np}^*$, see case (iii) above. Again, this is due to the presence of f . Indeed, from a formal point of view, by the estimates in (iii) it follows that, if $f = 0$, then to have sharp profiles one needs $c < 0$, while c is always positive in this case.

We point out that, if $c_{p,l}^* > c_{n,r}^*$ and φ is the profile corresponding to $c = c_{np}^* = c_{p,l}^*$ given by Theorem 2.2, then $\varphi'(\xi_\beta^-) \neq \varphi'(\xi_\beta^+)$. The same conclusion holds if $c_{p,l}^* < c_{n,r}^*$ and $c = c_{np}^* = c_{n,r}^*$. Both alternatives $c_{p,l}^* = c_{n,r}^*$ and $c_{p,l}^* \neq c_{n,r}^*$ can indeed occur: explicit examples are shown in Example 5.3. This suggests that Theorems 2.1 and 2.2 produce two separate families of solutions. Moreover, in Example 5.2, we show that the thresholds c_{pn}^* of Theorem 2.1 and c_{np}^* of Theorem 2.2 are *essentially* different, in the sense that taking opposite diffusivities do *not* produce necessarily $c_{pn}^* = c_{np}^*$. Roughly speaking, this is due essentially because (2.4)–(2.6) and (2.12)–(2.13) are unrelated estimates.

Also notice the different role played by the sub-thresholds $c_{p,r}^*$, $c_{n,l}^*$ and $c_{p,l}^*$, $c_{n,r}^*$: the former two discriminate the *existence* of the semi-wavefronts, the latter two the *regularity*. Indeed, the estimates (2.4), (2.6) concern the equilibrium points 0, 1 of g , while the estimates (2.12), (2.13) only concern the point β where D vanishes. The values of φ' at ξ_α or ξ_β : $\varphi'(\xi_\alpha)$ is uniquely determined (being possibly $-\infty$), while we can have $\varphi'(\xi_\beta^-) \neq \varphi'(\xi_\beta^+)$. Moreover, in the case (D_{pn}), a profile may be sharp only if $c = c_{pn}^*$; in case (D_{np}), a profile can be sharp also if $c > c_{np}^*$. As a consequence of these facts, items (i)–(iii) in the two theorems are similar but far from being symmetric.

3 Wavefronts with positive to negative diffusivities

In this section we assume condition (D_{pn}). First, we discuss the existence of semi-wavefronts to 0 and from 1 in Proposition 3.1 and in Proposition 3.2, respectively. Then, by pasting their profiles, we provide the proof of Theorem 2.1.

In the next proposition $s_-(\alpha, c)$ is given by (2.8).

Proposition 3.1. *Assume (f), (g), (D_{pn}), (2.2)₁. Then, there exists $c_{p,r}^*$ satisfying (2.4) such that Equation (1.1) has strict semi-wavefronts to 0, connecting α to 0, if and only if $c \geq c_{p,r}^*$. If φ is the non-increasing profile of one of them, then*

$$\varphi'(\xi) < 0 \quad \text{for any } 0 < \varphi(\xi) < \alpha. \quad (3.1)$$

For $c > c_{p,r}^*$, there exists $\beta(c) < 0$ such that every profile is uniquely determined (up to space shifts) by the value

$$(D(\varphi)\varphi)'(\xi_\alpha^+) =: \ell \in [\beta(c), 0], \quad (3.2)$$

where $\xi_\alpha \in \mathbb{R}$ is such that $(\xi_\alpha, +\infty)$ is the maximal-existence interval of φ .

If φ_ℓ is the profile satisfying (3.2), then $\varphi'_\ell(\xi_\alpha^+) = -\infty$ if $\ell \in [\beta(c), 0)$, while

$$\varphi'_0(\xi_\alpha^+) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha, c)} & \text{if } \dot{D}(\alpha) < 0 \text{ or } c > h(\alpha), \\ -\infty & \text{if } \dot{D}(\alpha) = 0 \text{ and } c \leq h(\alpha). \end{cases} \quad (3.3)$$

Proof. We refer to Figure 5, solid lines. The existence of these semi-wavefronts, the estimate of $c_{p,r}^*$ and condition (3.1) are discussed in [6, Theorem 2.2]. The proof of [6, Theorem 2.2]

also includes the equivalence between the presence of a semi-wavefront with speed c and the solvability of the problem $(1.4)_1$ for the same c , with $z(\varphi) = D(\varphi)\varphi'$. Conditions (3.2) and (3.3) then come, respectively, from [6, Proposition 5.1] and [6, (9.2)] (see also [6, Remark 9.2]) which are directly formulated for problem $(1.4)_1$. \square

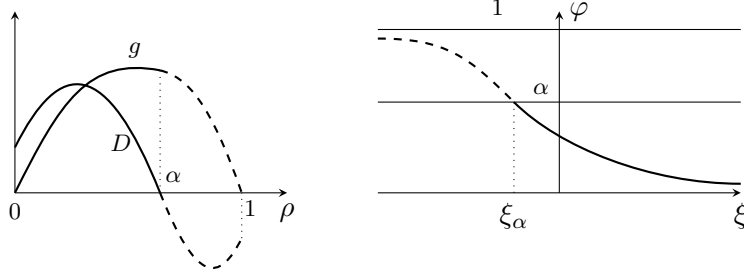


Figure 5: D satisfies (D_{pn}) : Left: the plots of D , g in $[0, \alpha]$ (solid lines) and in $[\alpha, 1]$ (dashed lines); right, a corresponding profile.

Remark 3.1. Every semi-wavefront φ given in Proposition 3.1 satisfies $(D(\varphi)\varphi')(\xi_0^-) = 0$, (see e.g. [6, formula (9.19)]), where $\xi_0 = \sup \{ \xi > \xi_\alpha : \varphi(\xi) > 0 \} \in (\xi_\alpha, +\infty]$.

Proposition 3.2. Assume (f), (g), (D_{pn}) and $(2.2)_2$. There exists $c_{n,l}^* \in \mathbb{R}$ satisfying (2.6) such that Equation (1.1) admits strict semi-wavefronts from 1, connecting 1 to α , with speed c , if and only if $c \geq c_{n,l}^*$. If φ is the non-increasing profile of one of such fronts, then it holds that

$$\varphi'(\xi) < 0 \quad \text{if } \alpha < \varphi(\xi) < 1. \quad (3.4)$$

For $c > c_{n,l}^*$, there exists $\gamma(c) > 0$ such that every profile is uniquely determined (up to space shifts) by the value

$$(D(\varphi)\varphi')(\xi_\alpha^-) =: s \in [0, \gamma(c)], \quad (3.5)$$

where $\xi_\alpha \in \mathbb{R}$ is such that $(-\infty, \xi_\alpha)$ is the maximal-existence interval of φ .

If φ_s is the profile satisfying (3.5), then $\varphi_s'(\xi_\alpha^-) = -\infty$ if $s \in (0, \gamma(c)]$, while

$$\varphi_0'(\xi_\alpha^-) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha, c)} & \text{if } \dot{D}(\alpha) < 0 \text{ or } c > h(\alpha), \\ -\infty & \text{if } \dot{D}(\alpha) = 0 \text{ and } c \leq h(\alpha). \end{cases} \quad (3.6)$$

Proof. We refer to Figure 5, dashed lines. Take, for $\varphi \in [0, 1]$,

$$\bar{D}(\varphi) := -D(1 - \varphi), \quad \bar{g}(\varphi) := g(1 - \varphi), \quad \bar{f}(\varphi) := f(1) - f(1 - \varphi). \quad (3.7)$$

Set $\bar{\alpha} := 1 - \alpha$. Then $\bar{D}, \bar{g}, \bar{f}$ satisfy (D_{pn}) with $\bar{\alpha}$ replacing α , (g) , (f) and $(2.2)_1$. Set $\bar{h} = \dot{\bar{f}}$. We apply Proposition 3.1 to deduce that there exists a real value c^* satisfying

$$\max \left\{ \bar{h}(0) + 2\sqrt{D_+ \bar{D} \bar{g}(0)}, \sup_{(0, \bar{\alpha}]} \delta(\bar{f}, 0) \right\} \leq c^* \leq \sup_{(0, \bar{\alpha}]} \delta(\bar{f}, 0) + 2\sqrt{\sup_{(0, \bar{\alpha}]} \delta(Dg, 0)}$$

such that there exist strict semi-wavefronts connecting 0 to $\bar{\alpha}$, with speed c , if and only if $c \geq c^*$. In the formula above, we recall that $\delta(\cdot, \cdot)$ is defined in (2.3). Direct manipulations show that the above chain of inequalities coincides with (2.6), in virtue of (3.7). Fix $c \geq c^*$ and let $\varphi_{\bar{\alpha},0}$ be the profile of a semi-wavefront connecting $\bar{\alpha}$ to 0, having speed c , given by Proposition 3.1. Moreover, assume that $\varphi_{\bar{\alpha},0}$ is maximally defined in $(\xi_{\bar{\alpha}}, +\infty)$. Define then $\varphi_{1,\alpha} : (-\infty, -\xi_{\bar{\alpha}}) \rightarrow \mathbb{R}$ by

$$\varphi_{1,\alpha}(\xi) := 1 - \varphi_{\bar{\alpha},0}(-\xi) \quad \text{for } \xi < \xi_{\alpha} := -\xi_{\bar{\alpha}}. \quad (3.8)$$

Notice that, because $\varphi_{\bar{\alpha},0}$ connects $\bar{\alpha}$ to 0, then (3.8) implies

$$\lim_{\xi \rightarrow -\infty} \varphi_{1,\alpha}(\xi) = 1 \quad \text{and} \quad \lim_{\xi \rightarrow \xi_{\alpha}^-} \varphi_{1,\alpha}(\xi) = \alpha,$$

thus $\varphi_{1,\alpha}$ connects 1 to α . Also, from (3.8) and $\bar{D}(\varphi_{\bar{\alpha},0})\varphi'_{\bar{\alpha},0} \in L^1(\xi_{\bar{\alpha}}, \infty)$, we have $D(\varphi_{1,\alpha})\varphi'_{1,\alpha} \in L^1(-\infty, \xi_{\alpha})$. Since $\varphi_{\bar{\alpha},0}$ satisfies (1.2), with $\bar{D}, \bar{g}, \bar{h}$, in $(\xi_{\bar{\alpha}}, +\infty)$ for $c \geq c^*$, then direct computations show that $\varphi_{1,\alpha}$ satisfies (1.2) in $(-\infty, \xi_{\alpha})$ with the same c . Solutions are always meant in the sense of Definition 2.1.

Viceversa, let φ be a profile of a strict semi-wavefront of (1.1), connecting 1 to α associated to some $c \in \mathbb{R}$, defined maximally in $(-\infty, 0)$. By setting

$$\varphi_{\bar{\alpha},0}(\tau) := 1 - \varphi(-\tau) \quad \text{for } \tau > 0, \quad (3.9)$$

we obtain the profile of a strict semi-wavefront, from $\bar{\alpha}$ to 0, of Equation (1.2) (with \bar{D}, \bar{g} and \bar{f} as in (3.7)) associated to the speed c . By applying Proposition 3.1 we deduce that $c \geq c^*$. The first part of the statement is hence proved, with $c_{n,l}^* = c^*$ defined above.

Finally, (3.4)–(3.6) follow by (3.8) and Proposition 3.1, for $\gamma(c) := -\beta(c)$. \square

Remark 3.2. According to (3.8) and Remark 3.1, every semi-wavefront φ given in Proposition 3.2 satisfies $(D(\varphi)\varphi')(\xi_1^+) = 0$, where $\xi_1 := \inf\{\xi < \xi_{\alpha} : \varphi(\xi) < 1\} \in [-\infty, \xi_{\alpha})$.

Proof of Theorem 2.1. Let $c_{p,r}^*$ and $c_{n,l}^*$ be the thresholds given in Proposition 3.1 and 3.2, respectively, and define c_{pn}^* as in (2.7).

First, we take $c \geq c_{pn}^*$ and prove that there is a wavefront to equation (1.1) that satisfies (1.3) and has speed c . Associated to such a value of c , from Proposition 3.1, there exists a strict semi-wavefront φ_0 (according to the notation introduced in the statement of Proposition 3.1) to 0 with wave speed c , connecting α to 0, such that

$$(D(\varphi_0)\varphi'_0)(\xi_{\alpha}^+) = 0, \quad (3.10)$$

see (3.2). In (3.10), the value ξ_{α} , which is finite because we are considering a *strict* semi-wavefront, is such that φ_0 is maximally defined in $(\xi_{\alpha}, +\infty)$. Analogously, from Proposition 3.2, we have that there exists a semi-wavefront from 1 with wave speed c , which connects 1 to α and such that its profile ψ_0 realizes

$$(D(\psi_0)\psi'_0)(\xi_{\alpha}^-) = 0, \quad (3.11)$$

see (3.5). Here, we assumed that ψ_0 is maximally defined in $(-\infty, \xi_{\alpha})$; this is always possible unless of shifting ψ_0 . We refer to Remark 3.4 for the reasons of the choices of φ_0 and ψ_0 .

Let $\xi_1, \xi_0 \in \mathbb{R}$ be such that

$$\begin{aligned}\xi_0 &:= \sup \{ \xi > \xi_\alpha : \varphi_0(\xi) > 0 \} \in (\xi_\alpha, +\infty], \\ \xi_1 &:= \inf \{ \xi < \xi_\alpha : \psi_0(\xi) < 1 \} \in [-\infty, \xi_\alpha),\end{aligned}$$

see Figure 6. Define then the function $\varphi = \varphi(\xi)$ by

$$\varphi(\xi) = \begin{cases} \varphi_0(\xi), & \xi \geq \xi_\alpha, \\ \psi_0(\xi), & \xi < \xi_\alpha. \end{cases} \quad (3.12)$$

It is clear that φ is well-defined and continuous in $(-\infty, +\infty)$; moreover, φ is non-increasing and connects 1 to 0. Since both φ_0 and ψ_0 satisfy (1.2) in their domains, then φ satisfies (1.2) pointwise in $(-\infty, \xi_1) \cup (\xi_1, \xi_\alpha) \cup (\xi_\alpha, \xi_0) \cup (\xi_0, +\infty)$.

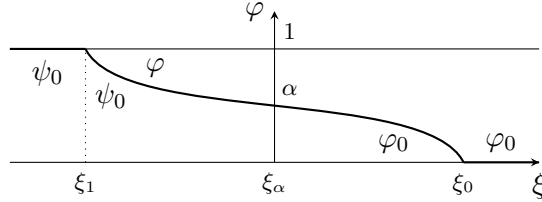


Figure 6: Construction of the profile φ in the case $\xi_0, \xi_1 \in \mathbb{R}$.

Formulas (3.10), (3.11) imply that $D(\varphi)\varphi'$ is continuously extended to $\xi = \xi_\alpha$; we have

$$(D(\psi_0)\psi_0')(\xi_1^+) = 0 \quad \text{and} \quad (D(\varphi_0)\varphi_0')(\xi_0^-) = 0. \quad (3.13)$$

Formula (3.13)₂ follows from Remark 3.1 applied to φ_0 and (3.13)₁ follows from Remark 3.2 applied to ψ_0 . Hence, we showed that $D(\varphi)\varphi' \in L^1_{\text{loc}}(-\infty, +\infty)$. It remains to show that φ is a solution of (1.2) in $(-\infty, +\infty)$, according to Definition 2.1. To this purpose, take $\zeta \in C_0^\infty(-\infty, \infty)$. Since φ is a distributional solution of (1.2) in both $(-\infty, \xi_\alpha)$ and $(\xi_\alpha, +\infty)$, then we can reduce to test (2.1) when $\text{supp } \zeta \subseteq [\xi_\alpha - \delta, \xi_\alpha + \delta] \subset (\xi_1, \xi_0)$, for some $\delta > 0$. Hence, we need to test whether

$$\int_{\xi_\alpha - \delta}^{\xi_\alpha + \delta} (D(\varphi)\varphi' - f(\varphi) + c\varphi)\zeta' - g(\varphi)\zeta \, d\xi = 0. \quad (3.14)$$

Clearly, the left-hand side of (3.14) equals

$$\begin{aligned} \int_{\xi_\alpha - \delta}^{\xi_\alpha} (D(\psi_0)\psi_0' - f(\psi_0) + c\psi_0)\zeta' - g(\psi_0)\zeta \, d\xi + \\ \int_{\xi_\alpha}^{\xi_\alpha + \delta} (D(\varphi_0)\varphi_0' - f(\varphi_0) + c\varphi_0)\zeta' - g(\varphi_0)\zeta \, d\xi. \end{aligned} \quad (3.15)$$

Let us focus on the former addend of (3.15). We have:

$$\begin{aligned}
& \int_{\xi_\alpha - \delta}^{\xi_\alpha} (D(\psi_0) \psi'_0 - f(\psi_0) + c\psi_0) \zeta' - g(\psi_0) \zeta d\xi = \\
& \lim_{\delta > \varepsilon \rightarrow 0^+} \int_{\xi_\alpha - \delta}^{\xi_\alpha - \varepsilon} (D(\psi_0) \psi'_0 - f(\psi_0) + c\psi_0) \zeta' - g(\psi_0) \zeta d\xi = \\
& \lim_{\delta > \varepsilon \rightarrow 0^+} \left((D(\psi_0) \psi'_0)(\xi_\alpha - \varepsilon) - f(\psi_0(\xi_\alpha - \varepsilon)) - c\psi_0(\xi_\alpha - \varepsilon) \right) \zeta(\xi_\alpha - \varepsilon) = \\
& (D(\psi_0) \psi'_0)(\xi_\alpha^-) \zeta(\xi_\alpha) - (f(\alpha) - c\alpha) \zeta(\xi_\alpha) = - (f(\alpha) - c\alpha) \zeta(\xi_\alpha). \quad (3.16)
\end{aligned}$$

Similar arguments involving the latter addend of (3.15) lead to

$$\begin{aligned}
& \int_{\xi_\alpha}^{\xi_\alpha + \delta} (D(\varphi_0) \varphi'_0 - f(\varphi_0) + c\varphi_0) \zeta' - g(\varphi_0) \zeta d\xi = \\
& - (D(\varphi_0) \varphi'_0)(\xi_\alpha^+) \zeta(\xi_\alpha) + (f(\alpha) - c\alpha) \zeta(\alpha) = (f(\alpha) - c\alpha) \zeta(\xi_\alpha). \quad (3.17)
\end{aligned}$$

Putting together (3.16) and (3.17) proves (3.14).

Conversely, we prove that if there exists a wavefront with speed c , whose profile φ is non-increasing and satisfies (1.2)-(1.3), then $c \geq c_{pn}^*$. Let $\xi_\alpha \in \mathbb{R}$ be such that $\varphi(\xi_\alpha) = \alpha$. Such a ξ_α obviously exists since φ is continuous and satisfies (1.3). Furthermore, we have

$$\{\varphi = \alpha\} = \{\xi_\alpha\}. \quad (3.18)$$

Indeed, otherwise there exists an open set $J \subset \{\varphi = \alpha\}$ where φ is constantly equal to α . Thus, in J , equation (1.2) for φ reduces to $g(\alpha) = 0$, which is clearly forbidden by (g). Then, we proved (3.18).

The function $\varphi_{\alpha,0} : (\xi_\alpha, +\infty) \rightarrow (0, \alpha)$, defined by $\varphi_{\alpha,0} = \varphi$, is a strict semi-wavefront to 0 with speed c , connecting α to 0, and the function $\varphi_{1,\alpha} : (-\infty, \xi_\alpha) \rightarrow (\alpha, 1)$, defined by $\varphi_{1,\alpha} = \varphi$, is a strict semi-wavefront from 1, with speed c , connecting 1 to α . From Propositions 3.1 and 3.2, both $c \geq c_{p,r}^*$ and $c \geq c_{n,l}^*$ must occur. Hence, $c \geq c_{pn}^*$, and the first part of Theorem 2.1 is proved.

To conclude the proof, we apply [6, Corollary 9.4] to $\varphi_{\alpha,0}$ (defined just above) and to $\varphi_{\bar{\alpha},0}$ (defined by (3.9)). \square

Remark 3.3 (Estimates for c_{pn}^*). We now explicitly provide estimates for the threshold c_{pn}^* in (2.7). Obviously, c_{pn}^* inherits the bounds, from above and below, for both $c_{p,r}^*$ and $c_{n,l}^*$. Such estimates are contained in Propositions 3.1 and 3.2. We hence have:

$$\begin{aligned}
c_{pn}^* & \geq \max \left\{ \sup_{(0,\alpha]} \delta(f, 0), \sup_{[\alpha,1)} \delta(f, 1), h(0) + 2\sqrt{D_+ Dg(0)}, h(1) + 2\sqrt{D_- Dg(1)} \right\}, \\
c_{pn}^* & \leq \max \left\{ \sup_{(0,\alpha]} \delta(f, 0) + 2\sqrt{\sup_{(0,\alpha]} \delta(Dg, 0)}, \sup_{[\alpha,1)} \delta(f, 1) + 2\sqrt{\sup_{[\alpha,1)} \delta(Dg, 1)} \right\}.
\end{aligned}$$

If f is identically zero, such estimates were given in [21, formula (14)].

Remark 3.4. It is worth emphasizing that, given c large enough, among the profiles in the families

$$\{\varphi_\ell : \ell \in [\beta(c), 0]\} \quad \text{and} \quad \{\psi_s : s \in [0, \gamma(c)]\},$$

given by Propositions 3.1 and 3.2 respectively, we can only benefit from φ_0 and ψ_0 to construct a wavefront as in Theorem 2.1. In particular, we can take advantage only of those profiles whose associated functions z (solving (1.4)) vanish at α^\pm . In all the other cases, the pasting of a profile φ_ℓ with a profile ψ_s does not provide a solution (according to the distributional sense of Definition 2.1) in a neighborhood of the matching point. Indeed, under these assumptions, (3.16) and (3.17) read respectively as

$$\int_{\xi_{\alpha-\delta}}^{\xi_\alpha} (D(\psi_s) \psi'_s - f(\psi_s) + c\psi_s) \zeta' - g(\psi_s) \zeta \, d\xi = [s + c\alpha - f(\alpha)] \zeta(\xi_\alpha), \quad (3.19)$$

$$\int_{\xi_\alpha}^{\xi_{\alpha+\delta}} (D(\varphi_\ell) \varphi'_\ell - f(\varphi_\ell) + c\varphi_\ell) \zeta' - g(\varphi_\ell) \zeta \, d\xi = [-\ell - c\alpha + f(\alpha)] \zeta(\xi_\alpha). \quad (3.20)$$

Thus, in place of (3.14), putting together (3.19) and (3.20) gives

$$\int_{\xi_{\alpha-\delta}}^{\xi_{\alpha+\delta}} (D(\varphi) \varphi' - f(\varphi) + c\varphi) \zeta' - g(\varphi) \zeta \, d\xi = (s - \ell) \zeta(\xi_\alpha), \quad (3.21)$$

which vanishes for each arbitrary test function ζ if and only if $s = \ell = 0$.

In the first instance, this seems to be suggested by the fact that if, formally,

$$z(\alpha) = D(\varphi(\xi_\alpha)) \varphi'(\xi_\alpha^-) = D(\alpha) \varphi'(\xi_\alpha^-) < 0,$$

then necessarily $\varphi'(\xi_\alpha^-) = -\infty$, because $D(\alpha) = 0$; the same remark holds if $w(\alpha) > 0$. Nonetheless, the failure of the pasting is not due to a possibly infinite derivative of the profile at the matching point; indeed, Theorem 2.1 (ii) shows that the wavefront φ can well have infinite slope at ξ_α , see profile φ^2 in Figure 3.

4 Wavefronts with negative to positive diffusivities

In this section we assume condition (D_{np}) and prove Theorem 2.2.

The existence and regularity of semi-wavefronts from 1, connecting 1 to β , was obtained essentially in [10, Theorem 2.7] in the case $D(1) > 0$ and in [8, Theorems 2.3 and 2.5] when $D(1) = 0$. We collect these results in the next proposition. The sharper estimate (2.12) for the threshold $c_{p,l}^*$ comes from [6, Corollary 5.4 and Remark 5.5]. The symbols $s_\pm(\alpha, c)$ are given by (2.8). We refer to Figure 7, solid lines.

Proposition 4.1. *Assume (f), (g), (D_{np}) . Then, for every $c \in \mathbb{R}$, Equation (1.1) has a (unique up to shifts) strict semi-wavefront solution from 1, connecting 1 to β , with speed c and profile φ defined in its maximal-existence interval $(-\infty, \xi_\beta)$, for some $\xi_\beta \in \mathbb{R}$. It holds that $\varphi' < 0$ if $\beta < \varphi < 1$. There exists $c_{p,l}^*$ satisfying (2.12) such that (2.11) holds true and we have*

$$\varphi'(\xi_\beta^-) = \begin{cases} \frac{g(\beta)}{s_-(\beta, c)} & \text{if } c > c_{p,l}^*, \\ \frac{g(\beta)}{s_+(\beta, c_{p,l}^*)} & \text{if } c = c_{p,l}^* \text{ and } \dot{D}(\beta) > 0, \\ -\infty & \text{if } c = c_{p,l}^* \text{ and } \dot{D}(\beta) = 0 \text{ or } c < c_{p,l}^*. \end{cases} \quad (4.1)$$

Moreover, the following results hold.

- (i) If $D(1) > 0$, then φ is classical.
- (ii) If $D(1) = 0$ and either $c > h(1)$ or $c = h(1)$ and $\dot{D}(1) < 0$, then φ is classical.
- (iii) If $D(1) = 0$ and $c < h(1)$, then φ is sharp at 1 (reached at some $\xi_1 < \xi_\beta$) with

$$\varphi'(\xi_1^+) = \begin{cases} \frac{h(1)-c}{\dot{D}(1)} < 0 & \text{if } \dot{D}(1) < 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

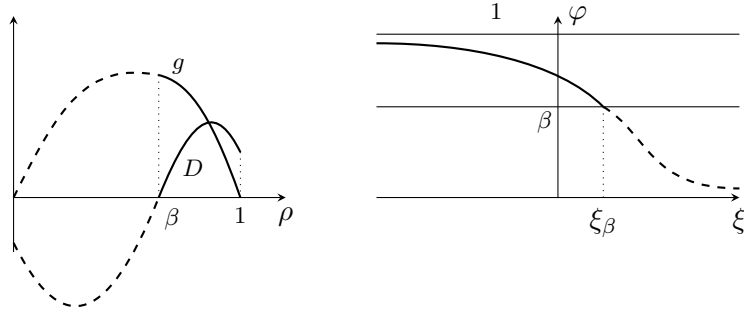


Figure 7: D satisfies (D_{np}) . Left: the plots of D , g in $[0, \beta]$ (dashed lines) and $[\beta, 1]$ (solid lines); right, a corresponding profile.

We discuss the existence of semi-wavefronts to 0 in the next proposition.

Proposition 4.2. *Assume (f), (g), (D_{np}) . Then, for every $c \in \mathbb{R}$ Equation (1.1) has a (unique up to space shifts) strict semi-wavefront solution to 0, connecting β to 0, with speed c and profile φ defined in its maximal-existence interval $(\xi_\beta, +\infty)$, for some $\xi_\beta \in \mathbb{R}$. It holds that $\varphi' < 0$ if $0 < \varphi < \beta$. In addition, there exists $c_{n,r}^* \in \mathbb{R}$ satisfying (2.14) and we have*

$$\varphi'(\xi_\beta^+) = \begin{cases} \frac{g(\beta)}{s_-(\beta, c)} & \text{if } c > c_{n,r}^*, \\ \frac{g(\beta)}{s_+(\beta, c_{n,r}^*)} & \text{if } c = c_{n,r}^* \text{ and } \dot{D}(\beta) > 0, \\ -\infty & \text{if } c < c_{n,r}^* \text{ or } c = c_{n,r}^* \text{ and } \dot{D}(\beta) = 0. \end{cases} \quad (4.2)$$

Moreover, the following results hold.

- (i) If $D(0) < 0$, then φ is classical.
- (ii) If $D(0) = 0$ and either $c > h(0)$ or $c = h(0)$ and $\dot{D}(0) < 0$ then φ is classical.
- (iii) If $D(0) = 0$ and $c < h(0)$ then φ is sharp at 0 (reached at some $\xi_0 > \xi_\beta$) with

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0)-c}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) < 0, \\ -\infty & \text{if } \dot{D}(0) = 0. \end{cases}$$

Proof. We refer to Figure 7, dashed lines. Let \bar{D} , \bar{g} and \bar{f} be defined by (3.7). We already observed in the proof of Proposition 3.2 that \bar{g} and \bar{f} still satisfy (g) and (f). In this case, instead, \bar{D} satisfies (D_{np}) with $\bar{\beta} := 1 - \beta$ replacing β . Hence, Proposition 4.1 applied to \bar{D} , \bar{g} and \bar{f} informs us that strict semi-wavefronts with speed $c \in \mathbb{R}$ connecting 1 to $\bar{\beta}$ exist for every c . Let $\varphi_{1,\bar{\beta}}$ be a profile of one of such fronts, defined in $(-\infty, \xi_{\bar{\beta}})$. Set $\xi_{\beta} := -\xi_{\bar{\beta}}$ and

$$\varphi_{\beta,0}(\xi) := 1 - \varphi_{1,\bar{\beta}}(-\xi) \quad \text{for } \xi > \xi_{\beta}. \quad (4.3)$$

With arguments analogous to those in the proof of Proposition 3.2, it turns out that $\varphi_{\beta,0}$ is the profile of a desired strict semi-wavefront of (1.1), connecting β to 0. Hence, the first part of the statement is proved. By Proposition 4.1 applied to \bar{D} , \bar{g} and \bar{f} , in the interval $[\bar{\beta}, 1]$, we obtain that there exists a real value, say \bar{c}^* , satisfying

$$\max \left\{ \sup_{(\bar{\beta}, 1]} \delta(\bar{f}, \bar{\beta}), \bar{h}(\bar{\beta}) + 2\sqrt{\dot{\bar{D}}(\bar{\beta})\bar{g}(\bar{\beta})} \right\} \leq \bar{c}^* \leq \sup_{(\bar{\beta}, 1]} \delta(\bar{f}, \bar{\beta}) + 2\sqrt{\sup_{\varphi \in (\bar{\beta}, 1]} \frac{1}{\varphi - \bar{\beta}} \int_{\bar{\beta}}^{\varphi} \frac{\bar{D}(s)\bar{g}(s)}{s} ds}$$

such that

$$\left(\bar{D}(\varphi_{1,\bar{\beta}})\varphi'_{1,\bar{\beta}} \right) (\xi_{\bar{\beta}}^-) = \begin{cases} 0 & \text{if } c \geq \bar{c}^*, \\ \ell < 0 & \text{if } c < \bar{c}^*, \end{cases}$$

and

$$\varphi'_{1,\bar{\beta}}(\xi_{\bar{\beta}}^-) = \begin{cases} \frac{\bar{g}(\bar{\beta})}{\bar{s}_-(\bar{\beta}, c)} & \text{if } c > \bar{c}^*, \\ \frac{\bar{g}(\bar{\beta})}{\bar{s}_+(\bar{\beta}, \bar{c}^*)} & \text{if } c = \bar{c}^* \text{ and } \dot{\bar{D}}(\bar{\beta}) > 0, \\ -\infty & \text{if } c = \bar{c}^* \text{ and } \dot{\bar{D}}(\bar{\beta}) = 0 \text{ or } c < \bar{c}^*. \end{cases} \quad (4.4)$$

Here, \bar{s}_{\pm} is defined as s_{\pm} in (2.8) but with \bar{D} , \bar{g} and \bar{h} instead of the non-subscripted ones. Define $c_{n,r}^* = \bar{c}^*$. The former of the two previous formulas yields to (2.14). Analogously, (4.4) implies (4.2). Finally, (i)–(iii) follow from the application of (i)–(iii) of Proposition 4.1 to $\varphi_{1,\bar{\beta}}$ and (4.3)–(3.7). The proof is then concluded. \square

Proof of Theorem 2.2. For each $c \in \mathbb{R}$, Propositions 4.1, 4.2 provide the existence of a strict semi-wavefront $\varphi_{1,\beta}$ from 1, connecting 1 to β , and a strict semi-wavefront $\varphi_{\beta,0}$ to 0, connecting β to 0. Let c_{np}^* be as in (2.15) and take $c \geq c_{np}^*$. Proposition 4.1 informs us that $\varphi_{1,\beta}$ satisfies (2.14)₁ while Proposition 4.2 implies that $\varphi_{\beta,0}$ realizes (2.11)₁. In addition, from the uniqueness up to shifts of both $\varphi_{1,\beta}$ and $\varphi_{\beta,0}$, we can suppose that their maximal-existence intervals are one the complement of the other and that the two of them have the finite extremum at the same $\xi_{\beta} \in \mathbb{R}$. Thus, by proceeding as in the proof of Theorem 2.1, we conclude that gluing together $\varphi_{1,\beta}$ and $\varphi_{\beta,0}$ at ξ_{β} produces the desired wavefront. This concludes the *if* part of the statement.

Suppose now that φ is a profile of a wavefront connecting 1 to 0 associated to some $c \in \mathbb{R}$. Necessarily, there exists a unique $\xi_{\beta} \in \mathbb{R}$ such that $\varphi(\xi_{\beta}) = \beta$. Let φ_+ be defined by $\varphi_+(\xi) = \varphi(\xi)$, for $\xi \in (-\infty, \xi_{\beta})$. Here, the index “+” stands for the positive sign of $D(\varphi_+)$

in its domain. The function φ_+ is a semi-wavefront from 1, connecting 1 to β . Analogously, φ_- defined by $\varphi_-(\xi) = \varphi(\xi)$, for $\xi \in (\xi_\beta, +\infty)$ is a semi-wavefront to 0, connecting β to 0. The function φ is a solution of Equation (1.2), according to Definition 2.1. Thus, if $\zeta \in C_0^\infty(-\infty, +\infty)$ is a test function with $\text{supp } \zeta \subseteq [\xi_\beta - \delta, \xi_\beta + \delta]$ then we have

$$\int_{\xi_\beta - \delta}^{\xi_\beta + \delta} (D(\varphi) \varphi' - f(\varphi) + c\varphi) \zeta' - g(\varphi) \zeta d\xi = 0. \quad (4.5)$$

Observe that both φ_+ and φ_- are classical solution of (1.2) in $(-\infty, \xi_\beta)$ and $(\xi_\beta, +\infty)$, respectively, because in these domains $\pm D(\varphi_\pm) > 0$. Since $\int_{\xi_\beta - \delta}^{\xi_\beta + \delta} = \lim_{\delta > \varepsilon \rightarrow 0^+} \int_{\xi_\beta - \delta}^{\xi_\beta - \varepsilon} + \int_{\xi_\beta + \varepsilon}^{\xi_\beta + \delta}$, from (4.5) and integration by parts we obtain

$$(D(\varphi_+) \varphi'_+) \left(\xi_\beta^- \right) \zeta(\xi_\beta) = 0 \quad \text{and} \quad (D(\varphi_-) \varphi'_-) \left(\xi_\beta^+ \right) \zeta(\xi_\beta) = 0. \quad (4.6)$$

Therefore, (2.11)₁ for φ_+ and (2.14)₁ for φ_- both hold true. As a consequence, by applying Proposition 4.1 to φ_+ and Proposition 4.2 to φ_- we deduce that $c \geq c_{p,l}^*$ and $c \geq c_{n,r}^*$, from which $c \geq c_{np}^*$.

Finally, the values of $\varphi'(\xi_\beta^\pm)$ follow from each possible combination of (4.1) applied to φ_+ and (4.2) applied to φ_- . Analogously, *Parts (i)–(iii)* follow from putting together (i)–(iii) of Proposition 4.1 applied to φ_+ and (i)–(iii) of Proposition 4.2 applied to φ_- . \square

Remark 4.1. In Remark 3.3 we deduced estimates from above and below for c_{pn}^* . Bounds for c_{np}^* of Theorem 2.2 can be obtained similarly, starting now from (2.12), (2.13) and (2.15).

Remark 4.2. From (2.16), we can make explicit the *a priori* regularity of φ at ξ_β depending on the value of c and the relative order between the thresholds (see also Figure 4). We have:

- (1) if $c > c_{np}^*$, we have $\varphi'(\xi_\beta^-) = \varphi'(\xi_\beta^+) \in (-\infty, 0)$;
- (2) if $c = c_{np}^* = c_{n,r}^* = c_{p,l}^*$, we have $\varphi'(\xi_\beta^+) = \varphi'(\xi_\beta^-) \in (-\infty, 0)$ if $\dot{D}(\beta) > 0$ and $\varphi'(\xi_\beta^\pm) = -\infty$ if $\dot{D}(\beta) = 0$;
- (3) if $c = c_{np}^* = c_{n,r}^* > c_{p,l}^*$, we have $\varphi'(\xi_\beta^-) > \varphi'(\xi_\beta^+)$, with $\varphi'(\xi_\beta^-) \in (-\infty, 0)$ and $\varphi'(\xi_\beta^+) \in [-\infty, 0)$, with $\varphi'(\xi_\beta^+) = -\infty$ if and only if $\dot{D}(\beta) = 0$;
- (4) if $c = c_{np}^* = c_{p,l}^* > c_{n,r}^*$, we have $\varphi'(\xi_\beta^+) > \varphi'(\xi_\beta^-)$, with $\varphi'(\xi_\beta^+) \in (-\infty, 0)$ and $\varphi'(\xi_\beta^-) \in [-\infty, 0)$, with $\varphi'(\xi_\beta^-) = -\infty$ if and only if $\dot{D}(\beta) = 0$.

5 Examples

In this section we provide some examples about Theorems 2.1 and 2.2.

Example 5.1. This example shows that (2.9)₂ can indeed occur when a front goes from a positive- towards a negative-diffusivity region and the term f is not identically zero. Consider D , g and f defined by

$$D(\varphi) = \begin{cases} \varphi \left(\varphi - \frac{1}{2} \right)^2 & \text{if } \varphi \in [0, 1/2], \\ -(1 - \varphi) \left(\frac{1}{2} - \varphi \right)^2 & \text{if } \varphi \in (1/2, 1], \end{cases} \quad g(\varphi) = \begin{cases} \varphi & \text{if } \varphi \in [0, 1/2], \\ 1 - \varphi & \text{if } \varphi \in (1/2, 1], \end{cases}$$

and

$$f(\varphi) = \begin{cases} \frac{\varphi^3}{3} + \frac{3}{4}\varphi^2 - \frac{1}{2}\varphi & \text{if } \varphi \in [0, 1/2], \\ \frac{\varphi^3}{3} - \frac{7}{4}\varphi^2 + 2\varphi - \frac{5}{8} & \text{if } \varphi \in (1/2, 1]. \end{cases}$$

Note, with these choices, D satisfies (D_{pn}) with $\alpha = 1/2$ and $\dot{D}(\alpha) = 0$ while (g) and (f) hold for g and f . Moreover, $h(\alpha) = 1/2$. From direct inspection, the function $z(\varphi) := \varphi(\varphi - 1/2)$, for $0 \leq \varphi \leq 1/2$, satisfies $(1.4)_1$ with $c = 0 < h(\alpha)$. By integrating the formal identity $z(\varphi) = D(\varphi)\varphi'$, the profile $\varphi_{\alpha,0}$ of a semi-wavefront connecting α to 0, with speed $c = 0$, can be determined. In particular, the following Cauchy problem:

$$\varphi' = \frac{1}{\varphi - 1/2} \quad \text{and} \quad \varphi(0) = \frac{1}{4},$$

is solved by $\varphi_{\alpha,0}(\xi) := \frac{1}{2} - \sqrt{\frac{1}{16} + 2\xi}$, for any $\xi \geq -\frac{1}{32}$, and $\varphi > 0$ for $\xi < \frac{3}{32}$. Since $g(0) = 0$, by setting $\varphi_{\alpha,0}(\xi) = 0$ for $\xi \geq \frac{3}{32}$ we have that $\varphi_{\alpha,0} : (-\frac{1}{32}, +\infty) \rightarrow [0, 1/2)$ is the desired wave profile associated to the speed $c = 0$. Moreover, with the notations of Theorem 2.1, $\xi_\alpha = -\frac{1}{32}$ and $\varphi'_{\alpha,0}(\xi_\alpha^+) = -\infty$. Similar arguments lead to conclude that

$$\varphi(\xi) := \begin{cases} 1 & \text{if } \xi \leq -5/32, \\ \frac{1}{2} + \sqrt{-1/16 - 2\xi} & \text{if } -5/32 < \xi < -1/32, \\ \frac{1}{2} - \sqrt{1/16 + 2\xi} & \text{if } -1/32 \leq \xi < 3/32, \\ 0 & \text{if } \xi \geq 3/32, \end{cases}$$

is the profile of a (sharp at both 0 and 1) wavefront of (1.1) with speed $c = 0$, such that $\varphi'(\xi_\alpha^\pm) = -\infty$. Observe, from *Part (ii)* of Theorem 2.1 it must occur $c_{pn}^* = 0$.

Example 5.2. We show that c_{pn}^* in Theorem 2.1 and c_{np}^* in Theorem 2.2 are *essentially* different: opposite diffusivities do *not* produce necessarily $c_{pn}^* = c_{np}^*$. To this aim, define

$$g(\varphi) := \begin{cases} \varphi^2 & \text{if } \varphi \in [0, 1/2], \\ (1 - \varphi)^2 & \text{if } \varphi \in (1/2, 1], \end{cases} \quad f(\varphi) := \begin{cases} \varphi^2(\varphi - 1) & \text{if } \varphi \in [0, 1/2], \\ \varphi(1 - \varphi)^2 - 1/4 & \text{if } \varphi \in (1/2, 1]. \end{cases}$$

The functions g and f satisfy (g) and (f). Let D_1 and D_2 be defined by

$$D_1(\varphi) := \begin{cases} \varphi(1/2 - \varphi) & \text{if } \varphi \in [0, 1/2], \\ -(1 - \varphi)(\varphi - 1/2) & \text{if } \varphi \in (1/2, 1], \end{cases} \quad D_2(\varphi) = -D_1(\varphi) \quad \text{for } \varphi \in [0, 1].$$

With these choices, D_1 satisfies (D_{pn}) with $\alpha = 1/2$ and D_2 satisfies (D_{np}) with $\beta = 1/2$. From (2.4) and $h(0) = \dot{f}(0) = 0$ we have $c_{p,r}^* \geq 0$. Direct computations show that the function $z = z(\varphi)$ defined by

$$z(\varphi) := \varphi^2(\varphi - 1/2), \quad \varphi \in [0, 1/2], \quad (5.1)$$

solves $\dot{z} = h - D_1 g/z$ in $(0, 1/2)$, $z < 0$ in $(0, 1/2)$ and $z(0) = 0$. From the proof of [6, Theorem 2.2] we have $c_{p,r}^* \leq 0$. Then, $c_{p,r}^* = 0$ follows at once by (2.4). Also, the symmetry of the coefficients implies that $c_{n,l}^* = c_{p,r}^* = 0$. Thus,

$$c_{pn}^* = 0.$$

Observe that we actually need to involve z in (5.1) since none of the intervals given by (2.4) and (2.6), even in the sharper form involving (2.5), reduce to the point $\{0\}$.

Starting from the formal identity $\dot{z}(\varphi) = D(\varphi)\varphi'$, (5.1), we can compute the profile φ of the (unique up to shifts) wavefront associated to c_{pn}^* in the current case. We have:

$$\varphi(\xi) = \begin{cases} \frac{1}{4}e^{-\xi} & \xi > \log(1/2), \\ 1 - e^\xi & \xi \leq \log(1/2), \end{cases}$$

where in the interval $\xi \leq \log(1/2)$ we make use of the symmetry of the problem.

On the other hand, consider now D_2 , g and f together. We have

$$c_{p,l}^* \geq \max \left\{ \sup_{(\frac{1}{2}, 1]} \delta(f, 1/2), h(1/2) + 2\sqrt{\dot{D}_2(1/2)g(1/2)} \right\} > 0,$$

since $h(1/2) = -1/4$ and $2\sqrt{\dot{D}_2(1/2)g(1/2)} = 1/\sqrt{2}$. It follows necessarily that

$$c_{np}^* \geq c_{p,l}^* > 0.$$

Then, we proved that replacing D , which satisfies (D_{pn}) , with $-D$, which then satisfies (D_{np}) , one can get $c_{np}^* > c_{pn}^*$.

Example 5.3. In Theorem 2.2 the case $c_{p,l}^* \neq c_{n,r}^*$ reveals the existence of unusual *non-regular fronts*, where $\varphi = \beta$, while in the case $c_{p,l}^* = c_{n,r}^*$ this is not possible. We now show an example in either case.

First, assume that D and g satisfy (D_{np}) -(g) and are such that Dg is convex in $(0, \beta)$ and concave in $(\beta, 1)$. For instance, we can take $\beta = \frac{1}{2}$, $D(\varphi) = \varphi - 1/2$ and $g(\varphi) = \varphi(1 - \varphi)$. We plainly have

$$\sup_{\varphi \in [0, \beta]} \frac{1}{\beta - \varphi} \int_{\varphi}^{\beta} \frac{(Dg)(s)}{s - \beta} ds = \dot{D}(\beta)g(\beta) \quad \text{and} \quad \sup_{\varphi \in (\beta, 1]} \frac{1}{\varphi - \beta} \int_{\beta}^{\varphi} \frac{(Dg)(s)}{s - \beta} ds = \dot{D}(\beta)g(\beta).$$

Suppose also that $f = 0$ in $[0, 1]$. Under these assumptions, inequalities (2.12) and (2.13) become indeed equalities:

$$c_{n,r}^* = c_{p,l}^* = c_{np}^* = 2\sqrt{\dot{D}(\beta)g(\beta)}.$$

Second, consider $D \in C^1[0, 1]$ such that $D < 0$ in $(0, 1/2)$ and $D(\varphi) = (\varphi - 1/2)^3$, for $\varphi \in [1/2, 1]$; assume that g satisfies (g) with $g(\varphi) = 1 - \varphi$, for $\varphi \in [1/2, 1]$, and let f be defined by

$$f(\varphi) := \begin{cases} 0 & \text{if } \varphi \in [0, \frac{1}{2}), \\ \left(\varphi - \frac{1}{2}\right)^2 \left(\varphi - \frac{3}{2}\right) & \text{if } \varphi \in [\frac{1}{2}, 1]. \end{cases}$$

For $1/2 \leq \varphi \leq 1$, set

$$z(\varphi) := \left(\varphi - \frac{1}{2}\right)^2 (\varphi - 1).$$

Such a z solves (1.4)₂ with $\alpha = 1/2$ and $c = 0$ in $(1/2, 1)$. Moreover, since it also holds $z(1/2) = 0$, then $0 \geq c_{p,l}^*$ (see e.g. the proof of [6, Theorem 2.2]). The left-hand side of (2.12) implies that $c_{p,l}^* \geq 0$, because $h(1/2) = \dot{f}(1/2) = 0$. Thus, $c_{p,l}^* = 0$. On the other hand, if we have $h = 0$ constantly in $(0, 1/2)$, then $c_{n,r}^* > 0$ (as already observed in Introduction, for the case of non-negative D we refer to [6, Remark 6.4] and reference therein; the case when D is negative is treated similarly). Thus, in this case we have

$$c_{np}^* = c_{n,r}^* > c_{p,l}^*.$$

Similarly, one can provide examples where $c_{np}^* = c_{p,l}^* > c_{n,r}^*$.

6 The case when D vanishes more than once

In this final section we briefly outline how to cope with the case when D changes sign more than once, focusing on the simplest of these cases. More precisely we consider one of the following assumptions, see Figure 8:

(D_{pnp}) $D \in C^1[0, 1]$, $D > 0$ in $(0, \alpha) \cup (\beta, 1)$ and $D < 0$ in (α, β) , with $\alpha < \beta$;

(D_{npn}) $D \in C^1[0, 1]$, $D < 0$ in $(0, \beta) \cup (\alpha, 1)$ and $D > 0$ in (β, α) , with $\beta < \alpha$.

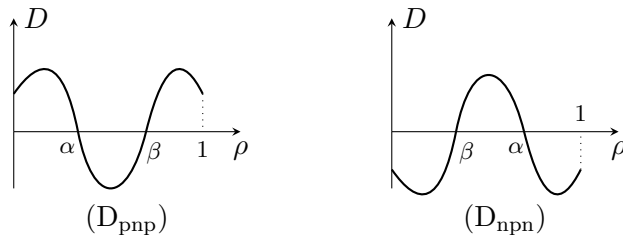


Figure 8: Typical plots of the functions D .

About the case (D_{pnp}) we refer to [14, 20], where g is, respectively, monostable and bistable; [3] for the case (D_{pnp}) and monostable g , but with a specific quadratic diffusivity D . The case (D_{npn}) has never been considered.

Theorem 6.1. *Assume (f), (D_{pnp}), (g) and (2.2)₁. Then, there exists $c_{pnp}^* \in \mathbb{R}$ such that Equation (1.1) admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying (1.3), if and only if $c \geq c_{pnp}^*$.*

Theorem 6.2. *Assume (f), (D_{npn}), (g) and (2.2)₂. Then, there exists $c_{npn}^* \in \mathbb{R}$ such that Equation (1.1) admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying (1.3), if and only if $c \geq c_{npn}^*$.*

In both theorems one can easily deduce more informations on the profiles, as in Theorems 2.1 and 2.2. We leave these details to the reader.

Proof of Theorem 6.1. We divide our problem in three sub-problems corresponding to the three connected components of $\{D \neq 0\}$. To the intervals $(0, \alpha)$ and $(\beta, 1)$ we apply

Propositions 3.1 and 4.1, respectively. Indeed, the results of Sections 3 and 4 hold under lighter assumptions, involving only that part of the conditions corresponding to the interval under consideration.

Let $c_{p,r}^*$ and $c_{p,l}^*$ be the thresholds appearing in Propositions 3.1 and 4.1, respectively. Hence, associated to the same speed c , both a semi-wavefront connecting α and 0, with profile $\varphi_{\alpha,0}$ satisfying (3.10) and a semi-wavefront connecting β to 1, whose profile is $\varphi_{1,\beta}$ satisfying (2.11)₁, are given, if and only if $c \geq \max\{c_{p,r}^*, c_{p,l}^*\}$.

We claim that there exists $c_{\alpha\beta}^* \in \mathbb{R}$ such that the following holds: if and only if $c \geq c_{\alpha\beta}^*$, Equation (1.1) admits a strict TW, connecting β to α , whose non-increasing profile $\varphi_{\beta,\alpha} \in (\alpha, \beta)$ is defined in some interval (ξ_β, ξ_α) , where $-\infty < \xi_\beta < \xi_\alpha < +\infty$ are such that $\varphi_{\beta,\alpha}(\xi_\beta^+) = \beta$ and $\varphi_{\beta,\alpha}(\xi_\alpha^-) = \alpha$ and also

$$\left(D(\varphi_{\beta,\alpha})\varphi'_{\beta,\alpha}\right)(\xi_\beta^+) = 0 = \left(D(\varphi_{\beta,\alpha})\varphi'_{\beta,\alpha}\right)(\xi_\alpha^-).$$

Then by gluing together $\varphi_{1,\beta}$, $\varphi_{\beta,\alpha}$ and $\varphi_{\alpha,0}$ (modulo shifts) we obtain the front of (1.1) satisfying (1.3), in virtue of (6), as in the proof of Theorem 2.1. Clearly, we define

$$c_{pnp}^* = \max\left\{c_{p,r}^*, c_{p,l}^*, c_{\alpha\beta}^*\right\}.$$

This proves the *if* part of Theorem 6.1. Viceversa, assume that a wavefront φ of (1.1), associated to some speed c , is given. As in the proof of the *only if* parts of Theorems 2.1 and 2.2, φ is decomposed into a semi-wavefront which connects 1 to β whose profile satisfies (2.11), a strict TW connecting β to α and a strict semi-wavefront connecting α to 0. By Propositions 3.1 and 3.2, the *only if* part of our claim implies $c \geq c_{pnp}^*$. Thus, Theorem 6.1 follows from the claim. We refer to Figure 9.

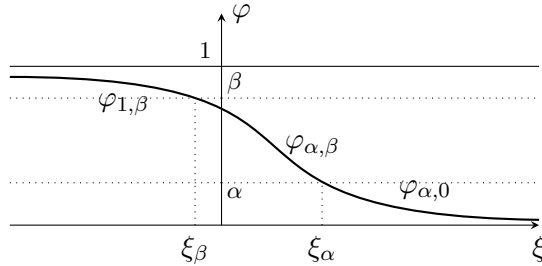


Figure 9: The pasting of the profiles.

We prove the claim. Let \bar{D} , \bar{g} , \bar{f} and $\bar{\alpha}$ be as in (3.7) and let \bar{h} be the derivative of \bar{f} , which gives $\bar{h}(\varphi) = h(1 - \varphi)$. Set $\bar{\beta} = 1 - \beta$. Let c be a real value and consider the following equation in the unknown $\psi = \psi(\xi) \in [\bar{\beta}, \bar{\alpha}]$:

$$\left(\bar{D}(\psi)\psi'\right)' + (c - \bar{h}(\psi))\psi' + \bar{g}(\psi) = 0. \quad (6.1)$$

We are interested in a non-increasing ψ satisfying (6.1) in some interval $(\xi_{\bar{\alpha}}, \xi_{\bar{\beta}})$, with $-\infty < \xi_{\bar{\alpha}} < \xi_{\bar{\beta}} < +\infty$, such that $\psi(\xi_{\bar{\alpha}}^+) = \bar{\alpha}$, $\psi(\xi_{\bar{\beta}}^-) = \bar{\beta}$ and

$$\left(\bar{D}(\psi)\psi'\right)(\xi_{\bar{\alpha}}^+) = 0 = \left(\bar{D}(\psi)\psi'\right)(\xi_{\bar{\beta}}^-).$$

By using a standard argument (see e.g. [24]) based on the monotonicity of ψ and the definition of $z(\psi) := \bar{D}(\psi)\psi'$, the existence of a ψ as above, but with not necessarily finite $\xi_{\bar{\alpha}}$ and $\xi_{\bar{\beta}}$, is equivalent to the solvability of the problem

$$\begin{cases} \dot{z}(\psi) = \bar{h}(\psi) - c - \frac{\bar{D}(\psi)\bar{g}(\psi)}{z(\psi)}, & \psi \in (\bar{\beta}, \bar{\alpha}), \\ z(\psi) < 0 & \psi \in (\bar{\beta}, \bar{\alpha}), \\ z(\bar{\beta}) = z(\bar{\alpha}) = 0. \end{cases}$$

Since $\bar{D} > 0$ in $(\bar{\beta}, \bar{\alpha})$, $\bar{D}(\bar{\beta}) = \bar{D}(\bar{\alpha}) = 0$, $\bar{g} > 0$ in $[\bar{\beta}, \bar{\alpha}]$, by applying [6, Proposition 4.2], such a problem is solvable if and only if $c \geq c^*$, for some $c^* \in \mathbb{R}$. Also, since (by arguing as in [6, Lemma 9.1] and [10, proof of Theorem 2.5] where analogous assumptions were considered) we have

$$\lim_{\psi \rightarrow \bar{\alpha}^-} \frac{\bar{D}(\psi)}{z(\psi)} \in \mathbb{R} \quad \text{and} \quad \lim_{\psi \rightarrow \bar{\beta}^+} \frac{\bar{D}(\psi)}{z(\psi)} \in \mathbb{R},$$

then $\xi_{\bar{\alpha}}$ and $\xi_{\bar{\beta}}$ are finite because ψ is bounded and $\psi'(\xi) < 0$ for ξ which tends to either $\xi_{\bar{\alpha}}^+$ or $\xi_{\bar{\beta}}^-$. Finally, setting $c_{\alpha\beta}^* := c^*$ concludes the proof of the claim since the desired $\varphi_{\beta,\alpha}$ exists if and only if there exists ψ , by $\varphi_{\beta,\alpha}(\xi) := 1 - \psi(-\xi)$. \square

Remark 6.1. The analog of Theorem 6.1 when $f = 0$ was given in the first part of [14, Theorem 4.2] with slightly different notation. We point out that in [14] wavefronts are necessarily classical at 1 (called classical or sharp of type (I), there). Instead, when $D(1) = 0$ and $c < h(1)$ the wavefronts of Theorem 6.1 can be sharp at 1, too, by applying Part (iii) of Proposition 4.1 to $\varphi_{1,\beta}$ of the proof of Theorem 6.1. Note, $c < h(1)$ cannot happen if $f = 0$.

We also observe that, as in [14, Theorem 4.2], wavefronts are sharp at 0 only if $D(0) = 0$ and $c = c_{np}^*$. Moreover, if $D(0) = 0$ and $c = c_{np}^*$, generalizing [14, (4.1)], it is possible to provide a sufficient condition on f and Dg in order to make wavefronts of Theorem 6.1 always classical at 0.

Proof of Theorem 6.2. We proceed in the spirit of the proof of Theorem 6.1. We consider separately the intervals where D has constant sign. From Proposition 3.2 we deduce that a semi-wavefront connecting 1 to α , with speed c and profile satisfying (3.5) with $s = 0$, exists if and only if $c \geq c_{n,l}^*$. Proposition 4.2 implies that there exists a semi-wavefront connecting β to 0 with speed c and profile satisfying (2.14)₁ if and only if $c \geq c_{n,r}^*$. About the interval (β, α) , we argue as in the proof of Theorem 6.1, but considering Equation (1.2), directly, rather than Equation (6.1). We have that a strict TW connecting α to β with speed c and profile $\varphi_{\alpha,\beta} : (\xi_\alpha, \xi_\beta) \rightarrow (\beta, \alpha)$ satisfying $\varphi_{\alpha,\beta}(\xi_\alpha^+) = \alpha$, $\varphi_{\alpha,\beta}(\xi_\beta^-) = \beta$ and such that $(D(\varphi_{\alpha,\beta})\varphi_{\alpha,\beta})'(\xi)$ tends to 0 if either $\xi \rightarrow \xi_\alpha^+$ and $\xi \rightarrow \xi_\beta^-$, exists if and only if $c \geq c_{\beta\alpha}^*$. To conclude the proof, we set $c_{npn}^* := \max\{c_{n,l}^*, c_{n,r}^*, c_{\beta\alpha}^*\}$. \square

Remark 6.2. Similarly to Remark 6.1, we can deduce information on the regularity of wavefronts given in Theorem 6.2.

Acknowledgments

The authors are members of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM) and acknowledge financial support from this institution.

References

- [1] D. G. Aronson, H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.
- [2] L. Bao, Z. Zhou. Traveling wave in backward and forward parabolic equations from population dynamics. *Discrete Contin. Dyn. Syst. Ser. B*, 19(6):1507–1522, 2014.
- [3] L. Bao, Z. Zhou. Traveling wave solutions for a one dimensional model of cell-to-cell adhesion and diffusion with monostable reaction term. *Discrete Contin. Dyn. Syst. Ser. S*, 10(3):395–412, 2017.
- [4] N. Bellomo, M. Delitala, V. Coscia. On the mathematical theory of vehicular traffic flow. I. Fluid dynamic and kinetic modelling. *Math. Models Methods Appl. Sci.*, 12(12):1801–1843, 2002.
- [5] N. Bellomo, C. Dogbe. On the modeling of traffic and crowds: a survey of models, speculations, and perspectives. *SIAM Rev.*, 53(3):409–463, 2011.
- [6] D. Berti, A. Corli, L. Malaguti. Uniqueness and nonuniqueness of fronts for degenerate diffusion-convection reaction equations. *Electron. J. Qual. Theory Differ. Equ.*, Paper No. 66, 34, 2020.
- [7] L. Bruno, A. Tosin, P. Tricerri, F. Venuti. Non-local first-order modelling of crowd dynamics: a multidimensional framework with applications. *Appl. Math. Model.*, 35(1):426–445, 2011.
- [8] A. Corli, L. di Ruvo, L. Malaguti. Sharp profiles in models of collective movements. *NoDEA Nonlinear Differential Equations Appl.*, 24(4):Paper No. 40, 31 pp., 2017.
- [9] A. Corli, L. di Ruvo, L. Malaguti, M. D. Rosini. Traveling waves for degenerate diffusive equations on networks. *Netw. Heterog. Media*, 12(3):339–370, 2017.
- [10] A. Corli, L. Malaguti. Semi-wavefront solutions in models of collective movements with density-dependent diffusivity. *Dyn. Partial Differ. Equ.*, 13(4):297–331, 2016.
- [11] A. Corli, L. Malaguti. Viscous profiles in models of collective movement with negative diffusivity. *Z. Angew. Math. Phys.*, 70(2):Art. 47, 22 pp., 2019.
- [12] A. Corli, L. Malaguti. Wavefronts in traffic flows and crowds dynamics. *Lecture Notes INDAM*, to appear, 2020.
- [13] D. A. DiCarlo, R. Juanes, L. Tara, T. P. Witelski. Nonmonotonic traveling wave solutions of infiltration into porous media. *Water Resources Res.*, 44:1–12, 2008.
- [14] L. Ferracuti, C. Marcelli, F. Papalini. Travelling waves in some reaction-diffusion-aggregation models. *Adv. Dyn. Syst. Appl.*, 4(1):19–33, 2009.
- [15] M. Garavello, K. Han, B. Piccoli. *Models for vehicular traffic on networks*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2016.
- [16] M. Garavello, B. Piccoli. *Traffic flow on networks*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.
- [17] B. H. Gilding, R. Kersner. *Travelling waves in nonlinear diffusion-convection reaction*. Birkhäuser Verlag, Basel, 2004.

- [18] D. Horstmann, K. J. Painter, H. G. Othmer. Aggregation under local reinforcement: from lattice to continuum. *European J. Appl. Math.*, 15(5):546–576, 2004.
- [19] B. S. Kerner, V. V. Osipov. *Autosolitons*. Kluwer Academic Publishers Group, Dordrecht, 1994.
- [20] M. Kuzmin, S. Ruggerini. Front propagation in diffusion-aggregation models with bi-stable reaction. *Discrete Contin. Dyn. Syst. Ser. B*, 16(3):819–833, 2011.
- [21] P. K. Maini, L. Malaguti, C. Marcelli, S. Matucci. Diffusion-aggregation processes with monostable reaction terms. *Discrete Contin. Dyn. Syst. Ser. B*, 6(5):1175–1189, 2006.
- [22] P. K. Maini, L. Malaguti, C. Marcelli, S. Matucci. Aggregative movement and front propagation for bi-stable population models. *Math. Models Methods Appl. Sci.*, 17(9):1351–1368, 2007.
- [23] L. Malaguti, C. Marcelli. Sharp profiles in degenerate and doubly degenerate Fisher-KPP equations. *J. Differential Equations*, 195(2):471–496, 2003.
- [24] L. Malaguti, C. Marcelli. Finite speed of propagation in monostable degenerate reaction-diffusion-convection equations. *Adv. Nonlinear Stud.*, 5(2):223–252, 2005.
- [25] C. Marcelli, F. Papalini. A new estimate of the minimal wave speed for travelling fronts in reaction-diffusion-convection equations. *Electron. J. Qual. Theory Differ. Equ.*, 10:1–13, 2018.
- [26] P. Nelson. Synchronized traffic flow from a modified Lighthill-Whitham model. *Phys. Review E*, 61:R6052–R6055, 2000.
- [27] V. Padrón. Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation. *Trans. Amer. Math. Soc.*, 356(7):2739–2756, 2004.
- [28] M. D. Rosini. *Macroscopic models for vehicular flows and crowd dynamics: theory and applications*. Springer, Heidelberg, 2013.