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Research paper

Instantaneous Kinematics and Singularity Analysis of Spatial Multi-DOF Mechanisms Based on the Locations of the Instantaneous Screw Axes

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ABSTRACT

Multi-degree-of-freedom (multi-DOF) mechanisms generate single-DOF mechanisms by locking all their generalized coordinates but one. Here, the superposition principle is used to state a relationship between spatial multi-DOF mechanisms' instantaneous kinematics (IK) and the IK of the single-DOF mechanisms they generate. Firstly, the relationship between the instantaneous screw axes (ISAs) of a multi-DOF mechanism and the ISAs of the single-DOF mechanisms, it generates, is found; then, it is used for its singularity analysis. In particular, the IK model of a generic multi-DOF spatial mechanism is written through the ISA locations and, successively, it is studied to identify all the singular configurations of the multi-DOF mechanism through the analysis of the single-DOF mechanisms it generates. The results are a technique for the determination of ISAs' locations in multi-DOF spatial mechanisms and a singularity-analysis technique, for the same mechanism types, based on the singularity analysis of single-DOF spatial mechanisms. Eventually, the proposed techniques are applied to a case study. As far as this author is aware, both these results are presented for the first time in the literature.

1. Introduction

Mechanisms' instantaneous kinematics (IK) is ruled by a linear model [1,2], named instantaneous input-output relationship (IOR), that is homogeneous both in the actuated-joint rates (inputs of the model) and in the components of the output-link twist (outputs of the model) provided that the mechanism constraints are time-independent (scleronomic) constraints that are either holonomic constraints or non-holonomic constraints whose dependence on motion-variables' rates is linear (i.e, they are first-order non-holonomic constraints) [3]. Since the vast majority of mechanisms satisfies these conditions usually, if it is not explicitly specified, the term "mechanism" refers to a mechanism with scleronomic constraints that are either holonomic or first-order non-holonomic; this non-declared convention is adopted in the present paper, too.

Any physical phenomenon that is modelled by a linear and homogeneous system of equations can be studied by means of the superposition principle [4]. In the context of mechanisms' IK, such a principle allows the computation of the output-link twist of a multi-degree-of-freedom (DOF) mechanism by summing up the effects, on the same twist, obtained by actuating one actuator at a time. Since a multi-DOF mechanism with all the actuated joints locked, but one, is a single-DOF mechanism, each term of this summation can be computed by using techniques that are specifically conceived for single-DOF mechanisms. Such techniques can take advantage from

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Nomenclature

DOF	degree of freedom
Hel	helical motion
Tra	translational motion
IFKP	instantaneous forward-kinematics problem
IIKP	instantaneous inverse-kinematics problem
IK	instantaneous kinematics
IOR	instantaneous input-output relationship
ISA	instantaneous screw axis
l	DOF number of the multi-DOF mechanism
a_k	coefficient that multiplies the input rate in the IOR (i.e., Eq. (1)) of the k-th single-DOF mechanism generated from an l-DOF mechanism
b_k	coefficient that multiplies the output rate in the IOR (i.e., Eq. (1)) of the k-th single-DOF mechanism generated from an l-DOF mechanism
q_k	generalized coordinate (input variable) of the k-th single-DOF mechanism and, at the same time, k-th generalized coordinate of the l-DOF mechanism with k=1, …, l
s_k	output (secondary) variable of the k-th single-DOF mechanism
ISA_{ji}	instantaneous screw axis (Fig. 1) of an Hel relative motion between links j and i in the l-DOF mechanism
$\mathrm{ISA}_{ji}^{(\mathrm{k})}$	instantaneous screw axis (Fig. 1) of an Hel relative motion between links j and i in the k-th single-DOF mechanism
A_{ji}	a point lying on $\mathrm{ISA}_{\mathrm{ji}}$
$\mathrm{A}_{ji}^{(\mathrm{k})}$	a point lying on $\mathrm{ISA}_{ji}^{(\mathrm{k})}$
$\mathbf{u}_{ji}$	unit vector parallel to $\mathrm{ISA}_{\mathrm{ji}}$
$\mathbf{u}_{ji}^{(\mathrm{k})}$	unit vector parallel to $\mathrm{ISA}_{ji}^{(\mathrm{k})}$
p_{ji}	pitch of an Hel relative motion between links j and i in the l-DOF mechanism
$p_{ji}^{(\mathrm{k})}$	pitch of an Hel relative motion between links j and i in the k-th single-DOF mechanism
$\omega_{ji}\mathbf{u}_{ji}$	angular velocity of an Hel relative motion between links j and i in the l-DOF mechanism; ω_{ji} is its signed magnitude
$\omega_{ji}^{(\mathrm{k})}\mathbf{u}_{ji}^{(\mathrm{k})}$	angular velocity of an Hel relative motion between links j and i in the k-th single-DOF mechanism; $\omega_{ji}^{(\mathrm{k})}$ is its signed magnitude
$\boldsymbol{\tau}_{ji}$	unit vector parallel to the translation direction of a Tra relative motion between links j and i in the l-DOF mechanism
$\boldsymbol{\tau}_{ji}^{(\mathrm{r})}$	unit vector parallel to the translation direction of a Tra relative motion between links j and i in the r-the single-DOF mechanism
$\nu_{ji}\boldsymbol{\tau}_{ji}$	translation velocity of a Tra relative motion between links j and i in the l-DOF mechanism; ν_{ji} is its signed magnitude
$\nu_{ji}^{(\mathrm{r})}\boldsymbol{\tau}_{ji}^{(\mathrm{r})}$	translation velocity of a Tra relative motion between links j and i in the r-th single-DOF mechanism; $\nu_{ji}^{(\mathrm{r})}$ is its signed magnitude
(P, **w**)	line passing through point P and parallel to unit vector **w**; according to these notations, (A_{ji}, $\mathbf{u}_{ji}$) is another way to denote ISA_{ji}
P	position vector locating the generic point P in a Cartesian reference
${}^{i}\mathbf{v}_{P\vert j}$	velocity of point P in the ji relative motion (i.e., with the point fixed to link j and the velocity measured from link i) of the l-DOF mechanism
${}^{i}\mathbf{v}_{P\vert j}^{(\mathrm{k})}$	velocity of point P in the ji relative motion (i.e., with the point fixed to link j and the velocity measured from link i) of the k-th single-DOF mechanism
$\mathbf{n}_{ijg}^{(\mathrm{k})}$	$\left(=\frac{\mathbf{u}_{ji}^{(\mathrm{k})}\times\mathbf{u}_{jg}^{(\mathrm{k})}}{\Vert\mathbf{u}_{ji}^{(\mathrm{k})}\times\mathbf{u}_{jg}^{(\mathrm{k})}\Vert}\right)$ unit vector of the common normal to $\mathrm{ISA}_{ji}^{(\mathrm{k})}$ and $\mathrm{ISA}_{jg}^{(\mathrm{k})}$ (Fig. 2)
$y_{ji,jg}^{(\mathrm{k})}$	$(=(Q_{jg}^{(\mathrm{k})}-Q_{ji}^{(\mathrm{k})})\cdot\mathbf{n}_{ijg}^{(\mathrm{k})}=(A_{jg}^{(\mathrm{k})}-A_{ji}^{(\mathrm{k})})\cdot\mathbf{n}_{ijg}^{(\mathrm{k})})$ signed distance between $\mathrm{ISA}_{ji}^{(\mathrm{k})}$ and $\mathrm{ISA}_{jg}^{(\mathrm{k})}$ (Fig. 2)
${}^{i}\widehat{\$}_{P\vert j}$	$\left(=\begin{pmatrix}\mathbf{u}_{ji}\\ p_{ji}\mathbf{u}_{ji}+(A_{ji}-P)\times\mathbf{u}_{ji}\end{pmatrix}\right)$ unit screw of the ji relative motion in the l-DOF mechanism when referred to point P
${}^{i}\widehat{\$}_{P\vert j}^{(\mathrm{k})}$	$\left(=\begin{pmatrix}\mathbf{u}_{ji}^{(\mathrm{k})}\\ p_{ji}^{(\mathrm{k})}\mathbf{u}_{ji}^{(\mathrm{k})}+(A_{ji}^{(\mathrm{k})}-P)\times\mathbf{u}_{ji}^{(\mathrm{k})}\end{pmatrix}\right)$ unit screw of the ji relative motion in the k-th single-DOF mechanism when referred to point P

the fact that, in single-DOF mechanisms, the locations of the instantaneous screw axes (ISAs) depend only on the mechanism configuration, which provides a geometric representation of the instantaneous motion of the mechanism. In the literature, a number of algorithms that directly [5] or indirectly [6–10] determine the ISA locations and relate them to the mechanism configuration have been presented for single-DOF spatial mechanisms.

The ISA location of the output-link motion together with the associated pitch fully describes the instantaneous motion of the same link and, when related to the mechanism configuration, becomes a geometric tool to study mechanism's IK. Indeed, for instance, such a relationship allows to visualize (e.g., on a CAD system) how the ISA moves during the mechanism motion, which is a graphic information of great interest during mechanism design, especially if it can be easily updated to take into account geometric variations of mechanism's links [11,12].

The analytic/geometric relationship between the ISA location of the output-link motion in a multi-DOF mechanism and the locations of the same ISA in the single-DOF mechanisms generated from it by locking all the actuated joints, but one, gives the connection between such an ISA and the configuration of the multi-DOF mechanism. Therefore, finding this relationship is relevant for an in-depth IK analysis of multi-DOF spatial mechanisms. Moreover, such a relationship is indeed an alternative expression of the IOR of the multi-DOF spatial mechanism and can be used for its singularity analysis, too.

By considering mechanisms as input-output systems [2,13–16], singularity analysis is definable as the determination of the mechanism configurations (singularities) that make mechanism's IOR fail in stating a one-to-one correspondence between actuated-joint rates and output-link twist. This definition allows the identification of three main types of singularities [13]: (i) those, usually named "serial singularities", where the actuated-joint rates are not uniquely determinable for an arbitrarily assigned output-link twist, (ii) those, usually named "parallel singularities", where the output-link twist is not uniquely determinable for arbitrarily assigned actuated-joint rates, and (iii) those where both the above-defined conditions occur. According to these definitions, the output link cannot move along any direction when starting from a serial singularity (i.e., it has a reduction of its instantaneous DOFs), which brings the conclusion that the serial-singularity locus coincides with output-link's workspace boundaries; vice versa, when starting from a parallel singularity, the output link can have extra instantaneous motions that are not controllable by the actuators, which, if the mechanism has less than six DOF, may also imply that the output link gains extra DOFs and changes its motion type [15,16]. Focusing on the difference between the instantaneous mobility (elementary DOFs) and its "full-cycle" mobility (finite DOFs), Hunt [17,18] identified "uncertainty" configurations, where the elementary DOFs are more than the finite DOFs, and "stationary" configurations, where the elementary DOFs are less than the finite DOFs. Roughly speaking (see [18] pp. 302-303), stationary and uncertainty configurations correspond to the above-defined serial and parallel singularities, respectively.

The duality between IK and statics, which the virtual work principle states [19], allows the demonstration that, at a serial singularity, some loads that the output link applies along some directions to the external world are not controllable through the actuators and might be even infinite; whereas, at a parallel singularity, small (even elementary) external loads applied to the output link along some directions need infinite generalized torques in some of the actuated joints to be equilibrated. As a consequence, singularities must be identified during design and avoided during functioning, that is, singularity analysis is a mandatory step to implement at the design stage, which justifies the ample literature (see, for instance, Refs. [20–31]) on mechanisms' singularity analysis.

In [32], for single-DOF spatial mechanisms, this author has provided the exhaustive enumeration both of the explicit expressions of all the possible IORs through their ISA locations and of the geometric/analytic conditions on ISAs that occur at mechanism's singular configurations (singularities). In the present paper, the two enumerations deduced in [32] are exploited, firstly, to determine the general explicit expression of the above-mentioned "alternative" IOR of a multi-DOF spatial mechanism, which allows the determination of ISAs' locations in a multi-DOF mechanism through the locations of the same ISAs in the single-DOF mechanisms generated from it, and, then, to relate singularity conditions of multi-DOF spatial mechanisms to the singularities of the single-DOF spatial mechanisms generated from them. Eventually, the effectiveness of the deduced relationships and singularity conditions is proved by applying them to a case study.

As far as this author is aware, both these results are presented for the first time in the literature. They, over providing a deeper comprehension of the IK of multi-DOF spatial mechanisms, are also of practical interest during mechanism design since they give geometric representations needed for designers.

The present paper is organized as follows. After having reminded the necessary background materials and introduced the adopted notations, section 2, firstly, addresses the IOR deduction and the ISA determination, and, then, presents the novel singularity-analysis

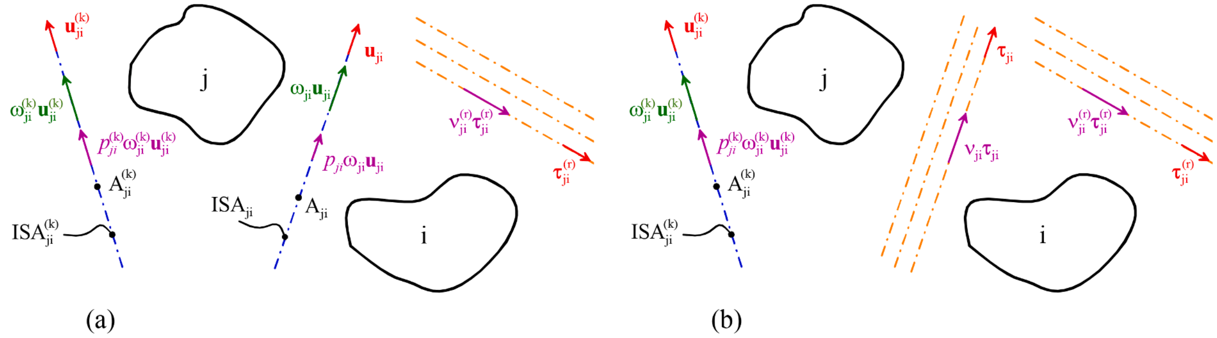

Figure 1. Two generic links of a l-DOF spatial mechanism with link indices i and j whose relative motion is (Hel and Tra stand for helical motion and translational motion, respectively): (a) Hel in the l-DOF mechanism and in the k-th single-DOF mechanism, but Tra in the r-th single-DOF mechanism, and (b) Tra in the l-DOF mechanism and in the r-th single-DOF mechanism, but Hel in the k-th single-DOF mechanism.

technique based on these tools. Section 3 illustrates the application of the proposed methodology to a case study. Section 4 discusses the obtained results and, eventually, section 5 draws the conclusions.

2. Materials and Methods

Hereafter, $l\in\{\mathbb{N}|\ l>1\}$ and $\mathbf{q}=(q_1,\cdots,q_l)^T$ will denote the DOF number and the l-tuple collecting all the generalized coordinates, respectively, of a generic l-DOF mechanism with m links and the phrase "n-th single-DOF mechanism" will refer to the single-DOF mechanism generated from the l-DOF mechanism by locking all the generalized coordinates but q_n. Fig. 1 shows two generic links of a l-DOF spatial mechanism with link indices i and j; in Fig. 1a, their relative motion is a helical motion (Hel) in the l-DOF mechanism and in the k-th single-DOF mechanism, whereas, it is a translational motion (Tra) in the r-th single-DOF mechanism; differently, in Fig. 1b, their relative motion is Tra in the l-DOF mechanism and in the r-th single-DOF mechanism, whereas, it is Hel in the k-th single-DOF mechanism.

With reference to Fig. 1a, ISA$_{ji}$ (ISA$_{ji}^{(\mathrm{k})}$), p_{ji} ($p_{ji}^{(\mathrm{k})}$), and ω_{ji} ($\omega_{ji}^{(\mathrm{k})}$) denote the instantaneous screw axis, the pitch and the angular-velocity's signed magnitude, respectively, of the motion of link j with respect to link i, hereafter named ji relative motion, in the l-DOF mechanism (in the k-th single-DOF mechanism). A$_{ji}$ (A$_{ji}^{(\mathrm{k})}$), $\mathbf{u}_{ji}$ ($\mathbf{u}_{ji}^{(\mathrm{k})}$) and $\omega_{ji}\mathbf{u}_{ji}$ ($\omega_{ji}^{(\mathrm{k})}\mathbf{u}_{ji}^{(\mathrm{k})}$) denote a point of ISA$_{ji}$ (ISA$_{ji}^{(\mathrm{k})}$), a unit vector parallel to ISA$_{ji}$ (ISA$_{ji}^{(\mathrm{k})}$), and the angular velocity of the ji relative motion, respectively, in the l-DOF mechanism (in the k-th single-DOF mechanism).

With reference to Fig. 1b, $\boldsymbol{\tau}_{ji}$ ($\boldsymbol{\tau}_{ji}^{(\mathrm{r})}$), ν_{ji} ($\nu_{ji}^{(\mathrm{r})}$), and $\nu_{ji}\boldsymbol{\tau}_{ji}$ ($\nu_{ji}^{(\mathrm{r})}\boldsymbol{\tau}_{ji}^{(\mathrm{r})}$) are a unit vector parallel to the translation velocity of the ji relative motion, the signed magnitude of the same velocity, and the translation velocity of the ji relative motion, respectively, in the l-DOF mechanism (in the r-th single-DOF mechanism).

From now on, the ordered couple (P, $\mathbf{w}$), where P is a point and $\mathbf{w}$ is a unit vector, will denote the line parallel to $\mathbf{w}$ and passing through point P (e.g., (A$_{ji}$, $\mathbf{u}_{ji}$) is an alternative way for denoting ISA$_{ji}$) and the pure rotation, which occurs when $\omega_{ji}\neq 0$ and $p_{ji}=0$ in the l-DOF mechanism ($\omega_{ji}^{(\mathrm{k})}\neq 0$ and $p_{ji}^{(\mathrm{k})}=0$ in the k-th single-DOF mechanism), will be considered only a special case of Hel (i.e., the one with $p_{ji}=0$ ($p_{ji}^{(\mathrm{k})}=0$)) and dealt with the formulas deduced for the Hel case. With these notations, there are only two types of ji relative motion (i.e., Hel and Tra) and, for the relative motion theorems [33], the following relationships hold

(a) if $\omega_{ji}\neq 0$ (Hel in the l-DOF mechanism), then A$_{ji}$=A$_{ij}$, $\mathbf{u}_{ji}=\mathbf{u}_{ij}$, $p_{ji}=p_{ij}$ and $\omega_{ji}=-\omega_{ij}$;
(b) if $\omega_{ji}^{(\mathrm{k})}\neq 0$ (Hel in the k-th single-DOF mechanism), then $\mathrm{A}_{ji}^{(\mathrm{k})}=\mathrm{A}_{ij}^{(\mathrm{k})}$, $\mathbf{u}_{ji}^{(\mathrm{k})}=\mathbf{u}_{ij}^{(\mathrm{k})}$, $p_{ji}^{(\mathrm{k})}=p_{ij}^{(\mathrm{k})}$ and $\omega_{ji}^{(\mathrm{k})}=-\omega_{ij}^{(\mathrm{k})}$;
(c) if $\omega_{ji}=0$ (Tra in the l-DOF mechanism), then $\boldsymbol{\tau}_{ji}=\boldsymbol{\tau}_{ij}$ and $\nu_{ji}=-\nu_{ij}$;
(d) if $\omega_{ji}^{(\mathrm{k})}=0$ (Tra in the k-th single-DOF mechanism), then $\boldsymbol{\tau}_{ji}^{(\mathrm{k})}=\boldsymbol{\tau}_{ij}^{(\mathrm{k})}$ and $\nu_{ji}^{(\mathrm{k})}=-\nu_{ij}^{(\mathrm{k})}$.

The IOR of the k-th single-DOF mechanism generated from an l-DOF mechanism can be written as follows [30]:

Table 1
Coefficients a_k and b_k of Eq. (1) and geometric/kinematic conditions on ISAs that are valid when the motion of both the input, i, and the output, o, links are referred to the same reference link, f.

Case	$\dot{q}_k$	$\dot{s}_k$	a_k	b_k	Geometric & Kinematic Conditions on ISAs
Hel-Hel(*)	$\omega_{if}^{(\mathrm{k})}$	$\omega_{of}^{(\mathrm{k})}$	$\mathbf{n}_{ofi}^{(\mathrm{k})}\cdot(\mathbf{u}_{fi}^{(\mathrm{k})}\times\mathbf{u}_{oi}^{(\mathrm{k})})$ or $(p_{oi}^{(\mathrm{k})}-p_{fi}^{(\mathrm{k})})\parallel\mathbf{u}_{of}^{(\mathrm{k})}\times\mathbf{u}_{fi}^{(\mathrm{k})}\parallel+y_{if,oi}^{(\mathrm{k})}(\mathbf{u}_{of}^{(\mathrm{k})}\cdot\mathbf{u}_{fi}^{(\mathrm{k})})$ or $\mathbf{u}_{if}^{(\mathrm{k})}\cdot(\mathbf{u}_{og}^{(\mathrm{k})}\times\mathbf{u}_{ig}^{(\mathrm{k})})$	$\mathbf{n}_{ofi}^{(\mathrm{k})}\cdot(\mathbf{u}_{of}^{(\mathrm{k})}\times\mathbf{u}_{oi}^{(\mathrm{k})})$ or $y_{of,oi}^{(\mathrm{k})}$ or $\mathbf{u}_{of}^{(\mathrm{k})}\cdot(\mathbf{u}_{og}^{(\mathrm{k})}\times\mathbf{u}_{ig}^{(\mathrm{k})})$	$\frac{\omega_{of}^{(\mathrm{k})}}{\omega_{if}^{(\mathrm{k})}}=\frac{(p_{oi}^{(\mathrm{k})}-p_{fi}^{(\mathrm{k})})\parallel\mathbf{u}_{of}^{(\mathrm{k})}\times\mathbf{u}_{fi}^{(\mathrm{k})}\parallel+y_{if,oi}^{(\mathrm{k})}(\mathbf{u}_{of}^{(\mathrm{k})}\cdot\mathbf{u}_{fi}^{(\mathrm{k})})}{y_{of,oi}^{(\mathrm{k})}}$
Hel-Tra	$\omega_{if}^{(\mathrm{k})}$	$\nu_{of}^{(\mathrm{k})}$	$-y_{if,oi}^{(\mathrm{k})}$	$\parallel\mathbf{u}_{if}^{(\mathrm{k})}\times\boldsymbol{\tau}_{of}^{(\mathrm{k})}\parallel$	$\begin{cases}\omega_{oi}^{(\mathrm{k})}=-\omega_{if}^{(\mathrm{k})}=\omega_{fi}^{(\mathrm{k})}\\ \mathbf{u}_{oi}^{(\mathrm{k})}=\mathbf{u}_{if}^{(\mathrm{k})}\end{cases}$
Tra-Hel	$\nu_{if}^{(\mathrm{k})}$	$\omega_{of}^{(\mathrm{k})}$	$\parallel\mathbf{u}_{of}^{(\mathrm{k})}\times\boldsymbol{\tau}_{if}^{(\mathrm{k})}\parallel$	$-y_{of,oi}^{(\mathrm{k})}$	$\begin{cases}\omega_{oi}^{(\mathrm{k})}=\omega_{of}^{(\mathrm{k})}\\ \mathbf{u}_{oi}^{(\mathrm{k})}=\mathbf{u}_{of}^{(\mathrm{k})}\end{cases}$
Tra-Tra(†)	$\nu_{if}^{(\mathrm{k})}$	$\nu_{of}^{(\mathrm{k})}$	$y_{jf,jo}^{(\mathrm{k})}\parallel\mathbf{u}_{jf}^{(\mathrm{k})}\times\boldsymbol{\tau}_{fi}^{(\mathrm{k})}\parallel$	$y_{jf,ji}^{(\mathrm{k})}\parallel\mathbf{u}_{jf}^{(\mathrm{k})}\times\boldsymbol{\tau}_{fo}^{(\mathrm{k})}\parallel$	$\begin{cases}\omega_{ji}^{(\mathrm{k})}=\omega_{jf}^{(\mathrm{k})}=-\omega_{oj}^{(\mathrm{k})}\\ \mathbf{u}_{ji}^{(\mathrm{k})}=\mathbf{u}_{jf}^{(\mathrm{k})}=\mathbf{u}_{oj}^{(\mathrm{k})}\end{cases}$

(*) Link g is any fourth link chosen so that $\mathbf{u}_{og}^{(\mathrm{k})}\neq\pm\mathbf{u}_{ig}^{(\mathrm{k})}$

(†) Link j is any fourth link chosen so that it performs a helical motion with respect to link f with an ISA$_{\mathrm{if}}^{(\mathrm{k})}$ that is neither parallel to $\boldsymbol{\tau}_{if}^{(\mathrm{k})}$ nor parallel to $\boldsymbol{\tau}_{of}^{(\mathrm{k})}$.

$$a_k(\mathbf{q})\,\dot{q}_k = b_k(\mathbf{q})\,\dot{s}_k \tag{1}$$

where $\dot{q}_k$ and $\dot{s}_k$ are respectively the input rate and the output rate of the IOR; whereas, a_k and b_k are two scalar functions of the mechanism configuration (i.e., of $\mathbf{q}$). The input $\dot{q}_k$ (output $\dot{s}_k$) may refer only to an helical motion or a translational motion between the input (output) link and a reference link. Accordingly, the possible cases are only four (the first (second) acronym refers to the input (output) motion): Hel-Hel, Hel-Tra, Tra-Hel, and Tra-Tra. In [32], the explicit expressions of a_k and b_k have been deduced for all the possible cases. Such expressions are reported in Tables 1 and 2, where, with reference to two generic ISAs, say $\mathrm{ISA}_{ji}^{(\mathrm{k})}(\equiv(\mathrm{A}_{ji}^{(\mathrm{k})},\mathbf{u}_{ji}^{(\mathrm{k})}))$ and $\mathrm{ISA}_{jg}^{(\mathrm{k})}(\equiv(\mathrm{A}_{jg}^{(\mathrm{k})},\mathbf{u}_{jg}^{(\mathrm{k})}))$, which share a common link index in the front subscript, the following additional definitions have been introduced (see Fig. 2; hereafter, an italic capital letter will denote the position vector of the point the capital letter refers to):

$$\mathbf{n}_{ijg}^{(\mathrm{k})} = \frac{\mathbf{u}_{ji}^{(\mathrm{k})}\times\mathbf{u}_{jg}^{(\mathrm{k})}}{\left\|\mathbf{u}_{ji}^{(\mathrm{k})}\times\mathbf{u}_{jg}^{(\mathrm{k})}\right\|};\; y_{ji,jg}^{(\mathrm{k})} = \left(Q_{jg}^{(\mathrm{k})}-Q_{ji}^{(\mathrm{k})}\right)\cdot\mathbf{n}_{ijg}^{(\mathrm{k})} = \left(A_{jg}^{(\mathrm{k})}-A_{ji}^{(\mathrm{k})}\right)\cdot\mathbf{n}_{ijg}^{(\mathrm{k})} \tag{2}$$

Eventually, it is worth reminding that the spatial version of the Aronhold-Kennedy theorem [5,6,8,34–37] states that "*In a system of three bodies the three ISAs of the three possible relative motions share a common normal.*" Fig. 2 shows such a geometric condition.

2.1. IOR Deduction and ISA Determination in a Multi-DOF Spatial Mechanism

The application of the superposition principle [4] to an l-DOF spatial mechanism with link o as output link, link f as reference link (frame) for the output motion, and q_k, k=1, …, l, as actuated-joint variables (inputs) yields the following relationships

$$\begin{cases} \omega_{of}\mathbf{u}_{of} = \displaystyle\sum_{k=1,l}\omega_{of}^{(\mathrm{k})}\mathbf{u}_{of}^{(\mathrm{k})} \\ {}^{f}\mathbf{v}_{B|o} = \displaystyle\sum_{k=1,l}{}^{f}\mathbf{v}_{B|o}^{(\mathrm{k})} \end{cases} \tag{3}$$

where B is a generic point fixed to link o, and ${}^{f}\mathbf{v}_{B|o}$ (${}^{f}\mathbf{v}_{B|o}^{(\mathrm{k})}$) is point-B's velocity in the of relative motion of the l-DOF mechanism (of the k-th single-DOF mechanism).

From now on, the further numerical assumption that, if $\omega_{of}\neq 0$ ($\omega_{of}^{(\mathrm{k})}\neq 0$), then $\nu_{of}=0$ ($\nu_{of}^{(\mathrm{k})}=0$), is introduced. Such an assumption together with the fact that a possible $\nu_{of}\neq 0$ ($\nu_{of}^{(\mathrm{k})}\neq 0$) may occur if and only if $\omega_{of}=0$ ($\omega_{of}^{(\mathrm{k})}=0$) makes it possible to write the following unique expressions of ${}^{f}\mathbf{v}_{B|o}$ and ${}^{f}\mathbf{v}_{B|o}^{(\mathrm{k})}$ that hold both for Tra and Hel

$${}^{f}\mathbf{v}_{B|o} = \omega_{of}\left[p_{of}\mathbf{u}_{of} + \left(A_{of}-B\right)\times\mathbf{u}_{of}\right] + \nu_{of}\boldsymbol{\tau}_{of} \tag{4a}$$

$${}^{f}\mathbf{v}_{B|o}^{(\mathrm{k})} = \omega_{of}^{(\mathrm{k})}\left[p_{of}^{(\mathrm{k})}\mathbf{u}_{of}^{(\mathrm{k})} + \left(A_{of}^{(\mathrm{k})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{k})}\right] + \nu_{of}^{(\mathrm{k})}\boldsymbol{\tau}_{of}^{(\mathrm{k})} \tag{4b}$$

which can be used in a program provided that the values of ω_{of} for Eq. (4a) and of $\omega_{of}^{(\mathrm{k})}$ for Eq. (4b) are previously computed.

The introduction of expressions (4) into system (3) gives

$$\begin{cases} \omega_{of}\mathbf{u}_{of} = \displaystyle\sum_{k=1,l}\omega_{of}^{(\mathrm{k})}\mathbf{u}_{of}^{(\mathrm{k})} \\ \omega_{of}\left[p_{of}\mathbf{u}_{of} + \left(A_{of}-B\right)\times\mathbf{u}_{of}\right] + \nu_{of}\boldsymbol{\tau}_{of} = \displaystyle\sum_{k=1,l}\left\{\omega_{of}^{(\mathrm{k})}\left[p_{of}^{(\mathrm{k})}\mathbf{u}_{of}^{(\mathrm{k})} + \left(A_{of}^{(\mathrm{k})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{k})}\right] + \nu_{of}^{(\mathrm{k})}\boldsymbol{\tau}_{of}^{(\mathrm{k})}\right\} \end{cases} \tag{5}$$

Since $\omega_{of}^{(\mathrm{k})}$ and $\nu_{of}^{(\mathrm{k})}$ are particular $\dot{s}_k$ of Eq. (1) (see Tables 1 and 2), they can be replaced in Eq. (5) through their expressions coming from Eq. (1). Such replacements transform Eq. (5) as follows

$$\omega_{of}\mathbf{u}_{of} = \sum_{j\in\left\{1,\cdots,l\ \middle|\ \omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\mathbf{u}_{of}^{(\mathrm{j})}\dot{q}_j \tag{6a}$$

$$\omega_{of}\left[p_{of}\mathbf{u}_{of} + \left(A_{of}-B\right)\times\mathbf{u}_{of}\right] + \nu_{of}\boldsymbol{\tau}_{of} = \sum_{j\in\left\{1,\cdots,l\ \middle|\ \omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})} + \left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j + \sum_{r\in\left\{1,\cdots,l\ \middle|\ \omega_{of}^{(\mathrm{r})}=0\right\}}\frac{a_r}{b_r}\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r \tag{6b}$$

where the explicit expressions of a_k and b_k for k=1,…, l, are deducible from Tables 1 and 2 according to the particular type of k-th single-DOF mechanism.

System (6) is the alternative IOR expression that relates the instantaneous kinematics of the l-DOF mechanism to the instantaneous kinematics of the single-DOF mechanisms it generates. Since the input rates, $\dot{q}_k$ for k=1,…, l, are indeed joint rates, the separation between rotational and translational input rates is obtainable from a simple inspection of the joints they refer to; whereas, the separation between rotational and translational output rates that is present in system (6) needs the analysis of all the single-DOF

Table 2
Coefficients a_k and b_k of Eq. (1) and geometric/kinematic conditions on ISAs that are valid when the motion of the input link, i, is referred to link f and the motion of the output link, o, is referred to link r.

Case	$\dot{q}_k$	$\dot{s}_k$	a_k	b_k	Geometric & Kinematic Conditions on ISAs
Hel-Hel	$\omega_{if}^{(k)}$	$\omega_{or}^{(k)}$	$\mathbf{u}_{of}^{(k)}\cdot(\mathbf{u}_{if}^{(k)}\times\mathbf{u}_{ir}^{(k)})^{(*)}$ or $(^{\dagger})$	$\mathbf{u}_{or}^{(k)}\cdot(\mathbf{u}_{of}^{(k)}\times\mathbf{u}_{ir}^{(k)})^{(*)}$ or $(^{\ddagger})$	$\begin{cases}\dfrac{\omega_{or}^{(k)}}{\omega_{if}^{(k)}}=\dfrac{\dfrac{\omega_{of}^{(k)}}{\omega_{if}^{(k)}}(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})-(\mathbf{u}_{if}^{(k)}\cdot\mathbf{u}_{oi})}{(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})-\dfrac{\omega_{ir}^{(k)}}{\omega_{or}^{(k)}}(\mathbf{u}_{ir}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})}\\ \textit{with}\\ \dfrac{\omega_{of}^{(k)}}{\omega_{if}^{(k)}}=\dfrac{(p_{oi}^{(k)}-p_{fi}^{(k)})\parallel\mathbf{u}_{of}^{(k)}\times\mathbf{u}_{fi}^{(k)}\parallel+y_{if,oi}^{(k)}(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{fi}^{(k)})}{y_{of,oi}^{(k)}}\\ \dfrac{\omega_{or}^{(k)}}{\omega_{ir}^{(k)}}=\dfrac{(p_{oi}^{(k)}-p_{ri}^{(k)})\parallel\mathbf{u}_{or}^{(k)}\times\mathbf{u}_{ir}^{(k)}\parallel+y_{ir,oi}^{(k)}(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{ir}^{(k)})}{y_{or,oi}^{(k)}}\end{cases}$
Hel-Tra	$\omega_{if}^{(k)}$	$\nu_{or}^{(k)}$	$\dfrac{(\mathbf{u}_{og}^{(k)}\times\mathbf{u}_{ig}^{(k)})\cdot[(\mathbf{u}_{if}^{(k)}\times\mathbf{u}_{of}^{(k)})\times\mathbf{u}_{oi}^{(k)}]}{\mathbf{u}_{of}^{(k)}\cdot(\mathbf{u}_{og}^{(k)}\times\mathbf{u}_{ig}^{(k)})}(^{\S})$ or $(^{**})$	$-\dfrac{\parallel\mathbf{u}_{ir}^{(k)}\times\boldsymbol{\tau}_{or}^{(k)}\parallel}{y_{ir,oi}^{(k)}}$	$\begin{cases}\omega_{oi}^{(k)}=\omega_{ri}^{(k)}=-\omega_{ir}^{(k)}\\ \mathbf{u}_{oi}^{(k)}=\mathbf{u}_{ri}^{(k)}=\mathbf{u}_{ir}^{(k)}\\ \omega_{of}^{(k)}=\omega_{rf}^{(k)}=-\omega_{fr}^{(k)}\\ \mathbf{u}_{of}^{(k)}=\mathbf{u}_{rf}^{(k)}=\mathbf{u}_{fr}^{(k)}\end{cases}$
Tra-Hel	$\nu_{if}^{(k)}$	$\omega_{or}^{(k)}$	$-\dfrac{\parallel\mathbf{u}_{of}^{(k)}\times\boldsymbol{\tau}_{if}^{(k)}\parallel}{y_{of,oi}^{(k)}}$	$\dfrac{(\mathbf{u}_{og}^{(k)}\times\mathbf{u}_{ig}^{(k)})\cdot[(\mathbf{u}_{or}^{(k)}\times\mathbf{u}_{ir}^{(k)})\times\mathbf{u}_{oi}^{(k)}]}{\mathbf{u}_{ir}^{(k)}\cdot(\mathbf{u}_{og}^{(k)}\times\mathbf{u}_{ig}^{(k)})}(^{\S})$ or $(^{\dagger\dagger})$	$\begin{cases}\omega_{oi}^{(k)}=\omega_{of}^{(k)}\\ \mathbf{u}_{oi}^{(k)}=\mathbf{u}_{of}^{(k)}\\ \omega_{ir}^{(k)}=\omega_{fr}^{(k)}=-\omega_{rf}^{(k)}\\ \mathbf{u}_{ir}^{(k)}=\mathbf{u}_{fr}^{(k)}=\mathbf{u}_{rf}^{(k)}\end{cases}$
Tra-Tra	$\nu_{if}^{(k)}$	$\nu_{or}^{(k)}$	$\parallel\mathbf{u}_{oi}^{(k)}\times\boldsymbol{\tau}_{if}^{(k)}\parallel y_{rf,of}^{(k)}$	$-\parallel\mathbf{u}_{oi}^{(k)}\times\boldsymbol{\tau}_{or}^{(k)}\parallel y_{rf,ri}^{(k)}$	$\begin{cases}\omega_{if}^{(k)}=\omega_{or}^{(k)}=0\\ \omega_{oi}^{(k)}=\omega_{of}^{(k)}=\omega_{ri}^{(k)}=\omega_{rf}^{(k)}\\ \mathbf{u}_{oi}^{(k)}=\mathbf{u}_{of}^{(k)}=\mathbf{u}_{ir}^{(k)}=\mathbf{u}_{rf}^{(k)}\end{cases}$

(*) This expression holds if and only if $\mathbf{u}_{of}^{(k)}\neq\pm\mathbf{u}_{ir}^{(k)}$

(†) $a_k(\mathbf{q})=\{[(p_{oi}^{(k)}-p_{fi}^{(k)})\|\mathbf{u}_{of}^{(k)}\times\mathbf{u}_{fi}^{(k)}\|+y_{if,oi}^{(k)}(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{fi}^{(k)})](\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})-y_{of,oi}^{(k)}(\mathbf{u}_{if}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})\}[(p_{oi}^{(k)}-p_{ri}^{(k)})\|\mathbf{u}_{or}^{(k)}\times\mathbf{u}_{ir}^{(k)}\|+y_{ir,oi}^{(k)}(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{ir}^{(k)})]$

(‡) $b_k(\mathbf{q})=y_{of,oi}^{(k)}\{(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})[(p_{oi}^{(k)}-p_{ri}^{(k)})\|\mathbf{u}_{or}^{(k)}\times\mathbf{u}_{ir}^{(k)}\|+y_{ir,oi}^{(k)}(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{ir}^{(k)})]-y_{or,oi}^{(k)}(\mathbf{u}_{ir}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})\}$

(§) Link g is any fifth link chosen so that $\mathbf{u}_{og}^{(k)}\neq\pm\mathbf{u}_{ig}^{(k)}$

(**) $a_k(\mathbf{q})=\dfrac{(p_{if}^{(k)}-p_{oi}^{(k)})\parallel\mathbf{u}_{of}^{(k)}\times\mathbf{u}_{if}^{(k)}\parallel(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})+(\mathbf{u}_{if}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})y_{of,oi}^{(k)}-y_{if,oi}^{(k)}(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{if}^{(k)})(\mathbf{u}_{of}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})}{y_{of,oi}^{(k)}}$

(††) $b_k(\mathbf{q})=\dfrac{(p_{or}^{(k)}-p_{oi}^{(k)})\parallel\mathbf{u}_{ir}^{(k)}\times\mathbf{u}_{or}^{(k)}\parallel(\mathbf{u}_{ir}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})+(\mathbf{u}_{or}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})y_{ir,oi}^{(k)}-y_{or,oi}^{(k)}(\mathbf{u}_{ir}^{(k)}\cdot\mathbf{u}_{or}^{(k)})(\mathbf{u}_{ir}^{(k)}\cdot\mathbf{u}_{oi}^{(k)})}{y_{ir,oi}^{(k)}}$

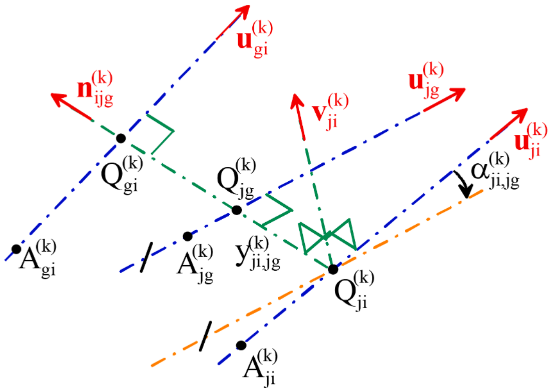

Figure 2. Geometric relationship the ISAs of the three possible relative motions among the three bodies g, i, and j must satisfy according to the spatial version of the Aronhold-Kennedy theorem.

mechanisms generated by the *l*-DOF mechanism for its determination.

In system (6), the motion of the output link is identified through ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)[1]), p_{of} and ω$_{of}$ in the case of helical motion and through ν_{of} and $\boldsymbol{\tau}_{of}$ in the case of translational motion[2]; whereas, all the other kinematic parameters that appear on the right-hand sides of Eqs. (6a) and (6b) (i.e., a_j, b_j, $A^{(j)}_{of}$, $\mathbf{u}^{(j)}_{of}$, $p^{(j)}_{of}$, a_r, b_r, and $\boldsymbol{\tau}^{(r)}_{of}$ with $j \in \left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of} \neq 0\right\}$ and $r \in \left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of} = 0\right\}$) uniquely depend on the mechanism configuration since they refer to single-DOF mechanisms. Therefore, system (6) states a relationship between ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)), p_{of} and ω$_{of}$ or ν_{of} and $\boldsymbol{\tau}_{of}$, on one side, and $\dot{q}_k$ for k=1,…, l, on the other side, which depends on the mechanism configuration.

The instantaneous forward-kinematics problem (IFKP), which is the determination of the instantaneous output motion compatible with assigned values of the actuated-joint rates (i.e., of $\dot{q}_k$ for k=1,…, l), is solvable through system (6), in which ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)), p_{of} and ω$_{of}$ or ν_{of} and $\boldsymbol{\tau}_{of}$ are the unknowns when solving the IFKP, as follows. Firstly, since the mechanism configuration and $\dot{q}_k$ for k=1,…, l are known in the IFKP, the right-hand sides of Eqs. (6a) and (6b) can be immediately computed. As a consequence, Eq. (6a) allows the direct computation of $\mathbf{u}_{of}$ and ω$_{of}$ through the following formulas

$$\mathbf{u}_{of} = \frac{\sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j}{\left\| \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j \right\|} \tag{7a}$$

$$\omega_{of} = \left(\sum_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j \right)\cdot \mathbf{u}_{of} = \frac{\left(\sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j \right)\cdot\left(\sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j \right)}{\left\| \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\mathbf{u}^{(j)}_{of}\dot{q}_j \right\|} \tag{7b}$$

Also, if Eq. (7b) yields a value of ω$_{of}$ different from zero (i.e., the *of* motion in the *l*-DOF mechanism is an helical motion), then ν_{of} =0 and Eq. (6b) becomes a linear system of three equations in three unknowns, that is, p_{of} plus the two scalar components of the projection of $(A_{of} - B)$ onto a plane perpendicular to $\mathbf{u}_{of}$ and passing through point B, say $(A^{*}_{of} - B) = x_{of}\mathbf{v}_{of} + y_{of}\mathbf{w}_{of}$ where $\mathbf{v}_{of}$ and $\mathbf{w}_{of}$ are two mutually orthogonal unit vectors that are also perpendicular to $\mathbf{u}_{of}$ with ($\mathbf{u}_{of}$, $\mathbf{v}_{of}$, $\mathbf{w}_{of}$) that is a right-handed system of unit vectors (Fig. 3). The solution of such a system provides the following explicit expressions of the unknowns

$$\begin{cases} p_{of} = \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \dfrac{a_j\left\{\mathbf{u}_{of}\cdot\left[p^{(j)}_{of}\mathbf{u}^{(j)}_{of} + \left(A^{(j)}_{of} - B\right)\times\mathbf{u}^{(j)}_{of}\right]\right\}}{b_j\omega_{of}}\dot{q}_j + \sum\limits_{r\in\left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of}=0\right\}} \dfrac{a_r\left(\mathbf{u}_{of}\cdot\boldsymbol{\tau}^{(r)}_{of}\right)}{b_r\omega_{of}}\dot{q}_r \\ x_{of} = -\left\{ \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \dfrac{a_j\left\{\mathbf{w}_{of}\cdot\left[p^{(j)}_{of}\mathbf{u}^{(j)}_{of} + \left(A^{(j)}_{of} - B\right)\times\mathbf{u}^{(j)}_{of}\right]\right\}\dot{q}_j}{b_j\omega_{of}} + \sum\limits_{r\in\left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of}=0\right\}} \dfrac{a_r\left(\mathbf{w}_{of}\cdot\boldsymbol{\tau}^{(r)}_{of}\right)}{b_r\omega_{of}}\dot{q}_r \right\} \\ y_{of} = \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \dfrac{a_j\left\{\mathbf{v}_{of}\cdot\left[p^{(j)}_{of}\mathbf{u}^{(j)}_{of} + \left(A^{(j)}_{of} - B\right)\times\mathbf{u}^{(j)}_{of}\right]\right\}}{b_j\omega_{of}}\dot{q}_j + \sum\limits_{r\in\left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of}=0\right\}} \dfrac{a_r\left(\mathbf{v}_{of}\cdot\boldsymbol{\tau}^{(r)}_{of}\right)}{b_r\omega_{of}}\dot{q}_r \end{cases} \tag{8}$$

Otherwise, if Eq. (7b) yields a value of ω$_{of}$ equal to zero (i.e., the *of* motion in the *l*-DOF mechanism is a translational motion), then Eq. (6b) allows the computation of ν_{of} and $\boldsymbol{\tau}_{of}$ through the explicit expressions

$$\boldsymbol{\tau}_{of} = \frac{\sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\left[p^{(j)}_{of}\mathbf{u}^{(j)}_{of} + \left(A^{(j)}_{of} - B\right)\times\mathbf{u}^{(j)}_{of}\right]\dot{q}_j + \sum\limits_{r\in\left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of}=0\right\}} \frac{a_r}{b_r}\boldsymbol{\tau}^{(r)}_{of}\dot{q}_r}{\left\| \sum\limits_{j\in\left\{1,\cdots,l \,\middle|\, \omega^{(j)}_{of}\neq 0\right\}} \frac{a_j}{b_j}\left[p^{(j)}_{of}\mathbf{u}^{(j)}_{of} + \left(A^{(j)}_{of} - B\right)\times\mathbf{u}^{(j)}_{of}\right]\dot{q}_j + \sum\limits_{r\in\left\{1,\cdots,l \,\middle|\, \omega^{(r)}_{of}=0\right\}} \frac{a_r}{b_r}\boldsymbol{\tau}^{(r)}_{of}\dot{q}_r \right\|} \tag{9a}$$

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[1] It is worth reminding that, since A$_{of}$ is any point of the line ISA$_{of}$ and $\mathbf{u}_{of}$ is a unit vector parallel to the same line, the location of ISA$_{of}$ through (A$_{of}$, $\mathbf{u}_{of}$) involves only four independent parameters.

[2] In the case of translational motion, the line at infinity (ideal line) that is the geometric locus of the points at infinity of the planes perpendicular to the translation direction (i.e., to $\boldsymbol{\tau}_{of}$) is the ISA.

$$\left[\begin{aligned}
\nu_{of} &= \boldsymbol{\tau}_{of}\cdot\left\{\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}+\left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j+\sum_{r\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{r})}=0\right\}}\frac{a_r}{b_r}\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r\right\}= \\
&= \frac{\left\{\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}+\left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j+\sum_{r\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{r})}=0\right\}}\frac{a_r}{b_r}\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r\right\}\cdot\left\{\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}+\left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j+\sum_{r\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{r})}=0\right\}}\frac{a_r}{b_r}\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r\right\}}{\left\|\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{j})}\neq 0\right\}}\frac{a_j}{b_j}\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}+\left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j+\sum_{r\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{r})}=0\right\}}\frac{a_r}{b_r}\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r\right\|}
\end{aligned}\right.\tag{9b}$$

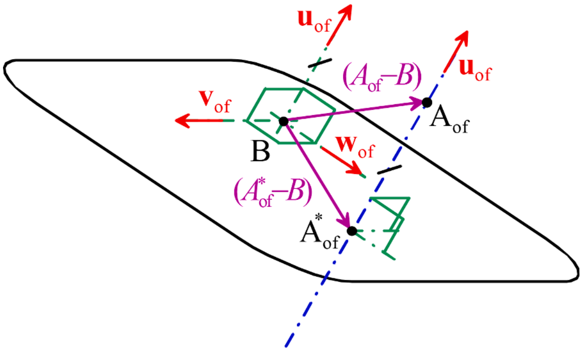

Figure 3. Projection $(A^{*}_{of}-B)=x_{of}\mathbf{v}_{of}+y_{of}\mathbf{w}_{of}$ of $(A_{of}-B)$ onto a plane perpendicular to $\mathbf{u}_{of}$ and passing through point B ($\mathbf{v}_{of}$ and $\mathbf{w}_{of}$ are two mutually orthogonal unit vectors that are also perpendicular to $\mathbf{u}_{of}$ with ($\mathbf{u}_{of}$, $\mathbf{v}_{of}$, $\mathbf{w}_{of}$) that is a right-handed system of unit vectors).

Formulas (7) and (8) relate the location of ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)) to the locations of the same ISA in the single-DOF mechanisms generated from the *l*-DOF mechanism. In particular, Eq. (7a) is writable in the following form (see Eqs. (1) and (5))

$$\mathbf{u}_{of}=\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(j)}\neq 0\right\}}\frac{\omega_{of}^{(\mathrm{j})}}{\omega_{of}}\mathbf{u}_{of}^{(\mathrm{j})} \tag{10}$$

which highlights that the direction of ISA$_{of}$ is somehow a “weighted” average of the directions of the same ISA in the single-DOF mechanisms generated from the *l*-DOF mechanism (i.e., of $\mathbf{u}_{of}^{(\mathrm{j})}$ for $j\in\{1,\cdots,l\;|\;\omega_{of}^{(\mathrm{j})}\neq 0\}$) with weights (i.e., $(\omega_{of}^{(\mathrm{j})}/\omega_{of})$ for $j\in\{1,\cdots,$ $l\;|\;\omega_{of}^{(\mathrm{j})}\neq 0\}$) whose summation is, in general, different from 1. The analysis of Eq. (10) reveals that, if, in the summation on the right-hand side, there are only two $\mathbf{u}_{of}^{(\mathrm{j})}$ for $j\in\{1,\cdots,l\;|\;\omega_{of}^{(\mathrm{j})}\neq 0\}$), say $\mathbf{u}_{of}^{(\mathrm{j}_1)}$ and $\mathbf{u}_{of}^{(\mathrm{j}_2)}$, that are not parallel to one another, the three unit vectors $\mathbf{u}_{of}$, $\mathbf{u}_{of}^{(\mathrm{j}_1)}$ and $\mathbf{u}_{of}^{(\mathrm{j}_2)}$ must be parallel to the planes perpendicular to $\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}$, which implies that common normals of any two out of the three lines ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)), ISA$_{of}^{(\mathrm{j}_1)}$(≡ (A$_{of}^{(\mathrm{j}_1)}$, $\mathbf{u}_{of}^{(\mathrm{j}_1)}$)) and ISA$_{of}^{(\mathrm{j}_2)}$(≡ (A$_{of}^{(\mathrm{j}_2)}$, $\mathbf{u}_{of}^{(\mathrm{j}_2)}$)) must be parallel to $\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}$.

2.2. *Identification of the Singularity Conditions*

System (6) is the alternative IOR to analyze for determining mechanism singularities. Such an IOR is rewritable as follows through simple algebraic manipulations:

$$\omega_{of}\mathbf{u}_{of}\prod_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(j)}\neq 0\right\}}b_j=\sum_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(j)}\neq 0\right\}}a_j\left(\prod_{n\in\left\{1,\cdots,l\;\middle|\;\left(\omega_{of}^{(n)}\neq 0\right)\&(n\neq j)\right\}}b_n\right)\mathbf{u}_{of}^{(\mathrm{j})}\dot{q}_j \tag{11a}$$

$$\left[\begin{array}{l}\left\{\omega_{of}\left[p_{of}\mathbf{u}_{of}+\left(A_{of}-B\right)\times\mathbf{u}_{of}\right]+\nu_{of}\boldsymbol{\tau}_{of}\right\}\prod\limits_{n=1,l}b_n=\sum\limits_{j\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{j})}\neq 0\right\}}a_j\left(\prod\limits_{\substack{n=1,l\\ \hline n\neq j}}b_n\right)\left[p_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}+\left(A_{of}^{(\mathrm{j})}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j})}\right]\dot{q}_j+\\ +\sum\limits_{r\in\left\{1,\cdots,l\;\middle|\;\omega_{of}^{(\mathrm{r})}=0\right\}}a_r\left(\prod\limits_{\substack{n=1,l\\ \hline n\neq r}}b_n\right)\boldsymbol{\tau}_{of}^{(\mathrm{r})}\dot{q}_r\end{array}\right. \tag{11b}$$

In the following part of this subsection, system (11) will be analyzed to identify all the mechanism singularities together with the relationships between the singularities of the *l*-DOF mechanism and the singularities of the single-DOF mechanisms it generates.

2.2.1. *Parallel Singularities*

By definition, a parallel singularity occurs if and only if the output-link twist is not uniquely determinable for arbitrarily assigned actuated-joint rates. The notations adopted to write system (11) identify the output-link twist through the parameters ISA$_{of}$ (≡(A$_{of}$, $\mathbf{u}_{of}$)), p_{of} and ω_{of} in the case of Hel or, alternatively, through ν_{of} and $\boldsymbol{\tau}_{of}$ in the case of Tra. Accordingly, system (11) identifies a parallel singularity when one or more of these parameters cannot be uniquely computed for at least one assigned set of actuated-joint rates' values (i.e., when the IFKP does not provide a unique solution through Eqs. (7) and (8) (through Eq. (9)) in the case of Hel (of Tra)).

The presence of the common factor $\prod_{j\in\{1,\cdots,l \,\mid\, \omega_{of}^{(j)}\neq 0\}} b_j$ in the left-hand side of Eq. (11a) and of the common factor $\prod_{n=1,l} b_n$ in the left-hand-side of Eq. (11b) brings to the conclusion that whatever be the values of the right-hand sides of Eqs. (11a) and (11b) (i.e., of the actuated-joint rates) the IFKP does not provide a unique solution if at least one coefficient b_k with $k\in\{1,\ldots, l\}$ is equal to zero. Moreover, since the division of Eq. (11a) (of Eq. (11b)) by one coefficient a_j with $j \in \{1,\cdots,l \,\mid\, \omega_{of}^{(j)} \neq 0\}$ (one coefficient a_k with $k\in\{1,\ldots, l\}$) is always possible and makes the inverse of that coefficient appear as a common factor at the left-hand side of Eq. (11a) (of Eq. (11b)), if at least one coefficient a_k with $k\in\{1,\ldots, l\}$ goes to infinity, the IFKP will not provide a unique solution. Accordingly, the conclusion is that

Statement 1. A parallel singularity of the l-DOF mechanism occurs if and only if either at least one coefficient b_k with $k\in\{1,\ldots, l\}$ is equal to zero or at least one coefficient a_k with $k\in\{1,\ldots, l\}$ goes to infinity.

Since a parallel singularity of the k-th single-DOF mechanism, $k\in\{1,\ldots, l\}$, occurs if and only if either b_k is equal to zero or a_k goes to infinity [32], Statement 1 immediately yields the following corollary

Corollary 1. The set of all the parallel singularities of an l-DOF mechanism is the union of the l sets of parallel singularities of the single-DOF mechanisms it generates.

2.2.2. Serial Singularities

By definition, a serial singularity occurs if and only if at least one of the actuated-joint rates, in the case $l \leq 6$, or of any subset of 6 actuated-joint rates, in the case $l > 6$, is not uniquely determinable for an arbitrarily assigned output-link twist and, in the case $l > 6$, arbitrarily assigned values of the actuated-joint rates not belonging to the above-said subset of 6 actuated-joint rates. The determination, in the case $l \leq 6$, of the actuated-joint rates for an arbitrarily assigned output-link twist or, in the case $l > 6$, of any subset of 6 actuated-joint rates for arbitrarily assigned output-link twist and actuated-joint rates not belonging to the subset is named instantaneous inverse-kinematics problem (IIKP) and requires the solution of system (11) where, now, the left-hand sides of Eqs. (11a) and (11b) are known, whereas all the $\dot{q}_k$ for $k=1,\ldots, l$, if $l \leq 6$, or only six $\dot{q}_k$ with $k\in\{1,\ldots, l\}$, if $l > 6$, are unknown.

The analysis of system (6), which has generated system (11) through algebraic manipulations, reveals that, when solving the IIKP, $\dot{q}_k$, for $k=1,\ldots, l$, cannot be uniquely determined if either a_k is equal to zero or b_k goes to infinity[3]. Since a serial singularity of the k-th single-DOF mechanism, $k\in\{1,\ldots, l\}$, occurs if and only if either a_k is equal to zero or b_k goes to infinity [32], the following statement holds

Statement 2. The union of the l sets of serial singularities of the single-DOF mechanisms generated from an l-DOF mechanism is a subset of the set of all the serial singularities of that l-DOF mechanism.

Over the serial singularities identified by Statement 2, there might only be serial singularities of the l-DOF mechanism where the coefficients a_k and b_k, for $k=1,\ldots, l$, are all different from zero and have finite values. For these other serial singularities, the following statement can be demonstrated

Statement 3. In the case that the coefficients a_k and b_k, for $k=1,\ldots, l$, are all different from zero and have finite values, a serial singularity occurs if and only if either (a) at least two unit screws ${}^{f}\widehat{\$}_{B|o}^{(j)} = \begin{pmatrix} \mathbf{u}_{of}^{(j)} \\ p_{of}^{(j)}\mathbf{u}_{of}^{(j)} + (A_{of}^{(j)} - B) \times \mathbf{u}_{of}^{(j)} \end{pmatrix}$ with $j \in \{1,\cdots,l \,\mid\, \omega_{of}^{(j)} \neq 0\}$ or (b) at least two vectors $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1,\cdots,l \,\mid\, \omega_{of}^{(r)} = 0\}$ are linearly dependent or (c) in the case $l > 3$, there are more than three $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1,\cdots,l \,\mid\, \omega_{of}^{(r)} = 0\}$.

Proof:

System (6) can be rewritten in the form of a unique 6-dimensional vector equation as follows

$$\omega_{of}\,{}^{f}\widehat{\$}_{B|o} + \nu_{of}\begin{pmatrix} \mathbf{0}_{3\times 1} \\ \boldsymbol{\tau}_{of} \end{pmatrix} = \sum_{j\in\left\{1,\cdots,l \,\middle|\, \omega_{of}^{(j)}\neq 0\right\}} \frac{a_{j}}{b_j}{}^{f}\widehat{\$}_{B|o}^{(j)}\dot{q}_j + \sum_{r\in\left\{1,\cdots,l \,\middle|\, \omega_{of}^{(r)}=0\right\}} \frac{a_r}{b_r}\begin{pmatrix} \mathbf{0}_{3\times 1} \\ \boldsymbol{\tau}_{of}^{(r)} \end{pmatrix}\dot{q}_r \tag{12}$$

where $\mathbf{0}_{3\times 1}$ is the null 3-dimensional vector; whereas, ${}^{f}\widehat{\$}_{B|o}$ and ${}^{f}\widehat{\$}_{B|o}^{(j)}$ are unit screws defined as follows

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[3] It is worth stressing that, since the zeroing of a b_k coefficient with $k\in\{1,\ldots, l\}$ identifies a parallel singularity (see statement 1), the coefficients b_k, $k=1,\ldots, l$, must be considered different from zero when looking for mechanism configurations that are only serial singularities.

$$ {}^{f}\widehat{\$}_{B|o} = \begin{pmatrix} \mathbf{u}_{of} \\ p_{of}\mathbf{u}_{of} + \left(A_{of} - B\right) \times \mathbf{u}_{of} \end{pmatrix};\; {}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o} = \begin{pmatrix} \mathbf{u}^{(\mathrm{j})}_{of} \\ p^{(\mathrm{j})}_{of}\mathbf{u}^{(\mathrm{j})}_{of} + \left(A^{(\mathrm{j})}_{of} - B\right) \times \mathbf{u}^{(\mathrm{j})}_{of} \end{pmatrix} \tag{13} $$

The analysis of Eq. (12) immediately reveals that if any subset of at-most-six unit screws ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ with $j \in \{1, \cdots, l \mid \omega^{(\mathrm{j})}_{of} \neq 0\}$ is constituted by linearly independent screws and the unit vectors $\boldsymbol{\tau}^{(\mathrm{r})}_{of}$ with $r \in \{1, \cdots, l \mid \omega^{(\mathrm{r})}_{of} = 0\}$ are linearly independent, which implies that there are no more than three $\boldsymbol{\tau}^{(\mathrm{r})}_{of}$ when $l > 3$, the solution of the IIKP is always unique no matter how the (at-most-six) unknown rates $\dot{q}_k$, for k=1,…, l, are chosen. Indeed, the coefficient matrix that multiplies the vector of the unknowns will always have full rank. As a consequence, if none of the conditions (a), (b) and (c), reported in statement 3, are satisfied, the mechanism configuration is not a serial singularity.

Differently, if two or more unit screws ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ with $j \in \{1, \cdots, l \mid \omega^{(\mathrm{j})}_{of} \neq 0\}$ or unit vectors $\boldsymbol{\tau}^{(\mathrm{r})}_{of}$ with $r \in \{1, \cdots, l \mid \omega^{(\mathrm{r})}_{of} = 0\}$ are linearly dependent (i.e., conditions (a) and/or (b) are satisfied) or there are more than three $\boldsymbol{\tau}^{(\mathrm{r})}_{of}$ with $r \in \{1, \cdots, l \mid \omega^{(\mathrm{r})}_{of} = 0\}$ when $l > 3$ (i.e., condition (c) is satisfied), choosing the rates $\dot{q}_k$ associated to those linearly-dependent unit screws and/or unit vectors as unknowns of the IIKP will make Eq. (12) a linear system with a rank-deficient matrix that multiplies the vector of the unknowns, which implies that the IIKP has not a unique solution. As a consequence, if at least one out of the conditions (a), (b) and (c), reported in statement 3, are satisfied, the mechanism configuration will be a serial singularity *Q.E.D.*

From a geometric perspective, a set of min{l, 6} unit screws ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ with $j \in \{1, \cdots, l \mid \omega^{(\mathrm{j})}_{of} \neq 0\}$ that are linearly dependent with a dimension equal to (min{l, 6}−n) implies the existence of an n-system of reciprocal screws that are still compatible with the motion type of the analyzed l-DOF mechanism. Also, a set of min{l, 3} unit vectors $\boldsymbol{\tau}^{(\mathrm{r})}_{of}$ with $r \in \{1, \cdots, l \mid \omega^{(\mathrm{r})}_{of} = 0\}$ that are linearly dependent with a dimension equal to (min{l, 3}−n) implies the existence of an n-dimensional system of vectors that are orthogonal to that set and are still compatible with the motion type of the analyzed l-DOF mechanism. These reciprocal screws and orthogonal vectors represent the restrictions on the range of the output motion (i.e., the *of* motion in the l-DOF mechanism) that arise at the serial singularity.

2.2.3. Type-(iii) Singularities

By definition, type-(iii) singularities are mechanism configurations where both the IFKP and IIKP have not a unique solution. Accordingly, if, at a given configuration, at least one parallel-singularity condition and at least one serial-singularity condition are satisfied, that configuration is a type-(iii) singularity.

Since statement 1 requires the existence of at least one coefficient b_k with k∈{1,…, l} equal to zero or at least one coefficient a_k with k∈{1,…, l} going to infinity when a parallel singularity occurs, type-(iii) singularities that satisfy statement 3 cannot be present. As a consequence, the following statement holds:

Statement 4. A type-(iii) singularity of an l-DOF mechanism occurs if and only if, at a given mechanism configuration, at least one single-DOF mechanism generated from that l-DOF mechanism is at a parallel singularity, while simultaneously, at least one single-DOF mechanism generated from that l-DOF mechanism is at a serial singularity.

Statement 4 implies the following corollary:

Corollary 2. The union of the l sets of type-(iii) singularities of the single-DOF mechanisms generated from an l-DOF mechanism is a subset of the set of all the type-(iii) singularities of that l-DOF mechanism.

2.3. Analytic Determination of the Singularities

The singularity conditions identified in subsection 2.2 are Statements 1, 2, 3, and 4. Statements 1 and 2 simply state that the singularities of the individual single-DOF mechanisms generated from the l-DOF mechanism are also singularities of the l-DOF mechanism itself. Furthermore, Statement 4 categorizes all type-(iii) singularities of the l-DOF mechanism as mechanism configurations where Statements 1 and 2 are simultaneously satisfied, regardless of whether they occur within the same single-DOF mechanism or across different single-DOF mechanisms. Therefore, the identification of singularities specified in Statements 1, 2, and 4 can be simplified to the determination of singularities within single-DOF mechanisms, as elaborated upon below. In contrast, Statement 3 is the sole singularity condition that pinpoints singularities that cannot be attributed to any single-DOF mechanism, requiring specific treatment.

2.3.1. Singularities Identified by Statements 1, 2 and 4

The identification of mechanism configurations that satisfy these statements must employ general techniques designed for conducting singularity analysis on single-DOF spatial mechanisms. In [32], the author proposed a geometric and analytical method for pinpointing singularities in single-DOF spatial mechanisms, which is grounded in the locations of ISAs. The geometric aspect of this method uses the exhaustive enumeration of the geometric relationships that appropriate ISAs satisfy at a singularity. Essentially, the geometric implementation of the method connects ISA locations to the mechanism's configuration and involves maneuvering the mechanism, represented within a 3D CAD software along with the ISAs, until the specific singularity condition that applies to it is encountered (for more details, refer to [32]). Differently, the analytic implementation of the same method consists in selecting the explicit expressions of the coefficients a_k and b_k that apply from Tables 1 and 2 and, then, analytically imposing the specific singularity condition.

The application of this approach to an l-DOF mechanism brings one to write the following equations, after having selected the right expressions of the coefficients in Tables 1 and 2,

$$a_k(q_1, \cdots, q_l) = 0 \qquad k = 1, \ldots, l \tag{14a}$$

$$b_k(q_1, \cdots, q_l) = 0 \qquad k = 1, \ldots, l \tag{14b}$$

$$a_k^{-1}(q_1, \cdots, q_l) = 0 \qquad k = 1, \ldots, l \tag{14c}$$

$$b_k^{-1}(q_1, \cdots, q_l) = 0 \qquad k = 1, \ldots, l \tag{14d}$$

and to identify as singularities of the l-DOF mechanism those configurations corresponding to $\mathbf{q}$ $(= (q_1, \cdots, q_l)^T)$ values that satisfy at least one out of Eqs. (14). In particular, Statement 1 is satisfied (i.e., a parallel singularity occurs) if either Eq. (14b) or Eq. (14c) are satisfied; also, Statement 2 is satisfied (i.e., a serial singularity occurs) if either Eq. (14a) or Eq. (14d) are satisfied; eventually, Statement 4 is satisfied (i.e., a type-(iii) singularity occurs) if at least one out of Eq. (14a) and Eq. (14d) and at least one out of Eq. (14a) and Eq. (14d) are satisfied for the same $\mathbf{q}$ value.

2.3.2. *Singularities Identified by Statement 3*

Checking whether a given mechanism configuration satisfies Statement 3 reduces itself to checking whether that configuration satisfies one out of singularity conditions (a), (b) and (c), reported in Statement 3.

The check of the possible satisfaction of condition (c) (i.e., in the case $l > 3$, there are more than three $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1, \cdots, l \mid \omega_{of}^{(r)} = 0\}$) is immediate. Indeed, it simply needs to verify whether Eq. (12) contains more than three $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1, \cdots, l \mid \omega_{of}^{(r)} = 0\}$.

Moreover, condition (b) (i.e., at least two vectors $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1, \cdots, l \mid \omega_{of}^{(r)} = 0\}$ are linearly dependent) refers only to the two cases in which there are (b1) only two or (b2) only three unit vectors $\boldsymbol{\tau}_{of}^{(r)}$ with $r \in \{1, \cdots, l \mid \omega_{of}^{(r)} = 0\}$ in Eq. (12). In the case that there are only two $\boldsymbol{\tau}_{of}^{(r)}$, say $\boldsymbol{\tau}_{of}^{(r_1)}(\mathbf{q})$ and $\boldsymbol{\tau}_{of}^{(r_2)}(\mathbf{q})$, in Eq. (12), condition (b) is satisfied if and only if those two unit vectors are parallel to one another. As a consequence, the locus of configurations that, in this case, satisfy condition (b) is analytically determinable by solving the following equation[4]

$$\boldsymbol{\tau}_{of}^{(r_1)}(\mathbf{q}) \times \boldsymbol{\tau}_{of}^{(r_2)}(\mathbf{q}) = 0 \tag{15}$$

Differently, in the case that there are only three $\boldsymbol{\tau}_{of}^{(r)}$, say $\boldsymbol{\tau}_{of}^{(r_1)}(\mathbf{q})$, $\boldsymbol{\tau}_{of}^{(r_2)}(\mathbf{q})$ and $\boldsymbol{\tau}_{of}^{(r_3)}(\mathbf{q})$, in Eq. (12), condition (b) is satisfied if and only if those three unit vectors are coplanar. As a consequence, the locus of configurations that, in this other case, satisfy condition (b) is analytically determinable by solving the following equation

$$\left[\boldsymbol{\tau}_{of}^{(r_1)}(\mathbf{q}) \times \boldsymbol{\tau}_{of}^{(r_2)}(\mathbf{q})\right] \cdot \boldsymbol{\tau}_{of}^{(r_3)}(\mathbf{q}) = 0 \tag{16}$$

The check of the possible satisfaction of condition (a) (i.e., at least two unit screws ${}^{f}\widehat{\$}_{B|o}^{(j)}$ with $j \in \{1, \cdots, l \mid \omega_{of}^{(j)} \neq 0\}$ are linearly dependent) is more complex than the other two and can be addressed as follows.

2.3.2.1. *Only 2 unit screws.*

If only two ${}^{f}\widehat{\$}_{B|o}^{(j)}$, say ${}^{f}\widehat{\$}_{B|o}^{(j_1)}$ and ${}^{f}\widehat{\$}_{B|o}^{(j_2)}$, are present in Eq. (12), imposing their linear dependency brings one to write (λ is a scalar coefficient that imposes the linear dependency)

$$\left\{\begin{array}{l} \mathbf{u}_{of}^{(j_1)} = \lambda \mathbf{u}_{of}^{(j_2)} \\ p_{of}^{(j_1)} \mathbf{u}_{of}^{(j_1)} + \left(A_{of}^{(j_1)} - B\right) \times \mathbf{u}_{of}^{(j_1)} = \lambda\left[p_{of}^{(j_2)} \mathbf{u}_{of}^{(j_2)} + \left(A_{of}^{(j_2)} - B\right) \times \mathbf{u}_{of}^{(j_2)}\right] \end{array}\right\rangle \Rightarrow \left(p_{of}^{(j_1)} - p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_1)} + \left(A_{of}^{(j_1)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_1)} = 0 \tag{17}$$

which, since the two vector terms on the left-hand side are perpendicular, yields (μ is a scalar coefficient)

$$\left\{\begin{array}{l} \left(p_{of}^{(j_1)} - p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_1)} = 0 \Rightarrow p_{of}^{(j_1)} = p_{of}^{(j_2)} \\ \left(A_{of}^{(j_1)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_1)} = 0 \Rightarrow A_{of}^{(j_1)} = A_{of}^{(j_2)} + \mu \mathbf{u}_{of}^{(j_1)} \end{array}\right. \tag{18}$$

---

[4] It is worth stressing that imposing the zeroing of a cross product (as vector Eq. (15) does) is a vector equation which corresponds to only two independent scalar equations. This can be easily realized by reminding that if the vector $\mathbf{x}_0$ satisfies the vector equation $\mathbf{a}\times\mathbf{x}=\mathbf{b}$ also $\mathbf{x}=\mathbf{x}_0+\lambda\mathbf{a}$ where λ is an arbitrary scalar coefficient satisfies it.

Equations (17) and (18) bring to the conclusion that, in this case, condition (a) is satisfied if and only if the two unit screws coincide with one another, that is, the two ISAs lies on the same line and the two pitches are equal.

2.3.2.2. Only 3 unit screws. If only three ${}^{f}\widehat{\$}_{B|o}^{(j)}$, say ${}^{f}\widehat{\$}_{B|o}^{(j_1)}$, ${}^{f}\widehat{\$}_{B|o}^{(j_2)}$ and ${}^{f}\widehat{\$}_{B|o}^{(j_3)}$, are present in Eq. (12) and none of them coincides with one of the remaining two, imposing their linear dependency brings one to write (λ_1 and λ_2 are two scalar coefficients that impose the linear dependency)

$$\mathbf{u}_{of}^{(j_3)} = \lambda_1 \mathbf{u}_{of}^{(j_1)} + \lambda_2 \mathbf{u}_{of}^{(j_2)} \tag{19a}$$

$$p_{of}^{(j_3)}\mathbf{u}_{of}^{(j_3)} + \left(A_{of}^{(j_3)} - B\right) \times \mathbf{u}_{of}^{(j_3)} = \lambda_1 \left[p_{of}^{(j_1)}\mathbf{u}_{of}^{(j_1)} + \left(A_{of}^{(j_1)} - B\right) \times \mathbf{u}_{of}^{(j_1)}\right] + \lambda_2 \left[p_{of}^{(j_2)}\mathbf{u}_{of}^{(j_2)} + \left(A_{of}^{(j_2)} - B\right) \times \mathbf{u}_{of}^{(j_2)}\right] \tag{19b}$$

Equation (19a) implies that the three ISAs associated to the three unit screws ${}^{f}\widehat{\$}_{B|o}^{(j_1)}$, ${}^{f}\widehat{\$}_{B|o}^{(j_2)}$ and ${}^{f}\widehat{\$}_{B|o}^{(j_3)}$ must be three lines parallel to a unique plane, that is, they satisfy the following analytic condition

$$\left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right) \cdot \mathbf{u}_{of}^{(j_3)} = 0 \tag{20}$$

Moreover the introduction of Eq. (19a) into Eq. (19b) yields

$$\lambda_1 \left[\left(p_{of}^{(j_3)} - p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)} + \left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \times \mathbf{u}_{of}^{(j_1)}\right] + \lambda_2 \left[\left(p_{of}^{(j_3)} - p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)} + \left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_2)}\right] = \mathbf{0} \tag{21}$$

whose dot products by $(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)})$ and by $\{[(A_{of}^{(j_3)} - A_{of}^{(j_1)}) \times \mathbf{u}_{of}^{(j_1)}] \times [(A_{of}^{(j_3)} - A_{of}^{(j_2)}) \times \mathbf{u}_{of}^{(j_2)}]\}$give

$$\lambda_1 \left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \cdot \left[\mathbf{u}_{of}^{(j_1)} \times \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)\right] + \lambda_2 \left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \cdot \left[\mathbf{u}_{of}^{(j_2)} \times \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)\right] = 0 \tag{22a}$$

$$\begin{aligned}&\lambda_1 \left(p_{of}^{(j_3)} - p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)} \cdot \left\{\left[\left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \times \mathbf{u}_{of}^{(j_1)}\right] \times \left[\left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_2)}\right]\right\} + \lambda_2 \left(p_{of}^{(j_3)} - p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)} \\ &\cdot \left\{\left[\left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \times \mathbf{u}_{of}^{(j_1)}\right] \times \left[\left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_2)}\right]\right\} \\ &= 0\end{aligned} \tag{22b}$$

Equations (22a) and (22b) admit values of (λ_1, λ_2) different from (0, 0) if and only if the following compatibility equation is satisfied

$$\begin{aligned}&\begin{vmatrix} \left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \cdot \left[\mathbf{u}_{of}^{(j_1)} \times \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)\right] & \left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \cdot \left[\mathbf{u}_{of}^{(j_2)} \times \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)\right] \\ \left(p_{of}^{(j_3)} - p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)} \cdot \left\{\left[\left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \times \mathbf{u}_{of}^{(j_1)}\right] \times \left[\left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_2)}\right]\right\} & \left(p_{of}^{(j_3)} - p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)} \cdot \left\{\left[\left(A_{of}^{(j_3)} - A_{of}^{(j_1)}\right) \times \mathbf{u}_{of}^{(j_1)}\right] \times \left[\left(A_{of}^{(j_3)} - A_{of}^{(j_2)}\right) \times \mathbf{u}_{of}^{(j_2)}\right]\right\} \end{vmatrix} \\ &= 0\end{aligned} \tag{23}$$

where $|(\cdot)|$ denotes the determinant of $(\cdot)$.

The conclusion is that, if only three ${}^{f}\widehat{\$}_{B|o}^{(j)}$ are present in Eq. (12) and none of them coincides with one of the remaining two, condition (a) is satisfied if and only if Eqs. (20) and (23) are simultaneously satisfied, that is, the singularity locus is constituted by the solutions of the system of these two equations.

2.3.2.3. Only 4 unit screws. If only four ${}^{f}\widehat{\$}_{B|o}^{(j)}$, say ${}^{f}\widehat{\$}_{B|o}^{(j_1)}$, ${}^{f}\widehat{\$}_{B|o}^{(j_2)}$, ${}^{f}\widehat{\$}_{B|o}^{(j_3)}$ and ${}^{f}\widehat{\$}_{B|o}^{(j_4)}$, are present in Eq. (12) with none of them that coincides with one of the remaining three and with no triplet of them that satisfies Eqs. (19) (i.e., that are linearly dependent), imposing their linear dependency brings one to write (μ_1, μ_2 and μ_3 are three scalar coefficients that impose the linear dependency)

$$\mathbf{u}_{of}^{(j_4)} = \mu_1 \mathbf{u}_{of}^{(j_1)} + \mu_2 \mathbf{u}_{of}^{(j_2)} + \mu_3 \mathbf{u}_{of}^{(j_3)} \tag{24a}$$

$$\left[\begin{aligned}&p_{of}^{(j_4)}\mathbf{u}_{of}^{(j_4)} + \left(A_{of}^{(j_4)} - B\right) \times \mathbf{u}_{of}^{(j_4)} = \mu_1 \left[p_{of}^{(j_1)}\mathbf{u}_{of}^{(j_1)} + \left(A_{of}^{(j_1)} - B\right) \times \mathbf{u}_{of}^{(j_1)}\right] + \mu_2 \left[p_{of}^{(j_2)}\mathbf{u}_{of}^{(j_2)} + \left(A_{of}^{(j_2)} - B\right) \times \mathbf{u}_{of}^{(j_2)}\right] + \\ &+\mu_3 \left[p_{of}^{(j_3)}\mathbf{u}_{of}^{(j_3)} + \left(A_{of}^{(j_3)} - B\right) \times \mathbf{u}_{of}^{(j_3)}\right]\end{aligned}\right. \tag{24b}$$

Since, by hypothesis, the triplet of unit vectors $(\mathbf{u}_{of}^{(j_1)}, \mathbf{u}_{of}^{(j_2)}, \mathbf{u}_{of}^{(j_3)})$ must be linearly independent, vector Eq. (24a) is always solvable and its solution provides the following explicit formulas for the coefficient μ_1, μ_2 and μ_3:

$$\mu_1 = \frac{\mathbf{u}_{of}^{(j_4)} \cdot \left(\mathbf{u}_{of}^{(j_2)} \times \mathbf{u}_{of}^{(j_3)}\right)}{\mathbf{u}_{of}^{(j_1)} \cdot \left(\mathbf{u}_{of}^{(j_2)} \times \mathbf{u}_{of}^{(j_3)}\right)};\ \mu_2 = \frac{\mathbf{u}_{of}^{(j_4)} \cdot \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_3)}\right)}{\mathbf{u}_{of}^{(j_2)} \cdot \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_3)}\right)};\ \mu_3 = \frac{\mathbf{u}_{of}^{(j_4)} \cdot \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)}{\mathbf{u}_{of}^{(j_3)} \cdot \left(\mathbf{u}_{of}^{(j_1)} \times \mathbf{u}_{of}^{(j_2)}\right)}. \tag{25}$$

The introduction of formulas (25) into Eq. (24a) and of the resulting relationship into Eq. (24b), after some rearrangements, yields

$$\left[\begin{array}{l}\left[\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)\right]\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_1)}\right)\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_1)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]-\left[\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)\right]\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_2)}\right)\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_2)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\left[\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)\right]\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_3)}\right)\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_3)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]=\mathbf{0}\end{array}\right. \tag{26}$$

Equation (26) is a vector equation (which corresponds to a system of three scalar equations) containing only scalar/vector terms that uniquely depend on the mechanism configuration[5] (i.e., on $\mathbf{q}$). The set of $\mathbf{q}$ values that solve vector Eq. (26) is the singularity locus of mechanism configurations that satisfy condition (a) in this case.

2.3.2.4. Only 5 unit screws. If only five ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j})}$, say ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j}_1)}$, ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j}_2)}$, ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j}_3)}$, ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j}_4)}$ and ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j}_5)}$, are present in Eq. (12) with none of them that coincides with one of the remaining four and with no triplet of them that satisfies Eqs. (19) (i.e., that are linearly dependent) and with no quartet of them that satisfies Eq. (26) (i.e., that are linearly dependent), imposing their linear dependency brings one to write (ε_1, ε_2, ε_3 and ε_4 are four scalar coefficients that impose the linear dependency[6]; by hypothesis, the triplet of unit vectors $(\mathbf{u}_{of}^{(\mathrm{j}_1)}, \mathbf{u}_{of}^{(\mathrm{j}_2)}, \mathbf{u}_{of}^{(\mathrm{j}_3)})$ must be linearly independent)

$$\mathbf{u}_{of}^{(\mathrm{j}_4)}=\mu_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\mu_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\mu_3\mathbf{u}_{of}^{(\mathrm{j}_3)}\Rightarrow\mu_1=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_1)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \mu_2=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_2)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \mu_3=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_3)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)} \tag{27a}$$

$$\mathbf{u}_{of}^{(\mathrm{j}_5)}=\rho_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\rho_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\rho_3\mathbf{u}_{of}^{(\mathrm{j}_3)}\Rightarrow\rho_1=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_1)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \rho_2=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_2)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \rho_3=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_3)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)} \tag{27b}$$

$$\mathbf{u}_{of}^{(\mathrm{j}_5)}=\varepsilon_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\varepsilon_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\varepsilon_3\mathbf{u}_{of}^{(\mathrm{j}_3)}+\varepsilon_4\mathbf{u}_{of}^{(\mathrm{j}_4)} \tag{27c}$$

$$\left[\begin{array}{l}p_{of}^{(\mathrm{j}_5)}\mathbf{u}_{of}^{(\mathrm{j}_5)}+\left(A_{of}^{(\mathrm{j}_5)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_5)}=\varepsilon_1\left[p_{of}^{(\mathrm{j}_1)}\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_1)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]+\varepsilon_2\left[p_{of}^{(\mathrm{j}_2)}\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_2)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\varepsilon_3\left[p_{of}^{(\mathrm{j}_3)}\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_3)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]+\varepsilon_4\left[p_{of}^{(\mathrm{j}_4)}\mathbf{u}_{of}^{(\mathrm{j}_4)}+\left(A_{of}^{(\mathrm{j}_4)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_4)}\right]\end{array}\right. \tag{27d}$$

The introduction of Eq. (27c) into Eq. (27d) transforms Eq. (27d) as follows

$$\left[\begin{array}{l}\varepsilon_1\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_1)}\right)\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_1)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]+\varepsilon_2\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_2)}\right)\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_2)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\varepsilon_3\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_3)}\right)\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_3)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]+\varepsilon_4\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_4)}\right)\mathbf{u}_{of}^{(\mathrm{j}_4)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_4)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_4)}\right]=\mathbf{0}\end{array}\right. \tag{28}$$

Also, the introduction of formulas (27a) and (27b) into Eq. (27c) transforms it as follows

$$(\varepsilon_1-\rho_1+\varepsilon_4\mu_1)\mathbf{u}_{of}^{(\mathrm{j}_1)}+(\varepsilon_2-\rho_2+\varepsilon_4\mu_2)\mathbf{u}_{of}^{(\mathrm{j}_2)}+(\varepsilon_3-\rho_3+\varepsilon_4\mu_3)\mathbf{u}_{of}^{(\mathrm{j}_3)}=\mathbf{0} \tag{29}$$

which, since, by hypothesis, the triplet of unit vectors $(\mathbf{u}_{of}^{(\mathrm{j}_1)}, \mathbf{u}_{of}^{(\mathrm{j}_2)}, \mathbf{u}_{of}^{(\mathrm{j}_3)})$ must be linearly independent, is satisfied if and only if the following relationships hold

$$\begin{cases}\varepsilon_1=\rho_1-\varepsilon_4\mu_1\\ \varepsilon_2=\rho_2-\varepsilon_4\mu_2\\ \varepsilon_3=\rho_3-\varepsilon_4\mu_3\end{cases} \tag{30}$$

Then, the introduction of formulas (27a) and (30) into vector Eq. (28) transforms it as follows

$$\varepsilon_4\mathbf{c}_1=\mathbf{c}_2 \tag{31}$$

with

$$\begin{array}{l}\mathbf{c}_1=\mu_1\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_1)}\right)\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_1)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]+\mu_2\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_2)}\right)\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_2)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\mu_3\left[\left(p_{of}^{(\mathrm{j}_4)}-p_{of}^{(\mathrm{j}_3)}\right)\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_4)}-A_{of}^{(\mathrm{j}_3)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]\end{array} \tag{32a}$$

---

[5] It is worth reminding that the unit screws ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j})}$ uniquely depend on the mechanism configuration, that is, they are ${}^{f}\widehat{\$}_{B|o}^{(\mathrm{j})}(\mathbf{q})$, since they only group two instantaneous kinematics vectors (i.e., $\omega_{of}^{(\mathrm{j})}\mathbf{u}_{of}^{(\mathrm{j})}$ and ${}^{f}\mathbf{v}_{B|o}^{(\mathrm{j})}$) divided by $\omega_{of}^{(\mathrm{j})}$ of a single-DOF mechanism, that is, they are six-tuples of velocity coefficients (see [5] for details).

[6] It is worth stressing that, if one of the coefficients that impose the linear dependency is equal to zero the case of n linearly dependent unit screws reduces itself to the case of (n – 1) linearly dependent unit screws.

$$\begin{array}{l}\mathbf{c}_2=\rho_1\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_1)}\right)\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_1)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]+\rho_2\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_2)}\right)\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_2)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\rho_3\left[\left(p_{of}^{(\mathrm{j}_5)}-p_{of}^{(\mathrm{j}_3)}\right)\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_5)}-A_{of}^{(\mathrm{j}_3)}\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]\end{array}\tag{32b}$$

where μ_{i} and $\rho_{\ i}$, for i=1,2,3, are given by formulas (27a) and (27b).

Equation (31) brings to the conclusion that vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ must be parallel to one another, that is, the following compatibility equation must hold

$$\mathbf{c}_1\times\mathbf{c}_2=0\tag{33}$$

The left-hand side of Eq. (33) is a function uniquely depending on the mechanism configuration (i.e., on $\mathbf{q}$) since $\mathbf{c}_1$ and $\mathbf{c}_2$ contain only vector/scalar parameters referable to single-DOF mechanisms. Vector Eq. (33) results in the nullification of a cross product, leading to a system of only two independent scalar equations in $\mathbf{q}$. The solutions to this system constitute the singularity locus, which represents the set of mechanism configurations where the five unit screws are linearly dependent.

2.3.2.5. Six or more unit screws. Six or more ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ can be present when $l\geq 6$. In this case, since the rank of the matrix that multiplies the actuated-joint rates in Eq. (12) can at most be six, it is sufficient to check whether all the subsets of six unit screws ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ are linearly independent. Each of these checks follows the same methodology that must be used when there are exactly six unit screws ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$. Such a methodology is illustrated below.

If only six ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$, say ${}^{f}\widehat{\$}^{(\mathrm{j}_1)}_{B|o}$, ${}^{f}\widehat{\$}^{(\mathrm{j}_2)}_{B|o}$, ${}^{f}\widehat{\$}^{(\mathrm{j}_3)}_{B|o}$, ${}^{f}\widehat{\$}^{(\mathrm{j}_4)}_{B|o}$, ${}^{f}\widehat{\$}^{(\mathrm{j}_5)}_{B|o}$ and ${}^{f}\widehat{\$}^{(\mathrm{j}_6)}_{B|o}$, are present in Eq. (12) with all the subsets of them containing less than six ${}^{f}\widehat{\$}^{(\mathrm{j})}_{B|o}$ that are linearly independent, imposing their linear dependency brings one to write (η_1, η_2, η_3, η_4 and η_5 are five scalar coefficients that impose the linear dependency)

$${}^{f}\widehat{\$}^{(\mathrm{j}_6)}_{B|o}=\eta_1{}^{f}\widehat{\$}^{(\mathrm{j}_1)}_{B|o}+\eta_2{}^{f}\widehat{\$}^{(\mathrm{j}_2)}_{B|o}+\eta_3{}^{f}\widehat{\$}^{(\mathrm{j}_3)}_{B|o}+\eta_4{}^{f}\widehat{\$}^{(\mathrm{j}_4)}_{B|o}+\eta_5{}^{f}\widehat{\$}^{(\mathrm{j}_5)}_{B|o}\tag{34}$$

which admits solutions for (η_1, η_2, η_3, η_4, η_5) if and only if the following relationship holds

$$\left|\begin{matrix}{}^{f}\widehat{\$}^{(\mathrm{j}_1)}_{B|o} & {}^{f}\widehat{\$}^{(\mathrm{j}_2)}_{B|o} & {}^{f}\widehat{\$}^{(\mathrm{j}_3)}_{B|o} & {}^{f}\widehat{\$}^{(\mathrm{j}_4)}_{B|o} & {}^{f}\widehat{\$}^{(\mathrm{j}_5)}_{B|o} & {}^{f}\widehat{\$}^{(\mathrm{j}_6)}_{B|o}\end{matrix}\right|=0\tag{35}$$

Even though Eq. (34) immediately yields Eq. (35) as a compatibility condition, reformulations of the compatibility condition are welcome. This is because Eq. (35) involves the determinant of a 6 × 6 matrix, which can be challenging to implement both analytically and numerically. One out of such possible reformulations is illustrated below.

Equation (34) and the fact that, by hypothesis, the triplet of unit vectors $(\mathbf{u}_{of}^{(\mathrm{j}_1)},\mathbf{u}_{of}^{(\mathrm{j}_2)},\mathbf{u}_{of}^{(\mathrm{j}_3)})$ must be linearly independent) brings one to write

$$\mathbf{u}_{of}^{(\mathrm{j}_4)}=\mu_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\mu_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\mu_3\mathbf{u}_{of}^{(\mathrm{j}_3)}\Rightarrow\mu_1=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_1)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \mu_2=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_2)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \mu_3=\frac{\mathbf{u}_{of}^{(\mathrm{j}_4)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_3)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}\tag{36a}$$

$$\mathbf{u}_{of}^{(\mathrm{j}_5)}=\rho_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\rho_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\rho_3\mathbf{u}_{of}^{(\mathrm{j}_3)}\Rightarrow\rho_1=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_1)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \rho_2=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_2)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \rho_3=\frac{\mathbf{u}_{of}^{(\mathrm{j}_5)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_3)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}\tag{36b}$$

$$\mathbf{u}_{of}^{(\mathrm{j}_6)}=\sigma_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\sigma_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\sigma_3\mathbf{u}_{of}^{(\mathrm{j}_3)}\Rightarrow\sigma_1=\frac{\mathbf{u}_{of}^{(\mathrm{j}_6)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_1)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_2)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \sigma_2=\frac{\mathbf{u}_{of}^{(\mathrm{j}_6)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_2)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right)};\ \sigma_3=\frac{\mathbf{u}_{of}^{(\mathrm{j}_6)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}{\mathbf{u}_{of}^{(\mathrm{j}_3)}\cdot\left(\mathbf{u}_{of}^{(\mathrm{j}_1)}\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right)}\tag{36c}$$

$$\mathbf{u}_{of}^{(\mathrm{j}_6)}=\eta_1\mathbf{u}_{of}^{(\mathrm{j}_1)}+\eta_2\mathbf{u}_{of}^{(\mathrm{j}_2)}+\eta_3\mathbf{u}_{of}^{(\mathrm{j}_3)}+\eta_4\mathbf{u}_{of}^{(\mathrm{j}_4)}+\eta_5\mathbf{u}_{of}^{(\mathrm{j}_5)}\tag{36d}$$

$$\left[\begin{array}{l}p_{of}^{(\mathrm{j}_6)}\mathbf{u}_{of}^{(\mathrm{j}_6)}+\left(A_{of}^{(\mathrm{j}_6)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_6)}=\eta_1\left[p_{of}^{(\mathrm{j}_1)}\mathbf{u}_{of}^{(\mathrm{j}_1)}+\left(A_{of}^{(\mathrm{j}_1)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_1)}\right]+\eta_2\left[p_{of}^{(\mathrm{j}_2)}\mathbf{u}_{of}^{(\mathrm{j}_2)}+\left(A_{of}^{(\mathrm{j}_2)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_2)}\right]+\\+\eta_3\left[p_{of}^{(\mathrm{j}_3)}\mathbf{u}_{of}^{(\mathrm{j}_3)}+\left(A_{of}^{(\mathrm{j}_3)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_3)}\right]+\eta_4\left[p_{of}^{(\mathrm{j}_4)}\mathbf{u}_{of}^{(\mathrm{j}_4)}+\left(A_{of}^{(\mathrm{j}_4)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_4)}\right]+\eta_5\left[p_{of}^{(\mathrm{j}_5)}\mathbf{u}_{of}^{(\mathrm{j}_5)}+\left(A_{of}^{(\mathrm{j}_5)}-B\right)\times\mathbf{u}_{of}^{(\mathrm{j}_5)}\right]\end{array}\right.\tag{36e}$$

The introduction of Eq. (36d) into Eq. (36e) transforms Eq. (36e) as follows

$$\left[\begin{array}{l}\eta_1\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_1)}\right)\times\mathbf{u}_{of}^{(j_1)}\right]+\eta_2\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_2)}\right)\times\mathbf{u}_{of}^{(j_2)}\right]+\\ +\eta_3\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_3)}\right)\mathbf{u}_{of}^{(j_3)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_3)}\right)\times\mathbf{u}_{of}^{(j_3)}\right]+\eta_4\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_4)}\right)\mathbf{u}_{of}^{(j_4)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_4)}\right)\times\mathbf{u}_{of}^{(j_4)}\right]+\\ +\eta_5\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_5)}\right)\mathbf{u}_{of}^{(j_5)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_5)}\right)\times\mathbf{u}_{of}^{(j_5)}\right]=\mathbf{0}\end{array}\right. \tag{37}$$

Also, the introduction of formulas (36a), (36b) and (36c) into Eq. (36d) transforms it as follows

$$(\eta_1-\sigma_1+\eta_4\mu_1+\eta_5\rho_1)\mathbf{u}_{of}^{(j_1)}+(\eta_2-\sigma_2+\eta_4\mu_2+\eta_5\rho_2)\mathbf{u}_{of}^{(j_2)}+(\eta_3-\sigma_3+\eta_4\mu_3+\eta_5\rho_3)\mathbf{u}_{of}^{(j_3)}=0 \tag{38}$$

which, since, by hypothesis, the triplet of unit vectors $(\mathbf{u}_{of}^{(j_1)},\mathbf{u}_{of}^{(j_2)},\mathbf{u}_{of}^{(j_3)})$ must be linearly independent, is satisfied if and only if the following relationships hold

$$\begin{cases}\eta_1=\sigma_1-\eta_4\mu_1-\eta_5\rho_1\\ \eta_2=\sigma_2-\eta_4\mu_2-\eta_5\rho_2\\ \eta_3=\sigma_3-\eta_4\mu_3-\eta_5\rho_3\end{cases} \tag{39}$$

Then, the introduction of formulas (36a), (36b) and (39) into vector Eq. (37) transforms it as follows

$$\eta_4\mathbf{d}_1+\eta_5\mathbf{d}_2=\mathbf{d}_3 \tag{40}$$

with

$$\begin{array}{l}\mathbf{d}_1=\mu_1\left[\left(p_{of}^{(j_4)}-p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)}+\left(A_{of}^{(j_4)}-A_{of}^{(j_1)}\right)\times\mathbf{u}_{of}^{(j_1)}\right]+\mu_2\left[\left(p_{of}^{(j_4)}-p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)}+\left(A_{of}^{(j_4)}-A_{of}^{(j_2)}\right)\times\mathbf{u}_{of}^{(j_2)}\right]+\\ \qquad+\mu_3\left[\left(p_{of}^{(j_4)}-p_{of}^{(j_3)}\right)\mathbf{u}_{of}^{(j_3)}+\left(A_{of}^{(j_4)}-A_{of}^{(j_3)}\right)\times\mathbf{u}_{of}^{(j_3)}\right]\end{array} \tag{41a}$$

$$\begin{array}{l}\mathbf{d}_2=\rho_1\left[\left(p_{of}^{(j_5)}-p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)}+\left(A_{of}^{(j_5)}-A_{of}^{(j_1)}\right)\right)\times\mathbf{u}_{of}^{(j_1)}]+\rho_2\left[\left(p_{of}^{(j_5)}-p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)}+\left(A_{of}^{(j_5)}-A_{of}^{(j_2)}\right)\times\mathbf{u}_{of}^{(j_2)}\right]+\\ \qquad+\rho_3\left[\left(p_{of}^{(j_5)}-p_{of}^{(j_3)}\right)\mathbf{u}_{of}^{(j_3)}+\left(A_{of}^{(j_5)}-A_{of}^{(j_3)}\right)\times\mathbf{u}_{of}^{(j_3)}\right]\end{array} \tag{41b}$$

$$\begin{array}{l}\mathbf{d}_3=\sigma_1\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_1)}\right)\mathbf{u}_{of}^{(j_1)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_1)}\right)\times\mathbf{u}_{of}^{(j_1)}\right]+\sigma_2\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_2)}\right)\mathbf{u}_{of}^{(j_2)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_2)}\right)\times\mathbf{u}_{of}^{(j_2)}\right]+\\ \qquad+\sigma_3\left[\left(p_{of}^{(j_6)}-p_{of}^{(j_3)}\right)\mathbf{u}_{of}^{(j_3)}+\left(A_{of}^{(j_6)}-A_{of}^{(j_3)}\right)\times\mathbf{u}_{of}^{(j_3)}\right]\end{array} \tag{41c}$$

where μ_I, ρ_{i} and σ_i, for i=1,2,3, are given by formulas (36a), (36b) and (36c).

The dot product of Eq (40) by ($\mathbf{d}_1\times\mathbf{d}_2$) yields the following compatibility equation

$$(\mathbf{d}_1\times\mathbf{d}_2)\cdot\mathbf{d}_3=0 \tag{42}$$

The left-hand side of Eq. (42) is a function that uniquely depends on the mechanism configuration (i.e., on $\mathbf{q}$) since $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ contain only vector/scalar parameters referable to single-DOF mechanisms. Therefore, Eq. (42) is an equation in $\mathbf{q}$ and its solutions constitute the singularity locus that collects the mechanism configurations where the six unit screws are linearly dependent.

3. Results

In this section, the effectiveness of the above-proposed techniques is assessed by applying them to the single-looped[7] $\underline{\mathrm{R}}$CCC$\underline{\mathrm{R}}$ spatial mechanism (Fig. 4), which is a 5-bar mechanism with two DOFs. The analysis of this spatial mechanism is of interest since it represents the spatial non-overconstrained version of the planar/spherical single-looped 5R mechanism. It can be employed as a reference scheme for manufacturing those planar/spherical linkages while mitigating the risk of jamming in the case that the planar/spherical geometric conditions are not fully met due to unavoidable manufacturing errors.

With reference to Fig. 4, the following choices have been introduced: link 1 is the frame, link 3 is the output link and the 31 relative motion is the output motion. In addition, (A_{ji}, $\mathbf{u}_{ji}$) with $ji\in\{21, 32, 43, 45, 51\}$ is at the same time the axis of the joint connecting links j and i and ISA$_{ji}$. Furthermore, h_j and $\mathbf{n}_j$, for j=1, …, 5, are the distance and common-normal' s unit vector, respectively, of the two joint axes at link j' s ends; whereas, d_{ji} with $ji\in\{21, 32, 43, 45, 51\}$ is the signed distance defined as follows: $d_{ji}=(Q'_{ji}-Q_{ji})\cdot\mathbf{u}_{ji}$. Eventually, θ_{ji} with $ji\in\{21, 32, 43, 45, 51\}$ is the angle counterclockwise with respect $\mathbf{u}_{ji}$ which $\mathbf{n}_i$ must rotate around the joint axis (A_{ji}, $\mathbf{u}_{ji}$) to coincide with $\mathbf{n}_j$.

The chosen actuated-joint variables are θ_{21} and θ_{51}, that is, $q_1=\theta_{21}$ and $q_2=\theta_{51}$ with $\mathbf{q}=(q_1, q_2)^{\mathrm{T}}$. Furthermore, the geometric constants of the frame are the lengths d_{21}, d_{51} and h_1, the unit vectors $\mathbf{n}_1$, $\mathbf{u}_{21}$ and $\mathbf{u}_{51}$ and the angle $\alpha_1=\arcsin((\mathbf{u}_{21}\times\mathbf{u}_{51})\cdot\mathbf{n}_1)$ between the axes of the two R-pairs; whereas, the geometric constants of mobile link j, for j=2, …, 5, are the distance h_j and the angle α_j between the two axes of the two joints at link j's ends. Eventually, the passive-joint variables are θ_{32}, d_{32}, θ_{43}, d_{43}, θ_{45} and d_{45}.

---

[7] R and C stand for revolute pair and cylindrical pair, respectively. The underscore indicates the actuated joint.

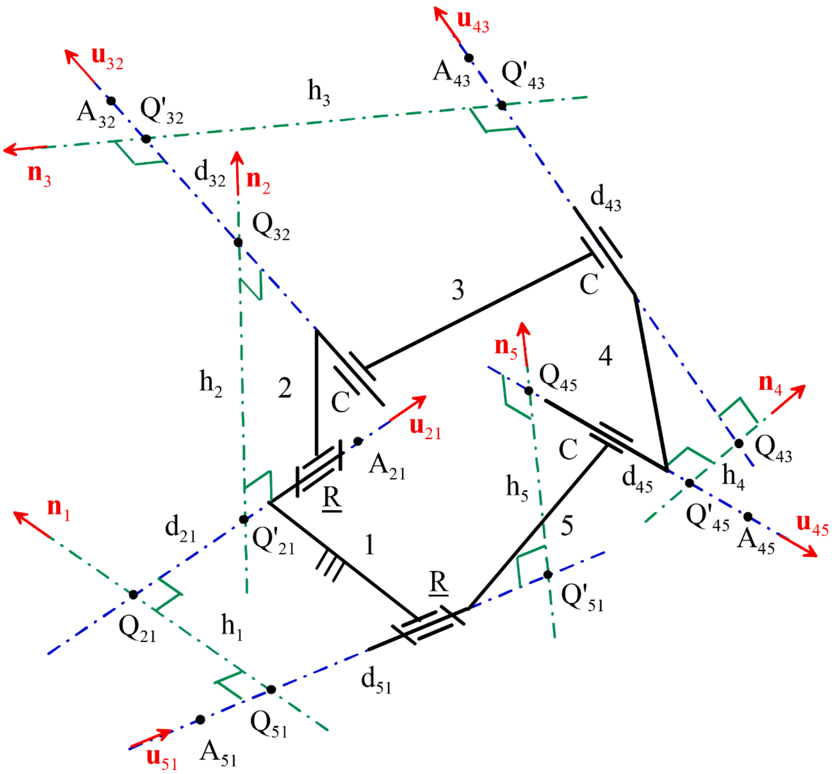

Figure 4. The single-looped $\underline{R}CCC\underline{R}$ spatial mechanism: sketch and notations.

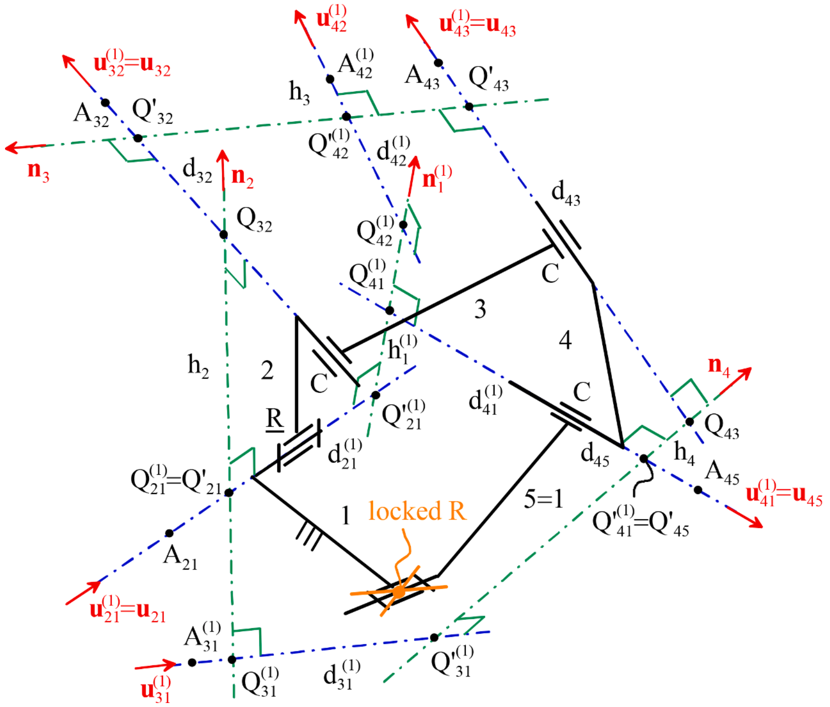

Figure 5. The 1st single-DOF mechanism, which is generated from the two-DOF mechanism shown in Fig. 4 by locking all the actuated-joint variables, but $q_1=\theta_{21}$ (the six blue lines are all the ISAs of this mechanism).

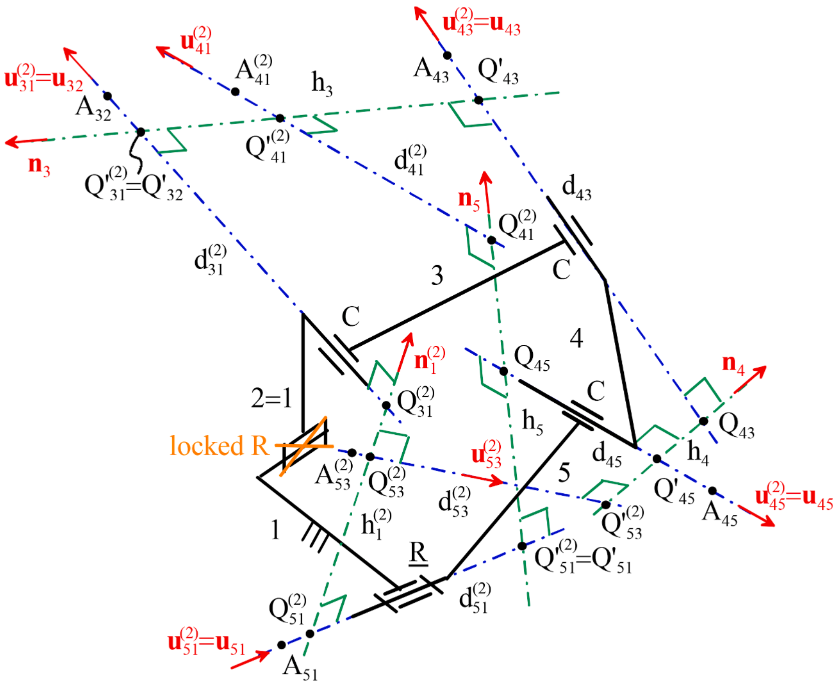

Figure 6. The 2nd single-DOF mechanism, which is generated from the two-DOF mechanism shown in Fig. 4 by locking all the actuated-joint variables, but $q_2=\theta_{51}$ (the six blue lines are all the ISAs of this mechanism).

The studied mechanism generates two single-DOF mechanisms since it is a 2-DOF spatial mechanism. Fig. 5 shows the 1st single-DOF mechanism, which is obtained by locking $q_2=\theta_{51}$, together with all its ISAs; whereas, Fig. 6 shows the 2nd single-DOF mechanism, which is obtained by locking $q_1=\theta_{21}$, together with all its ISAs. Both these single-DOF mechanisms are of $\underline{R}$CCC type. The ISAs reported in Figs. 5 and 6 have been determined through the geometric method presented in [5].

In the 1st single-DOF mechanism (Fig. 5), the input link is link 2, that is, i=2 in Tables 1 and 2, and the reference link for the motion of the input link is link 1, that is, f=1 in Tables 1 and 2, with $\dot{q}_1=\dot{\theta}_{21}=\omega_{21}=\omega_{21}^{(1)}$;whereas, the output link is link 3, that is, o=3 in Tables 1 and 2, and the reference link for the motion of the output link is link 1, that is, r=f=1 in Tables 1 and 2, with $\dot{s}_1=\dot{\theta}_{31}^{(1)}=\omega_{31}^{(1)}$. As a consequence, the explicit expressions of the coefficients a_1 and b_1 must be picked up from the Hel-Hel row of Table 1 since both the input and the output motions are Hel and r=f=1. Moreover, the availability of link 4 as a fourth link with $\mathbf{u}_{43}^{(1)}\neq\mathbf{u}_{42}^{(1)}$ brings one to select the following explicit expressions in that row:

$$a_1=\mathbf{u}_{21}^{(1)}\cdot\left(\mathbf{u}_{43}^{(1)}\times\mathbf{u}_{42}^{(1)}\right)\equiv\mathbf{u}_{21}\cdot\left(\mathbf{u}_{43}\times\mathbf{u}_{42}^{(1)}\right),\tag{43a}$$

$$b_1=\mathbf{u}_{31}^{(1)}\cdot\left(\mathbf{u}_{43}^{(1)}\times\mathbf{u}_{42}^{(1)}\right)\equiv\mathbf{u}_{31}^{(1)}\cdot\left(\mathbf{u}_{43}\times\mathbf{u}_{42}^{(1)}\right).\tag{43b}$$

where (see Fig. 5)

$$\mathbf{u}_{31}^{(1)}=\frac{\mathbf{n}_2\times\mathbf{n}_4}{\parallel\mathbf{n}_2\times\mathbf{n}_4\parallel},\ \mathbf{u}_{42}^{(1)}=\frac{\mathbf{n}_3\times\mathbf{n}_1^{(1)}}{\parallel\mathbf{n}_3\times\mathbf{n}_1^{(1)}\parallel}\tag{44}$$

with

$$\mathbf{n}_2=\frac{\mathbf{u}_{21}\times\mathbf{u}_{32}}{\parallel\mathbf{u}_{21}\times\mathbf{u}_{32}\parallel},\ \mathbf{n}_4=\frac{\mathbf{u}_{45}\times\mathbf{u}_{43}}{\parallel\mathbf{u}_{45}\times\mathbf{u}_{43}\parallel},\tag{45a}$$

$$\mathbf{n}_3=\frac{\mathbf{u}_{43}\times\mathbf{u}_{32}}{\parallel\mathbf{u}_{43}\times\mathbf{u}_{32}\parallel},\ \mathbf{n}_1^{(1)}=\frac{\mathbf{u}_{45}\times\mathbf{u}_{21}}{\parallel\mathbf{u}_{45}\times\mathbf{u}_{21}\parallel}.\tag{45b}$$

In the 2nd single-DOF mechanism (Fig. 6), the input link is link 5, that is, i=5 in Tables 1 and 2, and the reference link for the motion of the input link is link 1, that is, f=1 in Tables 1 and 2, with $\dot{q}_2=\dot{\theta}_{51}=\omega_{51}=\omega_{51}^{(1)}$; whereas, the output link is link 3, that is, o=3 in Tables 1 and 2, and the reference link for the motion of the output link is still link 1, that is, r=f=1 in Tables 1 and 2, with $\dot{s}_2=\dot{\theta}_{31}^{(2)}=\omega_{31}^{(2)}$. Accordingly, the explicit expressions of the coefficients a_2 and b_2 must be picked up from the Hel-Hel row of Table 1 since both the input and the output motions are Hel and r=f=1. Furthermore, the availability of link 4 as a fourth link with $\mathbf{u}_{43}^{(2)} \neq \mathbf{u}_{45}^{(2)}$ brings one to select the following explicit expressions in that row (see Fig. 6):

$$a_2 = \mathbf{u}_{51}^{(2)} \cdot \left(\mathbf{u}_{43}^{(2)} \times \mathbf{u}_{45}^{(2)}\right) \equiv \mathbf{u}_{51} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{45}), \tag{46a}$$

$$b_2 = \mathbf{u}_{31}^{(2)} \cdot \left(\mathbf{u}_{43}^{(2)} \times \mathbf{u}_{45}^{(2)}\right) \equiv \mathbf{u}_{32} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{45}). \tag{46b}$$

Moreover, Eqs. (43)–(46) make it possible to write system (6) (i.e., the IOR) of the studied mechanism in the following explicit form

$$\omega_{31}\mathbf{u}_{31} = \frac{a_1}{b_1}\mathbf{u}_{31}^{(1)}\dot{q}_1 + \frac{a_2}{b_2}\mathbf{u}_{31}^{(2)}\dot{q}_2 = \frac{a_1}{b_1}\frac{\mathbf{n}_2 \times \mathbf{n}_4}{\| \mathbf{n}_2 \times \mathbf{n}_4 \|}\dot{\theta}_{21} + \frac{a_2}{b_2}\mathbf{u}_{32}\dot{\theta}_{51} \tag{47a}$$

$$\left[\begin{array}{l} \omega_{31}[p_{31}\mathbf{u}_{31} + (A_{31} - B) \times \mathbf{u}_{31}] = \dfrac{a_1}{b_1}\left[p_{31}^{(1)}\mathbf{u}_{31}^{(1)} + \left(A_{31}^{(1)} - B\right) \times \mathbf{u}_{31}^{(1)}\right]\dot{q}_1 + \dfrac{a_2}{b_2}\left[p_{31}^{(2)}\mathbf{u}_{31}^{(2)} + \left(A_{31}^{(2)} - B\right) \times \mathbf{u}_{31}^{(2)}\right]\dot{q}_2 = \\ = \dfrac{a_1}{b_1}\left[p_{31}^{(1)}\dfrac{\mathbf{n}_2 \times \mathbf{n}_4}{\| \mathbf{n}_2 \times \mathbf{n}_4 \|} + \dfrac{\left(Q_{31}^{(1)} - B\right) \times (\mathbf{n}_2 \times \mathbf{n}_4)}{\| \mathbf{n}_2 \times \mathbf{n}_4 \|}\right]\dot{\theta}_{21} + \dfrac{a_2}{b_2}\left[p_{31}^{(2)}\mathbf{u}_{32} + (A_{32} - B) \times \mathbf{u}_{32}\right]\dot{\theta}_{51} \end{array}\right. \tag{47b}$$

where (see [5] for details)

$$Q_{31}^{(1)} = Q_{21}^{'} + \frac{(Q_{45}^{'} - Q_{21}^{'}) \cdot [\mathbf{n}_2 - (\mathbf{n}_4 \cdot \mathbf{n}_2)\mathbf{n}_4]}{1 - (\mathbf{n}_4 \cdot \mathbf{n}_2)^2}\mathbf{n}_2,\ Q_{31}^{'(1)} = Q_{45}^{'} + \frac{(Q_{21}^{'} - Q_{45}^{'}) \cdot [\mathbf{n}_4 - (\mathbf{n}_2 \cdot \mathbf{n}_4)\mathbf{n}_2]}{1 - (\mathbf{n}_2 \cdot \mathbf{n}_4)^2}\mathbf{n}_4, \tag{48a}$$

$$\left.\begin{array}{l} {}^{1}\mathbf{v}_{Q_{32}|3}^{(1)} = \omega_{31}^{(1)}\left[p_{31}^{(1)}\mathbf{u}_{31}^{(1)} + \left(Q_{31}^{(1)} - Q_{32}\right) \times \mathbf{u}_{31}^{(1)}\right] = \\ \quad = {}^{1}\mathbf{v}_{Q_{32}|2}^{(1)} + {}^{2}\mathbf{v}_{Q_{32}|3}^{(1)} = \\ \quad = \omega_{21}^{(1)}(Q_{21}^{'} - Q_{32}) \times \mathbf{u}_{21} + \dot{d}_{32}^{(1)}\mathbf{u}_{32} \end{array}\right\} \Rightarrow \left[\begin{array}{l} p_{31}^{(1)} = \dfrac{\omega_{21}^{(1)}}{\omega_{31}^{(1)}}[(Q_{21}^{'} - Q_{32}) \times \mathbf{u}_{21}] \cdot \mathbf{u}_{31}^{(1)} + \dfrac{\dot{d}_{32}^{(1)}}{\omega_{31}^{(1)}}\mathbf{u}_{32} \cdot \mathbf{u}_{31}^{(1)} = \\ \quad = \dfrac{\omega_{21}^{(1)}}{\omega_{31}^{(1)}}\left[(Q_{21}^{'} - Q_{32}) \times \mathbf{u}_{21} + \dfrac{\dot{d}_{32}^{(1)}}{\omega_{21}^{(1)}}\mathbf{u}_{32}\right] \cdot \mathbf{u}_{31}^{(1)} = \\ \quad = \dfrac{b_1}{a_1}\left[(Q_{21}^{'} - Q_{32}) \times \mathbf{u}_{21} + \dfrac{\dot{d}_{32}^{(1)}}{\omega_{21}^{(1)}}\mathbf{u}_{32}\right] \cdot \mathbf{u}_{31}^{(1)} \end{array}\right. \tag{48b}$$

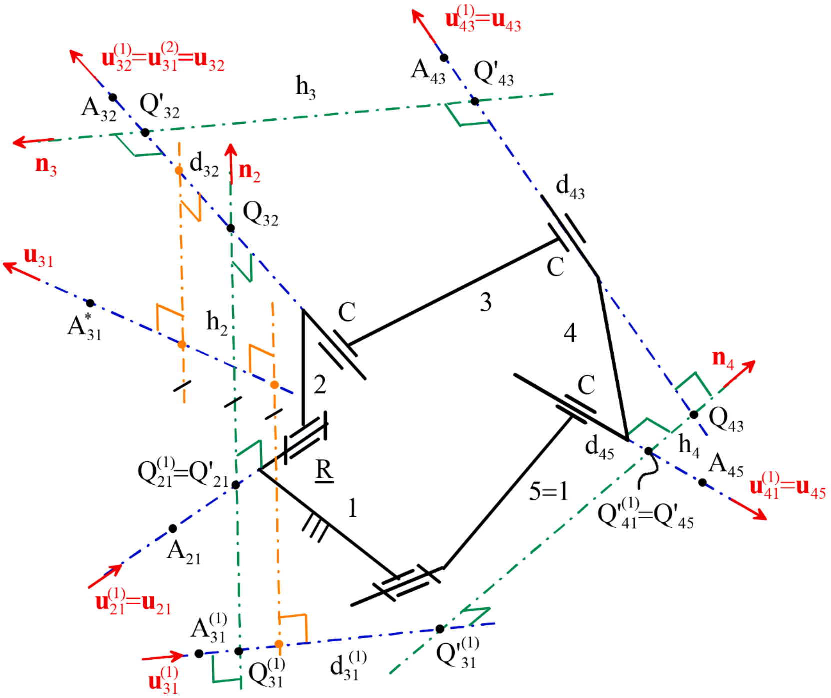

Figure 7. ISA$_{31}^{(1)}$, ISA$_{31}^{(2)}$($\equiv$ISA$_{32}$) and ISA$_{31}$ are all parallel to a unique plane that is perpendicular to $\mathbf{n}_2$.

$$ {}^{1}\mathbf{v}^{(2)}_{Q^{(2)}_{31}|3} = \omega^{(2)}_{31} p^{(2)}_{31} \mathbf{u}^{(2)}_{31} = \dot{d}^{(2)}_{31} \mathbf{u}^{(2)}_{31} \Rightarrow p^{(2)}_{31} = \frac{\dot{d}^{(2)}_{31}}{\omega^{(2)}_{31}} = \frac{\omega^{(2)}_{51}}{\omega^{(2)}_{31}} \frac{\dot{d}^{(2)}_{31}}{\omega^{(2)}_{51}} = \frac{b_2}{a_2} \frac{\dot{d}^{(2)}_{31}}{\omega^{(2)}_{51}} \tag{48c} $$

The explicit expressions of the velocity coefficients that appear in formulas (48b) and (48c) are deducible from Table 1 as follows (see Figs. 5 and 6)

$$ \left.\begin{array}{l} \textit{row Hel}-\textit{Tra of Table } 1 \\ \textit{with } i=1,\ r=f=2,\ o=3 \end{array}\right\} \Rightarrow \frac{\dot{d}^{(1)}_{32}}{\omega^{(1)}_{21}} = -\frac{\dot{d}^{(1)}_{32}}{\omega^{(1)}_{12}} = -\frac{a_{Hel-Tra}}{b_{Hel-Tra}} = \frac{y^{(1)}_{21,31}}{\| \mathbf{u}_{21} \times \mathbf{u}_{32} \|} = \frac{\left(Q'_{21} - Q^{(1)}_{31}\right) \cdot (\mathbf{u}_{21} \times \mathbf{u}_{32})}{(\mathbf{u}_{21} \times \mathbf{u}_{32}) \cdot (\mathbf{u}_{21} \times \mathbf{u}_{32})} \tag{49a} $$

$$ \left.\begin{array}{l} \textit{row Hel}-\textit{Tra of Table } 1 \\ \textit{with } i=5,\ r=f=1,\ o=3 \end{array}\right\} \Rightarrow \frac{\dot{d}^{(2)}_{31}}{\omega^{(2)}_{51}} = \frac{a_{Hel-Tra}}{b_{Hel-Tra}} = \frac{-y^{(2)}_{51,35}}{\| \mathbf{u}_{51} \times \mathbf{u}_{32} \|} = \frac{\left(Q^{(2)}_{53} - Q^{(2)}_{51}\right) \cdot (\mathbf{u}_{51} \times \mathbf{u}_{32})}{(\mathbf{u}_{51} \times \mathbf{u}_{32}) \cdot (\mathbf{u}_{51} \times \mathbf{u}_{32})} \tag{49b} $$

where

$$ Q^{(2)}_{51} = Q'_{51} + \frac{(Q'_{32} - Q'_{51}) \cdot [\mathbf{u}_{51} - (\mathbf{u}_{32} \cdot \mathbf{u}_{51})\mathbf{u}_{32}]}{1 - (\mathbf{u}_{32} \cdot \mathbf{u}_{51})^2} \mathbf{u}_{51},\ \mathbf{n}^{(2)}_{1} = \frac{\mathbf{u}_{51} \times \mathbf{u}_{32}}{\| \mathbf{u}_{51} \times \mathbf{u}_{32} \|},\ \mathbf{u}^{(2)}_{53} = \frac{\mathbf{n}^{(2)}_{1} \times \mathbf{n}_4}{\| \mathbf{n}^{(2)}_{1} \times \mathbf{n}_4 \|}, \tag{50a} $$

$$ Q^{(2)}_{53} = Q^{(2)}_{51} + \frac{\left(Q'_{45} - Q^{(2)}_{51}\right) \cdot \left[\mathbf{n}^{(2)}_{1} - \left(\mathbf{n}_4 \cdot \mathbf{n}^{(2)}_{1}\right)\mathbf{n}_4\right]}{1 - \left(\mathbf{n}_4 \cdot \mathbf{n}^{(2)}_{1}\right)^2} \mathbf{n}^{(2)}_{1}. \tag{50b} $$

The introduction of formulas (43), (46), (48) and, then, (49) into Eqs. (47a) and (47b) makes the coefficients that multiply $\dot{\theta}_{21}$ and $\dot{\theta}_{51}$ at the right-hand side of these equations explicitly expressed as functions of terms (i.e., vectors or scalar parameters) directly related to the configuration of the studied 2-DOF spatial mechanism, which can be determined either geometrically or analytically after its position analysis has been solved.

After having determined the explicit expression of the IOR (i.e., system (47)) of the studied multi-DOF mechanism, both its ISAs (i.e., only ISA$_{31}$ in this case) and its singularities can be determined as follows.

3.1. Determination of ISA$_{31}$

Formulas (7a) and (7b), when particularized to the case of Eq. (47a), yield

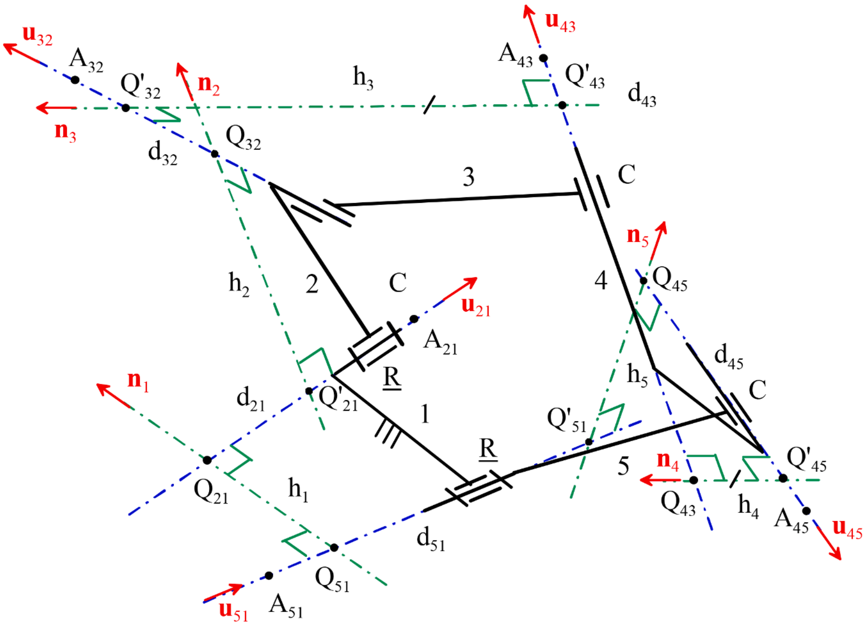

Figure 8. $\underline{\mathrm{R}}\mathrm{CCC}\underline{\mathrm{R}}$ mechanism at a configuration with $\mathbf{n}_3$ parallel to $\mathbf{n}_4$ (i.e., at a parallel singularity).

$$\mathbf{u}_{31} = \frac{\frac{a_1}{b_1}\frac{\mathbf{n}_2\times\mathbf{n}_4}{\|\mathbf{n}_2\times\mathbf{n}_4\|}\dot{\theta}_{21} + \frac{a_2}{b_2}\mathbf{u}_{32}\dot{\theta}_{51}}{\|\frac{a_1}{b_1}\frac{\mathbf{n}_2\times\mathbf{n}_4}{\|\mathbf{n}_2\times\mathbf{n}_4\|}\dot{\theta}_{21} + \frac{a_2}{b_2}\mathbf{u}_{32}\dot{\theta}_{51}\ \|} \tag{51a}$$

$$\omega_{31} = \left(\frac{a_1}{b_1}\frac{\mathbf{n}_2\times\mathbf{n}_4}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|}\dot{\theta}_{21} + \frac{a_2}{b_2}\mathbf{u}_{32}\dot{\theta}_{51}\right)\cdot\mathbf{u}_{31} \tag{51b}$$

Moreover, formulas (8), when particularized to the case of Eq. (47b), yield

$$\begin{cases} p_{31} = \left\{\dfrac{a_1}{b_1\omega_{31}}\left[p_{31}^{(1)}\dfrac{\mathbf{n}_2\times\mathbf{n}_4}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|} + \dfrac{\left(Q_{31}^{(1)} - B\right)\times(\mathbf{n}_2\times\mathbf{n}_4)}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|}\right]\dot{\theta}_{21} + \dfrac{a_2}{b_2\omega_{31}}\left[p_{31}^{(2)}\mathbf{u}_{32} + (A_{32} - B)\times\mathbf{u}_{32}\right]\dot{\theta}_{51}\right\}\cdot\mathbf{u}_{31} \\ x_{31} = -\left\{\dfrac{a_1}{b_1\omega_{31}}\left[p_{31}^{(1)}\dfrac{\mathbf{n}_2\times\mathbf{n}_4}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|} + \dfrac{\left(Q_{31}^{(1)} - B\right)\times(\mathbf{n}_2\times\mathbf{n}_4)}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|}\right]\dot{\theta}_{21} + \dfrac{a_2}{b_2\omega_{31}}\left[p_{31}^{(2)}\mathbf{u}_{32} + (A_{32} - B)\times\mathbf{u}_{32}\right]\dot{\theta}_{51}\right\}\cdot\mathbf{w}_{31} \\ y_{31} = \left\{\dfrac{a_1}{b_1\omega_{31}}\left[p_{31}^{(1)}\dfrac{\mathbf{n}_2\times\mathbf{n}_4}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|} + \dfrac{\left(Q_{31}^{(1)} - B\right)\times(\mathbf{n}_2\times\mathbf{n}_4)}{\|\ \mathbf{n}_2\times\mathbf{n}_4\ \|}\right]\dot{\theta}_{21} + \dfrac{a_2}{b_2\omega_{31}}\left[p_{31}^{(2)}\mathbf{u}_{32} + (A_{32} - B)\times\mathbf{u}_{32}\right]\dot{\theta}_{51}\right\}\cdot\mathbf{v}_{31} \end{cases} \tag{52}$$

where $(A_{31}^{*} - B) = x_{31}\mathbf{v}_{31} + y_{31}\mathbf{w}_{31}$ locates (see Fig. 3 with *of*=31) the intersection of ISA$_{31}$ with the plane perpendicular to $\mathbf{u}_{31}$ and passing through point B ($\mathbf{v}_{31}$ and $\mathbf{w}_{31}$ are two mutually orthogonal unit vectors that are also perpendicular to $\mathbf{u}_{31}$ with ($\mathbf{u}_{31}$, $\mathbf{v}_{31}$, $\mathbf{w}_{31}$) that is a right-handed system of unit vectors).

Equations (51a) and (52) locate ISA$_{31}$ through the computed values of (A$_{31}^{*}$, $\mathbf{u}_{31}$). From a geometric point of view, Eq. (47a) shows that $\mathbf{u}_{31}^{(1)}$, $\mathbf{u}_{31}^{(2)}$(=$\mathbf{u}_{32}$) and $\mathbf{u}_{31}$ must be coplanar, that is, the three lines ISA$_{31}^{(1)}$, ISA$_{31}^{(2)}$(≡ISA$_{32}$) and ISA$_{31}$ must be parallel to a unique plane. Such a geometric condition is shown in Fig. 7.

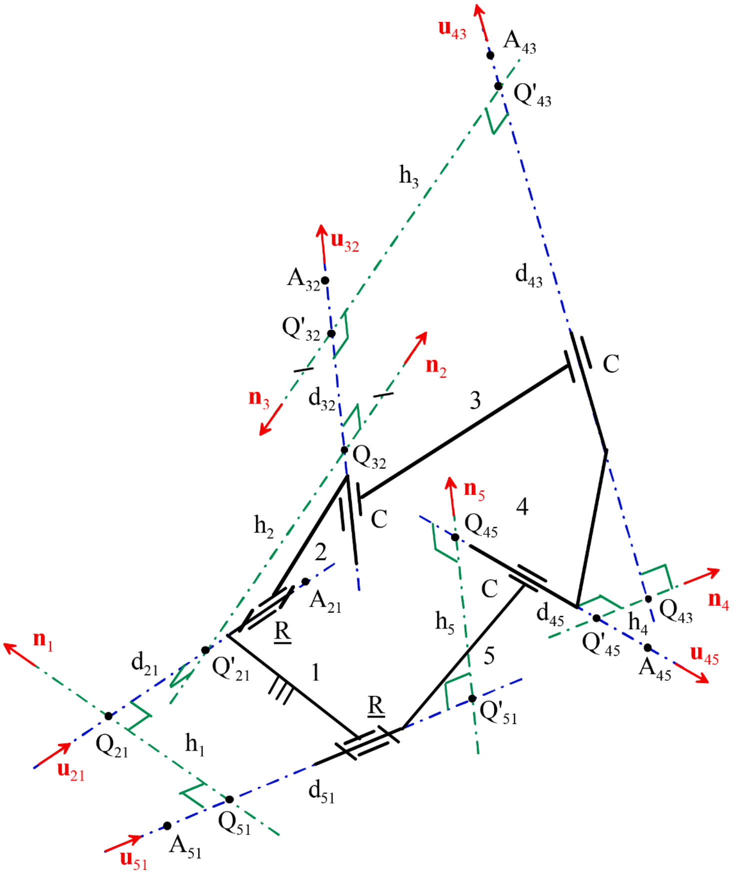

Figure 9. RCCCR mechanism at a configuration with $\mathbf{n}_2$ parallel to $\mathbf{n}_3$ (i.e., at a serial singularity that satisfies Eq. (53a)).

3.2. Singularity Analysis

Formulas (43) and (46) highlight that the coefficients a_k and b_k for k=1,2 have always finite values since they are mixed products of three unit vectors. As a consequence, their inverses, ${a_k}^{-1}$ and ${b_k}^{-1}$ for k=1,2, are never equal to zero and finding the singularities that are also singularities of the generated single-DOF mechanism (i.e., finding the solution of Eqs. (14)) reduces itself to finding the solutions of the following equations:

$$a_1 = \mathbf{u}_{21} \cdot \left(\mathbf{u}_{43} \times \mathbf{u}_{42}^{(1)}\right) = 0, \tag{53a}$$

$$b_1 = \mathbf{u}_{31}^{(1)} \cdot \left(\mathbf{u}_{43} \times \mathbf{u}_{42}^{(1)}\right) = 0, \tag{53b}$$

$$a_2 = \mathbf{u}_{51} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{45}) = 0, \tag{53c}$$

$$b_2 = \mathbf{u}_{32} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{45}) = 0. \tag{53d}$$

3.2.1. Parallel Singularities

According to Statement 1, the solutions either of Eq. (53b), which identifies all the parallel singularities of the 1st single-DOF mechanism, or of Eq. (53d), which identifies all the parallel singularities of the 2nd single-DOF mechanism, are all the parallel singularities of the studied 2-DOF mechanism.

From a geometric point of view, Eq. (53b) is satisfied if and only if $\mathbf{u}_{31}^{(1)}$, $\mathbf{u}_{43}$ and $\mathbf{u}_{42}^{(1)}$ are parallel to a unique plane. With reference to Fig. 5, since $\mathbf{n}_4$ is the unit vector of the common normal to $\text{ISA}_{31}^{(1)}(\equiv (\text{A}_{31}^{(1)},\mathbf{u}_{31}^{(1)}))$ and $\text{ISA}_{43}(\equiv (\text{A}_{43},\mathbf{u}_{43}))$, and $\mathbf{n}_3$ is the unit vector of the common normal to $\text{ISA}_{42}^{(1)}(\equiv (\text{A}_{42}^{(1)},\mathbf{u}_{42}^{(1)}))$ and $\text{ISA}_{43}(\equiv (\text{A}_{43},\mathbf{u}_{43}))$, Eq. (53b) is satisfied if and only if $\mathbf{n}_3$ and $\mathbf{n}_4$ are parallel to one another. Moreover, Eq. (53d) is satisfied if and only if $\mathbf{u}_{32}$, $\mathbf{u}_{43}$ and $\mathbf{u}_{45}$ are parallel to a unique plane. With reference to Fig. 6, since $\mathbf{n}_4$ is the unit vector of the common normal to $\text{ISA}_{45}(\equiv (\text{A}_{45},\mathbf{u}_{45}))$ and $\text{ISA}_{43}(\equiv (\text{A}_{43},\mathbf{u}_{43}))$, and $\mathbf{n}_3$ is the unit vector of the common normal to $\text{ISA}_{32}(\equiv (\text{A}_{32},\mathbf{u}_{32}))$ and $\text{ISA}_{43}(\equiv (\text{A}_{43},\mathbf{u}_{43}))$, also Eq. (53d) is satisfied if and only if $\mathbf{n}_3$ and $\mathbf{n}_4$ are parallel to one another. The conclusion is that, in the studied 2-DOF mechanism, the parallelism of $\mathbf{n}_3$ and $\mathbf{n}_4$ is the only geometric condition that identifies a parallel singularity. Since the directions of $\mathbf{n}_3$ and $\mathbf{n}_4$ are given by the geometry of links 3 and 4 (see Fig. 4), respectively, such a parallelism occurs if and only if the common normal of the joint axes at the ends of link 3 is parallel to the common normal of the joint axes at the ends of link 4. A mechanism configuration that satisfies such a condition is shown in Fig. 8.

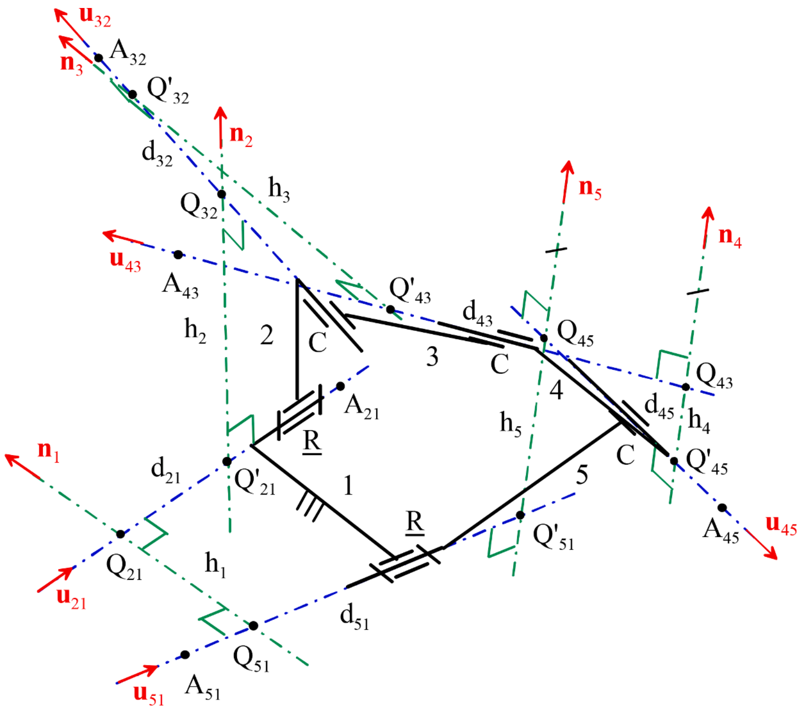

Figure 10. $\underline{\text{R}}$CCC$\underline{\text{R}}$ mechanism at a configuration with $\mathbf{n}_4$ parallel to $\mathbf{n}_5$ (i.e., at a serial singularity that satisfies Eq. (53c)).

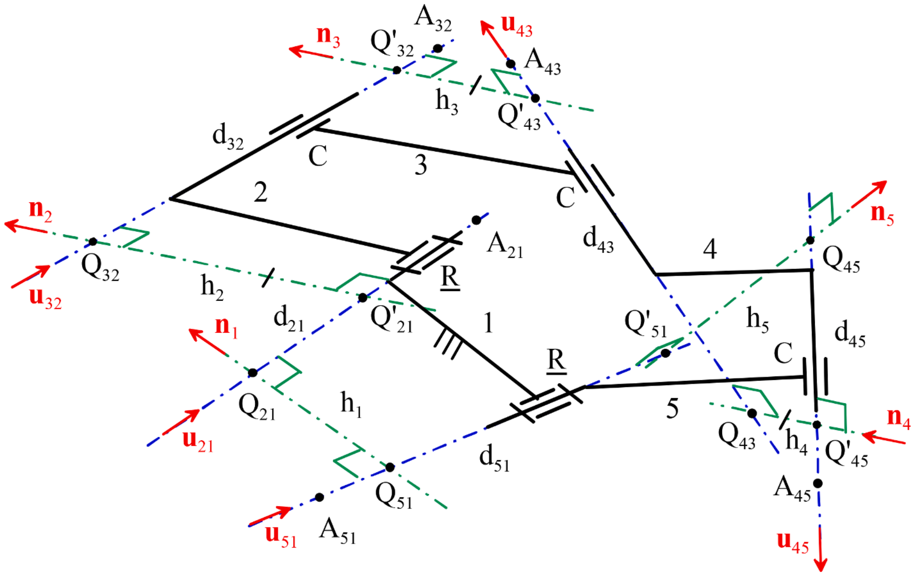

Figure 11. R̲CCCR̲ mechanism at a type-(iii) singularity with $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ that are all parallel.

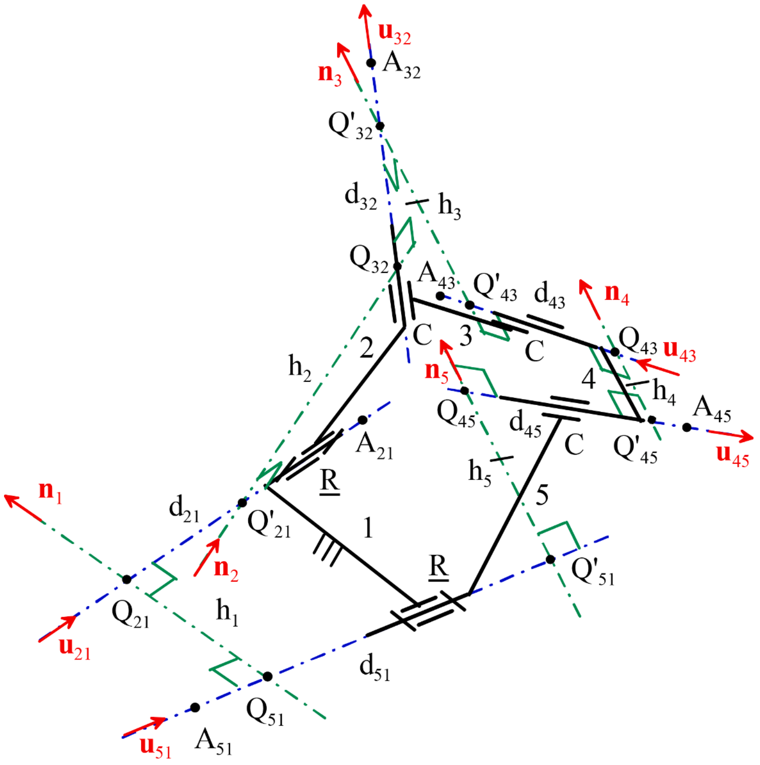

Figure 12. R̲CCCR̲ mechanism at a type-(iii) singularity with $\mathbf{n}_3$, $\mathbf{n}_4$ and $\mathbf{n}_5$ that are all parallel.

3.2.2. Serial Singularities

According to Statement 2, the solutions either of Eq. (53a), which identifies all the serial singularities of the 1st single-DOF mechanism, or of Eq. (53c), which identifies all the serial singularities of the 2nd single-DOF mechanism, are also serial singularities of the studied 2-DOF mechanism.

From a geometric point of view, Eq. (53a) is satisfied if and only if $\mathbf{u}_{21}$, $\mathbf{u}_{43}$ and $\mathbf{u}_{42}^{(1)}$ are parallel to a unique plane. With reference to Fig. 5, since $\mathrm{ISA}_{42}^{(1)}(\equiv(\mathrm{A}_{42}^{(1)},\mathbf{u}_{42}^{(1)}))$, $\mathrm{ISA}_{43}(\equiv(\mathrm{A}_{43},\mathbf{u}_{43}))$ and $\mathrm{ISA}_{32}(\equiv(\mathrm{A}_{32},\mathbf{u}_{32}))$ must be all parallel to a unique plane, such a condition is satisfied if and only if also $\mathbf{u}_{21}$ is parallel to the same plane, that is, when $\mathbf{n}_2$ and $\mathbf{n}_3$ are parallel to one another. Since the directions of $\mathbf{n}_2$ and $\mathbf{n}_3$ are given by the geometry of links 2 and 3 (see Fig. 4), respectively, such a parallelism occurs if and only if the common normal of the joint axes at the ends of link 3 is parallel to the common normal of the joint axes at the ends of link 2. A mechanism configuration that satisfies such a condition is shown in Fig. 9. Moreover, Eq. (53c) is satisfied if and only if $\mathbf{u}_{51}$, $\mathbf{u}_{43}$ and $\mathbf{u}_{45}$ are parallel to a unique plane. With reference to Fig. 4, since $\mathbf{n}_4$ is the unit vector of the common normal to ISA$_{45}$($\equiv$ (A$_{45}$,$\mathbf{u}_{45}$)) and ISA$_{43}$($\equiv$ (A$_{43}$,$\mathbf{u}_{43}$)), and $\mathbf{n}_5$ is the unit vector of the common normal to ISA$_{51}$($\equiv$ (A$_{51}$,$\mathbf{u}_{51}$)) and ISA$_{45}$($\equiv$ (A$_{45}$,$\mathbf{u}_{45}$)), Eq. (53c) is satisfied if and only if $\mathbf{n}_4$ and $\mathbf{n}_5$ are parallel to one another. Since the directions of $\mathbf{n}_4$ and $\mathbf{n}_5$ are given by the geometry of links 4 and 5 (see Fig. 4), respectively, such a parallelism occurs if and only if the common normal of the joint axes at the ends of link 4 is parallel to the common normal of the joint axes at the ends of link 5. A mechanism configuration that satisfies such a condition is shown in Fig. 10.

In addition, according to Statement 3 and the demonstration reported in subsection 2.3.2.1, further serial singularities occur if and only if ISA$_{31}^{(1)}$($\equiv$ (A$_{31}^{(1)}$, $\mathbf{u}_{31}^{(1)}$)) coincides with ISA$_{32}$($\equiv$ (A$_{32}$, $\mathbf{u}_{32}$) $\equiv$ (A$_{31}^{(2)}$, $\mathbf{u}_{31}^{(2)}$) $\equiv$ ISA$_{31}^{(2)}$) and $p_{31}^{(1)} = p_{31}^{(2)}$. With reference to Fig. 5, since ISA$_{31}^{(1)}$($\equiv$ (A$_{31}^{(1)}$,$\mathbf{u}_{31}^{(1)}$)) is the common normal to (Q$_{43}$, $\mathbf{n}_4$) and (Q$_{32}$, $\mathbf{n}_2$), a necessary condition for the occurrence of such a condition is that ISA$_{32}$($\equiv$ (A$_{32}$,$\mathbf{u}_{32}$)) be the common normal to (Q$_{43}$, $\mathbf{n}_4$) and (Q$_{32}$, $\mathbf{n}_2$). After having satisfied this necessary condition, the additional condition $p_{31}^{(1)} = p_{31}^{(2)}$ can be transformed into a further geometric condition by replacing $p_{31}^{(1)}$ and $p_{31}^{(2)}$ with their expressions given by Eq. (48b) and Eq. (48c), respectively, where the velocity coefficients are given by formulas (49a) and (49b); such a substitution yields

$$\frac{b_1}{a_1}\left[\left(Q'_{21}-Q_{32}\right)\times\mathbf{u}_{21}+\frac{\left(Q'_{21}-Q_{31}^{(1)}\right)\cdot\left(\mathbf{u}_{21}\times\mathbf{u}_{32}\right)}{\left(\mathbf{u}_{21}\times\mathbf{u}_{32}\right)\cdot\left(\mathbf{u}_{21}\times\mathbf{u}_{32}\right)}\mathbf{u}_{32}\right]\cdot\mathbf{u}_{31}^{(1)}=\frac{b_2}{a_2}\,\frac{\left(Q_{53}^{(2)}-Q_{51}^{(2)}\right)\cdot\left(\mathbf{u}_{51}\times\mathbf{u}_{32}\right)}{\left(\mathbf{u}_{51}\times\mathbf{u}_{32}\right)\cdot\left(\mathbf{u}_{51}\times\mathbf{u}_{32}\right)} \tag{54}$$

3.2.3. *Type-(iii) Singularities*

According to Statement 4, a type-(iii) singularity occurs if and only if at least one out of Eqs. (53b) and (53d) is satisfied simultaneously with at least one out of Eqs. (53a) and (53c). As proved in subsections 3.2.1 and 3.2.2, Eqs. (53b) and (53d) are satisfied if and only if $\mathbf{n}_3$ and $\mathbf{n}_4$ are parallel to one another; whereas, Eq. (53a) (Eq. (53c)) is satisfied if and only if $\mathbf{n}_2$ and $\mathbf{n}_3$ (if $\mathbf{n}_4$ and $\mathbf{n}_5$) are parallel to one another. As a consequence a type-(iii) singularity occurs either if $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ are all parallel (Fig. 11) or if $\mathbf{n}_3$, $\mathbf{n}_4$ and $\mathbf{n}_5$ are all parallel (Fig. 12).

4. Discussion

The case study, presented in section 3, has been addressed through a purely geometric approach to make clear all the steps of the proposed techniques. The geometric implementation is a specific feature of the proposed techniques and needs that the 3D model of the mechanism be generated into a 3D CAD system and, then, moved inside the same system till the sought-after geometric conditions are satisfied. Nevertheless, all the written relationships can be transformed into explicit equations, whose unknowns are the mechanism's generalized coordinates, by previously solving the closure equations of the mechanism. Such an alternative approach requires the numerical/analytical solution of the above-mentioned explicit equations and it is longer and less intuitive than the geometric one.

Since each single-DOF mechanism generated from the multi-DOF mechanism is related to one actuated-joint variable, the relationships, which the proposed techniques state between the multi-DOF mechanism and the single-DOF ones, highlight the role that each actuator plays in shaping the behavior of the multi-DOF-mechanism. Such pieces of information are relevant during the mechanism design; also, the availability of a geometric approach implementable through a 3D CAD system makes the evaluation of the changes in the mechanism geometry easier. For instance, a drawing like the one of Fig. 7 immediately highlights how ISA$_{31}$ ($\equiv$(A$_{31}^{*}$, $\mathbf{u}_{31}$)) changes its pose in response to changes in the poses of joints' axes. Of course, if, in some single-DOF mechanisms, the ISA determination were cumbersome (as it happens in indeterminate mechanisms [5]), the geometric implementation of the proposed techniques would become cumbersome, too. Nevertheless, even in these cases, the geometric approach can be implemented by adding the computation, non-implementable inside the CAD system, through an external pre-processor program that provides to the CAD system all the data necessary to draw the difficult-to-determine ISAs. For instance, the general Gough-Stewart platform [38], which is a 6-DOF 5-looped spatial mechanism of 6U$\underline{\text{P}}$S type, (i.e., where the output link (platform), *o*, is connected to its reference link (base), *r*, through six in-parallel kinematic chains (limbs) of Universal-$\underline{\text{P}}$rismatic-Spherical (U$\underline{\text{P}}$S) type (Fig. 13a)), generates 6 single-DOF mechanisms of 5US-U$\underline{\text{P}}$S type (Fig. 13b). Such a single-DOF mechanism can be seen as a 5US mechanism [5], which is an indeterminate mechanism, whose configuration is controlled through the actuated P pair of the unique unlocked U$\underline{\text{P}}$S limb, that is, the 5US-U$\underline{\text{P}}$S mechanism is a Tra-Hel mechanism where the ISA$_{or}^{(\text{k})}$ is determinable by solving the indeterminacy of the 5US single-DOF sub-mechanism with the techniques presented in the literature [5]. Thus, the determination of ISA$_{or}^{(\text{k})}$ can be done through the external pre-processor program; then, all the other ISAs of the Tra-Hel case (see Fig. 5 of Ref. [32]) can be determined inside the CAD system. Moreover, when the actuated P pairs are locked, both the 6U$\underline{\text{P}}$S and all the six 5US-U$\underline{\text{P}}$S, it generates, become the same 6US structure (Fig. 13c), and such 6US structure is really a structure (i.e., a parallel singularity does not occur) if and only if, in the *k*-th 5US-U$\underline{\text{P}}$S (Fig. 13b), the sixth US limb generated from the U$\underline{\text{P}}$S limb by locking the P pair forbids the single-DOF platform motion identified by the ISA$_{or}^{(\text{k})}$ and $p_{or}^{(\text{k})}$ that the 5US sub-mechanism imposes. Also, since the 6US structure is the same for all the six 5US-U$\underline{\text{P}}$S, if this condition is not satisfied by one out of the six 5US-U$\underline{\text{P}}$S, it will not be satisfied by all the six 5US-U$\underline{\text{P}}$S. As a consequence, the occurrence of a parallel singularity can be geometrically verified by considering only two 5US-U$\underline{\text{P}}$S, say the *k*-th one (Fig. 13b) and the

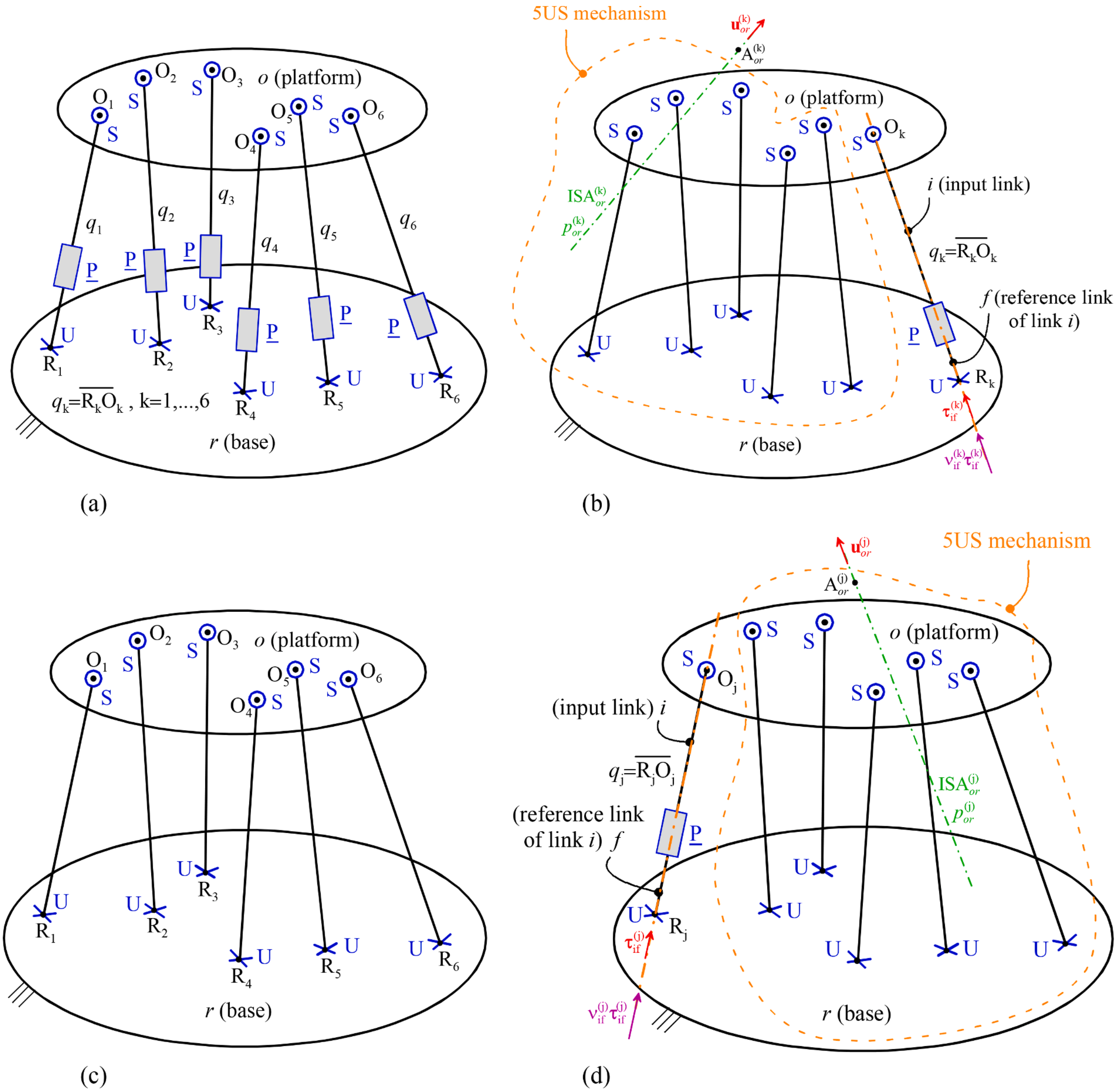

Figure 13. General Gough-Stewart mechanism: (a) general 6-DOF 5-looped mechanism of 6UPS type, (b) k-th single-DOF mechanism of 5US-UPS type generated from the 6UPS mechanism, (c) 6US structure generated by locking all the P pairs, and (d) j-th single-DOF mechanism of 5US-UPS type with $j\neq k$ generated from the 6UPS mechanism.

j-th one (Fig. 13d) with $j\neq k$, and checking that $\mathrm{ISA}_{or}^{(k)}$ and $p_{or}^{(k)}$ coincide with $\mathrm{ISA}_{or}^{(j)}$ and $p_{or}^{(j)}$, respectively, which is quite a simple check.

Regarding the singularity analysis, the comparison between the here-proposed ISA-based technique and the previously-proposed techniques that use screw theory deserves a specific discussion. The standard approach presented in the literature [18,39] builds a *screw-based Jacobian* by focusing on joints' axes. It, firstly, repeatedly writes the twist of the output motion (i.e., the *or* motion of Table 2) as summation of products between the joint rates and the screws associated to the joints belonging to the serial chains (limbs) that simultaneously connect the output link to the frame. Then, it eliminates all the passive-joint rates of each summation by exploiting the screws that are reciprocal to all the passive-joint screws of that summation [40].

The result is an IOR whose analysis allows the identification of the singularities. Such a technique, which does not require the determination of ISAs (apart from the immediately-known ones that are related to the joints), is mainly analytical and sometimes requires the computation of determinants of 6 × 6 matrices. Differently, the here-presented technique is centered on ISAs and on their role in instantaneous kinematics and, by relating the ISAs of the multi-DOF mechanism to the ones of the generated single-DOF mechanisms, directly provides geometric conditions that identify the singularities, which can also be exploited for analytic/numeric computation. Therefore, it is easier to use than the standard approach in a design context, which always needs to foresee the effects of structural changes. More recently [30,31], Slavutin et al. introduced the concept of *minimal parallel robot* and, by using ISAs and the spatial version of the Aronhold-Kennedy theorem (Fig. 2), identified the singularity conditions of the minimal parallel robots; then, since parallel mechanisms can be decomposed into minimal parallel robots, they proposed a singularity analysis technique based on this decomposition and on the fact that, if a substructure of a structure is at a singular configuration (i.e., can perform infinitesimal motions), the structure itself is at a singular configuration. This approach holds specifically for parallel mechanisms with limbs of type UPS or SPS or with limbs that are replaceable with such limb types. It shares with the here-presented one the tools (ISAs and the spatial version of the Aronhold-Kennedy theorem) adopted for the singularity analysis and that the singularities of the whole structure are obtained through the singularity analysis of substructures; nevertheless, since the two approaches refer to different substructures, they highlight different aspects of the sought-after singularities.

Eventually, it is worth stressing that the here-deduced results are indeed the extension to spatial multi-DOF mechanisms of those previously obtained by the author for planar [41] and spherical [42] multi-DOF mechanisms and together with those other results provides a complete set of geometric/analytic tools for studying the instantaneous kinematics of multi-DOF mechanisms. Such tools also integrate and extend the possible uses (see, for instance, [43,44]) of ISAs in the mobility analysis of mechanisms.

5. Conclusion

The instantaneous kinematics (IK) of a spatial mechanism is fully described by the locations of the instantaneous screw axes (ISAs) and, in single-DOF spatial mechanisms, such locations uniquely depend on the mechanism configuration. Also, for single-DOF spatial mechanisms, a previous paper [32] of this author has provided the exhaustive enumeration both of the explicit expressions of all the possible instantaneous input-output relationships (IORs) through their ISA locations and of the geometric/analytic conditions on ISAs that occur at mechanism's singular configurations (singularities).

Here, these previous results together with the fact that the superposition principle allows to relate the IK of a multi-DOF mechanism and the IK of the single-DOF mechanisms, generated from it by locking all the actuated joints but one, have been exploited to deduce the general explicit expression of an “alternative” IOR valid for any multi-DOF spatial mechanism. Moreover, such an “alternative” IOR has allowed the determination of ISAs' locations in a multi-DOF mechanism through the locations of the same ISAs in the single-DOF mechanisms generated from it, and, then, to relate singularity conditions of multi-DOF spatial mechanisms to the singularities of the single-DOF spatial mechanisms generated from them.

In particular, the IK model of a generic multi-DOF spatial mechanism has been written through the ISA locations and, then, it has been studied to identify all the singular configurations of the multi-DOF mechanism through the analysis of the single-DOF mechanisms it generates. The results are a technique for the determination of ISAs' locations in multi-DOF spatial mechanisms and a singularity-analysis technique, for the same mechanism types, based on the singularity analysis of single-DOF spatial mechanisms.

Both the proposed techniques are geometrically implementable by building the 3D model of the mechanism into a 3D CAD system and, then, by moving it inside the same system till the sought-after geometric conditions are satisfied. Eventually, the proposed techniques have been applied to a case study to prove their effectiveness. As far as this author is aware, both these results are presented for the first time in the literature.

CRediT authorship contribution statement

Raffaele Di Gregorio: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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