ON MORI CHAMBER AND STABLE BASE LOCUS DECOMPOSITIONS

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ABSTRACT. The effective cone of a Mori dream space admits two wall-and-chamber decompositions called Mori chamber and stable base locus decompositions. In general the former is a non trivial refinement of the latter. We investigate, from both the geometrical and the combinatorial viewpoints, the differences between these decompositions. Furthermore, we provide a criterion to establish whether the two decompositions coincide for a Mori dream space of Picard rank two, and we construct an explicit example of a Mori dream space of Picard rank two for which the decompositions are different, showing that our criterion is sharp. Finally, we classify the smooth toric 3-folds of Picard rank three for which the two decompositions are different.

CONTENTS

1. INTRODUCTION

Mori dream spaces, introduced by Y. Hu and S. Keel in [HK00], are varieties whose total coordinate ring, called the *Cox ring*, is finitely generated. The birational geometry of a Mori dream space is encoded in its cone of effective divisors together with a chamber decomposition on it, called *Mori chamber decomposition*. Two effective divisors lie in the interior of the same Mori chamber if there is an isomorphism between the target spaces of the corresponding dominant rational maps making the obvious triangular diagram commutative.

The birational geometry of a Mori dream spaces can also be described via the Variation of Geometric Invariant Theory of its Cox ring. As proven in [HK00], [ADHL15, Section 3.3.4], and [HKP06, Appendix A] in the case of complete toric varieties using the volume function, from this point of view GIT chambers correspond to Mori chambers.

The pseudo-effective cone of a projective variety with zero irregularity, so in particular of a Mori dream space, can be decomposed into chambers depending on the stable base locus of the corresponding linear series. Such decomposition, called *stable base locus decomposition*, in general is coarser than the Mori chamber decomposition.

The Mori theory of important classes of moduli spaces such as moduli of curves [Has05], [HH09], [HH13], Hilbert schemes of points on surfaces [BC13], [ABCH13], Kontsevich spaces of stable maps [Che08], [CC10], [CC11], spaces of complete forms [Hue15], [Mas18a], [Mas18b], and moduli spaces of parabolic bundles [Muk05], [AM16] have recently been studied in a series of papers.

In this paper, given a Mori dream space X, we aim to understand how far is the stable base locus decomposition of $\mathrm{Eff}(X)$ from determining its Mori chamber decomposition. In Section 3 we produce examples of Mori dream spaces for which the two decompositions are different and we interpret them both from the geometric and the combinatorial viewpoints.

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While producing examples of either non compact varieties or of varieties with Picard rank greater than or equal to three turns out to be fairly feasible, it is quite tricky to exhibit a normal $\mathbb{Q}$-factorial projective Mori dream space of Picard rank two for which the Mori chamber decomposition is a non trivial refinement of the stable base locus decomposition. In Example 3.6 we construct such a Mori dream space, and at the best of our knowledge this is the fist example of a projective variety displaying this particular behavior appearing in the literature.

Theorem 1.1. *Let Z be the toric variety with Cox ring $K[T_1,\dots,T_{11}]$ whose grading matrix and irrelevant ideal are the following*

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathcal{J}_{\mathrm{irr}}(Z) = \langle T_1, T_2 \rangle \cap \langle T_3, \dots, T_{11} \rangle$$

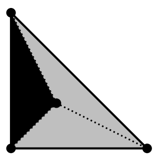
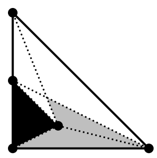
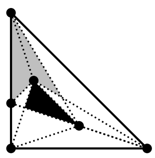
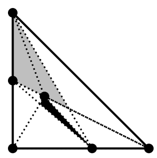
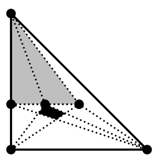
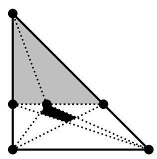
and let F, G be two general polynomials of degree $(2,2)$ in the T_i. Then the ring

$$\frac{K[T_1,\dots,T_{11}]}{(F,G)}$$

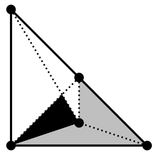
is the Cox ring of a projective normal $\mathbb{Q}$-factorial Mori dream space $X \subset Z$ of Picard rank two. Furthermore, the Mori chamber decomposition of $\operatorname{Eff}(X)$ consists of three chambers while its stable base locus decomposition consists of two chambers.

By Proposition 2.13 if X is a toric 3-fold such that the two decompositions differ inside the movable cone then there are at least five Mori chambers in the movable cone, and Examples 3.8, 3.9 show that Proposition 2.13 is sharp meaning that one can have that the two decompositions differ inside the movable cone with five chambers and with three chambers in dimension higher than three. In Section 4 we restrict to the smooth case, and we classify smooth toric 3-folds of Picard rank three such that their Mori chamber and stable base locus decomposition do not coincide.

Theorem 1.2. *Let X be a smooth toric 3-fold of Picard rank three such that its Mori chamber and stable base locus decomposition do not coincide. Then the Mori chamber decomposition of X is one of the seven types listed in the following table.*

grading matrix	*Effective cone*	*grading matrix*	*Effective cone*
$G_1 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 1 & 0 \\ \beta & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\alpha, \beta > 0$		$G_2 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 1 & 0 \\ \beta & \gamma & 1 & 0 & 0 & 1 \end{bmatrix}$ $\alpha, \beta, \gamma < 0$	
$G_3 := \begin{bmatrix} 1 & 0 & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\gamma > 0$		$G_4 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\beta > 0 > \gamma$	
$G_5 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\beta < 0 < \gamma$		$G_6 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ $\beta < \gamma - 1 < -1$	

$$G_7 := \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



Where, in the pictures, the black region is the semi-ample cone and the two gray regions are the GIT chambers sharing the same stable base locus. In particular, for any smooth toric 3-fold the Mori chamber and the stable base locus decomposition coincide inside the movable cone.

In Section 5 we focus on Mori dream spaces of Picard rank two. Recall that a Mori dream space can be recovered as a GIT quotient, with respect to a suitable polarization, of the spectrum of its Cox ring by a torus. In the fist part of Section 5, assuming that the Picard rank is two, we reach a simple description of the non semi-stable loci with respect to all the possible polarizations, and of the stable base loci of effective divisors in terms of the generators of the Cox ring. Thanks to these characterizations in Theorem 5.8 we get technical criteria on the non semi-stable loci aimed to establish whether the Mori chamber and the stable base locus decomposition of a given Mori dream space of Picard rank two coincide. As observed in Corollary 5.13 the irreducibility of the non semi-stable loci is a sufficient condition for the two decompositions to coincide.

In Theorem 5.14 we prove that under suitable inequalities, that need just the knowledge of the generators of the Cox ring in order to be checked, the two decompositions coincide. Furthermore, in Proposition 5.2 we get another criterion for the equality of the decompositions. The usefulness of these results lies in the fact that in general, even in Picard rank two, the stable base locus decomposition is considerably easier to compute than the Mori chamber decomposition.

Note that if X is a projective Mori dream space of Picard rank two we can fix a total order on the classes in the effective cone: $w \leqslant w'$ if w is on the left of w'. Given two convex cones λ, λ' contained in the effective cone we will write $\lambda \leqslant \lambda'$ if $w \leqslant w'$ for any $w \in \lambda$ and $w' \in \lambda$. Denote by $\{f_1, \dots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$ of X, and let $w_i = \deg(f_i)$ for any i.

The criteria in the Proposition 5.2, Theorem 5.14 and Corollary 5.15 can be summarized in the following statement.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two, $\{f_1, \dots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$, $w_i := \deg(f_i)$, and λ_A be the ample chamber of X. Denote by c the codimension of X into its canonical toric embedding* [ADHL15, Section 3.2.5]. *Define*

$$h^+ := \#\{f_i : w_i \geqslant \lambda_A\} \quad \text{and} \quad h^- := \#\{f_i : w_i \leqslant \lambda_A\}$$

If one of the following two conditions is satisfied

(i) all the generators of $\mathcal{R}(X)$ appear in the walls of the stable base locus decomposition of $\mathrm{Eff}(X)$*,*

(ii) $h^- > c$ and $h^+ > c$,

then the Mori chamber decomposition and the stable base locus decomposition of $\mathrm{Eff}(X)$ *coincide.*

In particular, if Z is a projective normal $\mathbb{Q}$-factorial toric variety with $\mathrm{rk}(\mathrm{Cl}(Z)) = 2$*, and $X \subseteq Z$ is a projective normal $\mathbb{Q}$-factorial Mori dream hypersurface such that $\imath^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism, then the Mori chamber and the stable base locus decompositions of both* $\mathrm{Eff}(Z)$ *and* $\mathrm{Eff}(X)$ *coincide.*

As observed in Remark 5.16, Theorem 1.1 shows that the bounds in Theorem 1.3 item (ii) can not be improved. Indeed in Theorem 1.1 we have $h^+ = c = 2$.

In Subsection 5.17 we show how Theorem 1.3 item (ii), together with the classification of Picard rank two varieties, with a torus action of complexity one in [FHN16, Theorem 1.1] immediately implies that the Mori chamber decomposition is equal to the stable base locus decomposition for this class of varieties. Note that, as shown by the examples in Subsection 5.17, Lemma 5.5 along with the proof of Theorem 5.8 provide a concrete method to compute Mori chamber decompositions.

We would like so stress that these results can also be useful in order to compute the Sarkisov factorization of a birational map $X \dashrightarrow Y$ between two $\mathbb{Q}$-factorial Fano varieties of Picard rank one. Indeed, if there exists a Mori dream space Z of Picard rank two admitting a dominant morphism $Z \to X$ then such a factorization is determined by a so called 2-ray game on Z, and such a 2-ray game is in turn determined by the Mori chamber decomposition of $\mathrm{Eff}(Z)$. We refer to [Cor95], [HM13], [AZ16], [Ahm17] for details on this topic and explicit examples.

Finally, in Section 6 we apply Theorem 1.3 item (i) to show that the Mori chamber decomposition of the blow-up $\mathbb{G}(r,n)_1$ of the Grassmannian $\mathbb{G}(r,n)$, parametrizing r-planes in $\mathbb{P}^n$, at a point coincides with its stable base locus decomposition, and can be described in terms of linear systems of hyperplanes containing the osculating spaces of $\mathbb{G}(r,n)$ at the blown-up point. This provides a positive answer to [MR18, Question 6.9].

All through the paper we will work over an algebraically closed field K of characteristic zero, and given a $\mathbb{Q}$-factorial Mori dream space X we will denote by $\mathrm{MCD}(X)$ and $\mathrm{SBLD}(X)$ respectively the Mori chamber decomposition and the stable base locus decomposition of its effective cone.

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2. Mori chamber and stable base locus decompositions

Let X be a normal projective variety over an algebraically closed field of characteristic zero. We denote by $N^1(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\mathrm{Nef}(X) \subset N^1(X)$ generated by classes of nef divisors. The *movable cone* of X is the convex cone $\mathrm{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X. The *effective cone* of X is the convex cone $\mathrm{Eff}(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions $\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}$.

We will denote by $N_1(X)$ be the real vector space of numerical equivalence classes of 1-cycles on X. The closure of the cone in $N_1(X)$ generated by the classes of irreducible curves in X is called is called the *Mori cone* of X, we will denote it by $\mathrm{NE}(X)$.

A class $[C] \in N_1(X)$ is called *moving* if the curves in X of class $[C]$ cover a dense open subset of X. The closure of the cone in $N_1(X)$ generated by classes of moving curves in X is called the *moving cone* of X and we will denote it by $\mathrm{mov}(X)$. We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

We say that a birational map $f : X \dashrightarrow X'$ to a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small $\mathbb{Q}$-factorial modification* if X' is $\mathbb{Q}$-factorial and f is an isomorphism in codimension one. If $f : X \dashrightarrow X'$ is a small $\mathbb{Q}$-factorial modification, then the natural pull-back map $f^* : N^1(X') \to N^1(X)$ sends $\overline{\mathrm{Mov}(X')}$ and $\mathrm{Eff}(X')$ isomorphically onto $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$, respectively. In particular, we have $f^*(\mathrm{Nef}(X')) \subset \overline{\mathrm{Mov}(X)}$.

Definition 2.1. A normal projective $\mathbb{Q}$-factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\mathrm{Pic}\,(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$,
- $\mathrm{Nef}\,(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and $\mathrm{Mov}\,(X) = \bigcup_i f_i^*(\mathrm{Nef}\,(X_i))$.

By [BCHM10, Corollary 1.3.2] smooth Fano varieties are Mori dream spaces. In fact, there is a larger class of varieties called log Fano varieties which are Mori dream spaces as well. By the work of M. Brion [Bri93] we have that $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [Per14, Section 4].

The collection of all faces of all cones $f_i^*(\mathrm{Nef}\,(X_i))$ in Definition 2.1 forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}\,(X_i))$ and $f_j^*(\mathrm{Nef}\,(X_j))$, meet along a facet, then there exist a normal projective variety Y, a small modification $\varphi : X_i \dashrightarrow X_j$, and $h_i : X_i \to Y$ and $h_j : X_j \to Y$ small birational morphisms of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 2.2. Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details.

There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form: $\mathcal{C}_i = \left\langle g_i^*\big(\mathrm{Nef}(Y_i)\big), \mathrm{Exc}(g_i)\right\rangle$. We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^\circ$ a *maximal chamber* of $\mathrm{Eff}(X)$.

Definition 2.3. Let X be a normal projective variety with finitely generated divisor class group $\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\mathrm{PDiv}(X)$, in particular $h^1(X, \mathcal{O}_X) = 0$. The *Cox sheaf* and *Cox ring* of X are defined as

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D) \qquad\qquad \mathcal{R}(X) := \Gamma(X, \mathcal{R})$$

Recall that $\mathcal{R}$ is a sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras, whose multiplication maps are discussed in [ADHL15, Section 1.4]. In case the divisor class group is torsion-free one can just take the direct sum over a subgroup of $\mathrm{WDiv}(X)$, isomorphic to $\mathrm{Cl}(X)$ via the quotient map, getting immediately a sheaf of $\mathcal{O}_X$-algebras. Denote by $\widehat{X}$ the relative spectrum of $\mathcal{R}$ and by $\overline{X}$ the spectrum of $\mathcal{R}(X)$. The $\mathrm{Cl}(X)$-grading induces an action of the quasi-torus $H_X := \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(X)]$ on both spaces. The inclusion $\mathcal{O}_X \to \mathcal{R}$ induces a good quotient $p_X : \widehat{X} \to X$ with respect to this action. Summarizing we have the following diagram

$$\begin{tikzcd} \widehat{X} \arrow[d, "p_X"'] & [-25pt] \subseteq \overline{X} \\ X & \end{tikzcd}$$

to which we will refer as the *Cox construction* of X. In case $\mathcal{R}(X)$ is a finitely generated algebra the complement of $\widehat{X}$ in the affine variety $\overline{X}$ has codimension $\geqslant 2$. This subvariety is the *irrelevant locus* and its defining ideal is the *irrelevant ideal* $\mathcal{J}_{\mathrm{irr}}(X) \subseteq \mathcal{R}(X)$.

Remark 2.4. By [HK00, Proposition 2.9] a normal and $\mathbb{Q}$-factorial projective variety X over an algebraically closed field K, with finitely generated Picard group is a Mori dream space if and only if $\mathcal{R}(X)$ is a finitely generated K-algebra. Furthermore, the following equality holds

$$\dim \mathcal{R}(X) = \dim(X) + \mathrm{rank}\,\mathrm{Cl}(X)$$

see for instance [ADHL15, Theorem 3.2.1.4].

Let X be a normal $\mathbb{Q}$-factorial projective variety, and let D be an effective $\mathbb{Q}$-divisor on X. The stable base locus $\mathbf{B}(D)$ of D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$\mathbf{B}(D) = \bigcap_{s>0} B(sD).$$

Since stable base loci do not behave well with respect to numerical equivalence, we will assume that $h^1(X, \mathcal{O}_X) = 0$ so that linear and numerical equivalence of $\mathbb{Q}$-divisors coincide.

Then numerically equivalent $\mathbb{Q}$-divisors on X have the same stable base locus, and the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of X can be decomposed into chambers depending on the stable base locus of the corresponding linear series called *stable base locus decomposition*, see [CdFG17, Section 4.1.3] for further details.

If X is a Mori dream space, satisfying then the condition $h^1(X, \mathcal{O}_X) = 0$, determining the stable base locus decomposition of $\mathrm{Eff}(X)$ is a first step in order to compute its Mori chamber decomposition.

Remark 2.5. Recall that two divisors D_1, D_2 are said to be *Mori equivalent* if $\mathbf{B}(D_1) = \mathbf{B}(D_2)$ and the following diagram of rational maps is commutative

$$\begin{tikzcd} & X \arrow[dl, dashed, "\phi_{D_1}"'] \arrow[dr, dashed, "\phi_{D_2}"] & \\ X(D_1) \arrow[rr, "\sim"'] & & X(D_2) \end{tikzcd}$$

where the horizontal arrow is an isomorphism. Therefore, the Mori chamber decomposition is a refinement of the stable base locus decomposition.

Let X be a Mori dream space with Cox ring $\mathcal{R}(X)$ and grading matrix Q. The matrix Q defines a surjection

$$Q\colon E \to \mathrm{Cl}(X)$$

from a free, finitely generated, abelian group E to the divisor class group of X. Denote by γ the positive quadrant of $E_{\mathbb{Q}} := E \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $e_1, \ldots, e_r$ be the canonical basis of $E_{\mathbb{Q}}$. Given a face $\gamma_0 \preceq \gamma$ we say that $i \in \{1, \ldots, r\}$ is a cone index of γ_0 if $e_i \in \gamma_0$. The face γ_0 is an $\mathfrak{F}$*-face* if there exists a point of $\overline{X} = \mathrm{Spec}(\mathcal{R}(X))$ whose i-th coordinate is non-zero exactly when i is a cone index of γ_0 [ADHL15, Construction 3.3.1.1]. The set of these points is denoted by $\overline{X}(\gamma_0)$.

Example 2.6. If $\mathcal{R}(X) = \frac{K[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4T_5\rangle}$ then $\gamma_0 = \mathrm{cone}(e_1, e_4)$ is an $\mathfrak{F}$-face and

$$\overline{X}(\gamma_0) = \{(x_1, 0, 0, x_4, 0) \in \overline{X} \;:\; x_1x_4 \neq 0\}$$

On the other hand, $\mathrm{cone}(e_1, e_3, e_4)$ is not an $\mathfrak{F}$-face.

Given the Cox construction of X we denote by $X(\gamma_0) \subseteq X$ the image of $\overline{X}(\gamma_0)$, and given an $\mathfrak{F}$-face γ_0 its image $Q(\gamma_0) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ is an *orbit cone* of X. The set of all orbit cones of X is denoted by Ω. Accordingly to [ADHL15, Definition 3.1.2.6] a class $w \in \mathrm{Cl}(X)$ defines the *GIT chamber*

$$\lambda(w) := \bigcap_{\{\omega\in\Omega\,:\,w\in\omega\}} \omega \tag{2.7}$$

If w is an ample class of X the corresponding GIT chamber is the semi-ample cone of X. The variety X can be reconstructed from the pair $(\mathcal{R}(X), \Phi)$ formed by the Cox ring together with a *bunch of cones*, consisting of certain subsets of the orbit cones [ADHL15, Definition 3.1.3.2]. According to [ADHL15, Example 3.1.3.6] every GIT chamber λ defines a *bunch of orbit cones*

$$\Phi(\lambda) := \{\omega \in \Omega \;:\; \omega^\circ \supseteq \lambda^\circ\}$$

Given a class $w \in \mathrm{Cl}(X)$ we denote by $\lambda^{\mathrm{sbl}}(w)$ the subset of $\mathrm{Cl}(X)_{\mathbb{Q}}$ consisting of all classes which have the same stable base locus of w.

Proposition 2.8. *Let X be a normal variety with finitely generated Cox ring, bunch of orbit cones Φ, and let $w \in \mathrm{Cl}(X)$ be a class of X. Then*

$$\lambda^{\mathrm{sbl}}(w) = \bigcap_{\{\omega\in\Phi\,:\,w\in\omega\}} \omega \cap \bigcap_{\{\omega\in\Phi\,:\,w\notin\omega\}} \omega^c$$

Proof. Recall that, according to [ADHL15, Construction 3.2.1.3], the set of relevant faces $\mathrm{rlv}(\Phi)$ is the set of faces of γ which are mapped by Q to elements of Φ. Each relevant face $\gamma_0 \preceq \gamma$ defines a subset $X(\gamma_0) \subseteq X$ consisting of all the points of X whose i-th Cox coordinate is non-zero exactly when i is a cone index of γ_0 [ADHL15, Construction 3.3.1.1]. By [ADHL15, Proposition 3.3.2.8] the stable base locus of a class w is the union

$$\mathbf{B}(w) := \bigcup_{\{\gamma_0\in\mathrm{rlv}(\Phi)\,:\,w\notin Q(\gamma_0)\}} X(\gamma_0) \tag{2.9}$$

Applying Q to the elements of the set $\{\gamma_0 \in \mathrm{rlv}(\Phi) \;:\; w \notin Q(\gamma_0)\}$ one gets the set $\{\omega \in \Phi \;:\; w \notin \omega\}$ and the former set is completely determined by the latter. We claim that two classes w, w' define the same stable base locus if and only if the following holds

$$\{\omega \in \Phi \;:\; w \in \omega\} = \{\omega \in \Phi \;:\; w' \in \omega\} \tag{2.10}$$

Clearly, if w, w' define the same stable base locus then (2.10) holds. Now, assume that there is an $\omega \in \Phi$ such that $w' \notin \omega$ and $w \in \omega$. Let $\gamma_0 \preceq \gamma$ be such that $Q(\gamma_0) = \omega$. It suffices to show that $X(\gamma_0)$ is not contained in the stable base locus of w. Indeed if $X(\gamma_1)$ is any strata which contains $X(\gamma_0)$ then $\gamma_0 \preceq \gamma_1$, so that $w \in \omega = Q(\gamma_0) \subseteq Q(\gamma_1)$. Thus $X(\gamma_1)$ does not appear in (2.9), and the claim is proved. Finally, the statement follows by observing that if

$$w' \in \bigcap_{\{\omega\in\Phi\,:\,w\in\omega\}} \omega \cap \bigcap_{\{\omega\in\Phi\,:\,w\notin\omega\}} \omega^c$$

then the cones which contain, respectively do not contain w' are the same of those which contain, respectively do not contain w. $\square$

Corollary 2.11. *Let X be a normal variety with finitely generated Cox ring, bunch of orbit cones Φ and let $w \in \mathrm{Cl}(X)$ be a class of X. Then the following inclusion holds*

$$\lambda(w) \subseteq \lambda^{\mathrm{sbl}}(w)$$

Proof. It is a direct consequence of Proposition 2.8 and of the following equalities

$$\lambda(w) = \bigcap_{\{\omega\in\Omega\,:\,w\in\omega\}} \omega = \bigcap_{\{\omega\in\Omega\,:\,w\in\omega\}} \omega \cap \bigcap_{\{\omega\in\Omega\,:\,w\notin\omega\}} \omega^c$$

where the first is by definition while the second is due to the fact that any two classes in the relative interior $\lambda(w)$ determine the same chamber. □

Proposition 2.12. *Let X be a normal variety with finitely generated Cox ring, bunch of orbit cones Φ, and $w_1, w_2 \in \mathrm{Cl}(X)$. Then the following are equivalent:*

(1) $\lambda^{\mathrm{sbl}}(w_1) = \lambda^{\mathrm{sbl}}(w_2)$*;*
(2) $\{\omega \in \Phi \,:\, w_1 \in \omega\} = \{\omega \in \Phi \,:\, w_2 \in \omega\}$*;*
(3) $\bigcap_{\{\omega\in\Phi\,:\,w_1\in\omega\}} \omega = \bigcap_{\{\omega\in\Phi\,:\,w_2\in\omega\}} \omega$*.*

Moreover, if $\Phi = \Phi(\lambda)$, with λ distinct from λ_1 and λ_2, then each of the above condition is implied by $\mathrm{cone}(\lambda \cup \lambda_1) \cap \mathring{\lambda}_2 \neq 0$ *and* $\mathrm{cone}(\lambda \cup \lambda_2) \cap \mathring{\lambda}_1 \neq 0$*.*

Proof. The equivalence of (1) with (2) is given by the proof of Proposition 2.8, and the implication (2) $\Rightarrow$ (3) is clear. To prove (3) $\Rightarrow$ (2) it suffices to observe that if there is an $\omega \in \Omega$ such that $w_1 \in \omega$ but $w_2 \notin \omega$, then w_2 would be contained in $\bigcap_{\{\omega\in\Phi\,:\,w_2\in\omega\}} \omega$ but not in $\bigcap_{\{\omega\in\Phi\,:\,w_1\in\omega\}} \omega$, so that the two sets will be different. Finally if $\mathrm{cone}(\lambda \cup \lambda_1) \cap \mathring{\lambda}_2 \neq 0$ holds then any orbit cone $\omega \in \Phi(\lambda)$ which contains λ_1 must intersect the interior of λ_2 so that $\lambda_2 \subseteq \omega$. Thus $\{\omega \in \Phi \,:\, w_1 \in \omega\} \subseteq \{\omega \in \Phi \,:\, w_2 \in \omega\}$ holds. Similarly one proves the opposite inclusion. □

Proposition 2.13. *Let X be a toric 3-fold such there are two Mori chambers in* $\mathrm{Mov}(X)$ *whose union gives a single stable base locus chamber. Then there are at least five Mori chambers inside* $\mathrm{Mov}(X)$*.*

Proof. Assume there are four Mori chambers inside $\mathrm{Mov}(X)$, and consider the chamber corresponding to $\mathrm{Nef}(X)$. If the three remaining chambers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are adjacent to $\mathrm{Nef}(X)$, and C_1, C_2, C_3 are the corresponding three distinct flipping curves, then by Nakamaye's theorem [Laz04, Theorem 10.3.5] we get that the stable base locus of a divisor in $\mathcal{C}_i$ is C_i, and hence the stable base loci of divisors in $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are distinct. Note that this argument works also for three or less chambers.

If $\mathcal{C}_1, \mathcal{C}_2$ are adjacent to $\mathrm{Nef}(X)$ and $\mathcal{C}_3$ is not then again Nakamaye's theorem yields that divisors in $\mathcal{C}_1, \mathcal{C}_2$ have as base loci two different irreducible curves. Furthermore, the base locus of a divisor in $\mathcal{C}_3$ is a curve with two components.

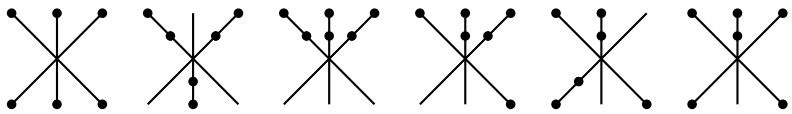
Now, let $\mathcal{C}_1, \ldots, \mathcal{C}_4$ be four consecutive maximal Mori chambers contained in the moving cone of a $\mathbb{Q}$-factorial toric threefold X of Picard rank n and let $X = X_1, \ldots, X_4$ be the corresponding birational models of X. Assume that $\mathcal{C}_1$ is the ample cone of X. Let $\tau_{ij} := \mathcal{C}_i \cap \mathcal{C}_j$ and let $\Gamma_{ij} \subseteq X_j$ be the new irreducible curve produced by the wall crossing of τ_{ij}. By the previous argument the two chambers with the same stable locus are $\mathcal{C}_3$ and $\mathcal{C}_4$. In order for $\mathcal{C}_3$ and $\mathcal{C}_4$ to have the same stable base locus on X the center $\Gamma_{34} \subseteq X_4$ of the small $\mathbb{Q}$-factorial modification given by the wall crossing of τ_{34} must be an irreducible curve which does not exists in X. This is possible only if this curve is the strict transform of the curve Γ_{12} created by the wall crossing of τ_{12}. In particular

$$\Gamma_{34} \sim \alpha\Gamma_{12} + \beta\Gamma_{23},$$

with $\alpha, \beta \in \mathbb{Q}$. If we denote by H_{ij} the linear span of the cone τ_{ij} in $\mathrm{Cl}_{\mathbb{Q}}(X)$, then the above discussion implies that the three hyperplanes H_{12}, H_{23}, H_{34} lie on a pencil. Each of these hyperplanes must contain at least $n-1$ of the $n+3$ classes of generators of the Cox ring. Assuming that the intersection $H_{12} \cap H_{23} \cap H_{34}$ contains r of these classes then we have the following inequality

$$3(n-1-r) + r \leqslant n+3$$

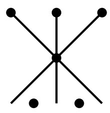
which implies $r \geqslant n-3$. If $r = n-3$ then each of the three hyperplanes must contain exactly two more classes of generators. Up to symmetries the possible configurations of the six classes on the three hyperplanes are the following.



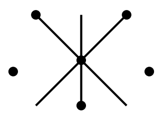
In the last four cases at least one of the facets τ_{ij} would be not contained in the interior of the moving cone (see Proposition [ADHL15, 3.3.2.3]). In the first two cases there would be at least six Mori chambers in the moving cone.

If $r = n-2$ then each of the three hyperplanes contains at least one more class. The possible configurations are the following, where in each case still two more generators have to be added to the picture.

In the first case it is possible to fulfill these conditions only if the two more classes are as in the picture below.



In this case the moving cone contains at least five Mori chambers, as shown in Example 3.8. In the last case, up to symmetry there is again one possibility for the open chambers where the two new classes can belong. This is displayed in the picture below.



However, in this last case we can have either five or three chambers in the movable cone.

If $r = n-1$ then again each of the three hyperplanes contains at least one more class, so that the above configurations still apply. Here we have to add only one more class to the picture and thus one of the three cones τ_{ij} would not be contained in the interior of the moving cone. $\square$

Examples 3.8 and 3.9 in the next section show that Proposition 2.13 is sharp in the sense that one can has Mori chamber decomposition distinct from the stable base locus decomposition inside the movable cone with five chambers and with three chambers in dimension higher than three.

3. Examples

In this section we give examples of varieties for which the Mori chamber and the stable base locus decomposition do not coincide, and we analyze this phenomenon from both the geometrical and the combinatorial point of view.

Example 3.1. (Birational viewpoint) Consider a plane $\Pi \subset \mathbb{P}^n$ and five general points $p_1, \dots, p_5 \in \Pi$. Let $f : X \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at $p_1, \dots, p_5$ with exceptional divisors $E_1, \dots, E_5$. Then the strict transform $\widetilde{\Pi} \subset X$ of Π is a del Pezzo surface of degree four. In particular $\widetilde{\Pi}$ is a Mori dream space.

Let $e_1, \dots, e_5$ be classes of a line in the exceptional divisors, and l the pull-back a a general line in $\mathbb{P}^n$. Let $\widetilde{C} \subset X$ be an irreducible curve. If $\widetilde{C}$ gets contracted by f the $\widetilde{C} \sim me_i$ with $m > 0$ for some $i \in \{1, \dots, 5\}$. Otherwise, we may write $\widetilde{C} \sim dl - m_1e_1 - \dots - m_5e_5$, that is $\widetilde{C}$ is the strict transform of a curve $C \subset \mathbb{P}^n$ of degree d having multiplicity m_i at p_i for $i = 1, \dots, 5$.

If $d < m_1 + \dots + m_5$ then $C \subset \Pi$ and $\widetilde{C} \subset \widetilde{\Pi}$. In this case we may write $\widetilde{C}$ as a linear combination with non-negative coefficients of $e_1, \dots, e_5, l - e_i - e_j, 2l - e_1 - \dots - e_5$ since these are the generators of $\mathrm{NE}(\widetilde{\Pi})$. If $d \geqslant m_1 + \dots + m_5$ then we may write

$$\widetilde{C} \sim m_1(l - e_1) + \dots m_5(l - e_5) + (d - m_1 - \dots - m_5)l$$

where $l - e_i = (l - e_i - e_j) + e_j$. Therefore,

$$\mathrm{NE}(X) = \langle e_i, l - e_i - e_j, 2l - e_1 - \dots - e_5 \rangle$$

Now, let $D \subset \mathbb{P}^n$ be the divisor given by the union of $n-2$ general hyperplanes containing Π. For the strict transform $\widetilde{D} \subset X$ of D we have $\widetilde{D} \sim (n-2)(H - E_1 - \dots - E_5)$, and

$$-K_X - \epsilon\widetilde{D} \sim (n+1-\epsilon(n-2))H - ((n-1)-\epsilon(n-2))(E_1 + \dots E_5)$$

where H is the pull-back of the hyperplane section of $\mathbb{P}^n$ via the blow-up morphism. Now, note that $(-K_X-\epsilon\widetilde{D})\cdot(l-e_i-e_j) = \epsilon(n-2)-n+3$, $(-K_X-\epsilon\widetilde{D})\cdot(2l-e_i-\dots-e_5) = 3\epsilon(n-2)-3n+7$, and $(-K_X-\epsilon\widetilde{D})\cdot e_i = n-1-\epsilon(n-2)$. Therefore, if $n \geqslant 3$ we have that $-K_X - \epsilon\widetilde{D}$ is ample for any $\frac{3n-7}{3n-6} < \epsilon\frac{n-1}{n-2}$.

Furthermore, since $\widetilde{D}$ is the union of $n-2$ smooth irreducible divisors intersecting transversally along the surface $\widetilde{\Pi}$ the pair $(X, \epsilon\widetilde{D})$ is klt for any $0 < \epsilon < 1$. Then, for any $\frac{3n-7}{3n-6} < \epsilon < 1$ the divisor $\epsilon\widetilde{D}$ induces a log Fano structure on X, and [BCHM10, Corollary 1.3.2] yields that X is a Mori dream space.

Now, consider the divisor $D_i \sim 2H - E_1 - \dots - \widehat{E_i} - \dots - E_4$ where the hat means that E_i does no appear in the expression of D_i. Note that D_i is nef, and since $D_i^n > 0$ it is also big. Therefore, D_i is semi-ample and big. Now, consider a curve $\widetilde{C} \subset X$

$$\widetilde{C} \sim \alpha_{1,2}(l-e_1-e_2) + \dots + \alpha_{4,5}(l-e_4-e_5) + \beta\widetilde{C}_{1,\dots,5} + \gamma_1 e_1 + \dots + \gamma_5 e_5$$

where $\widetilde{C}_{1,\dots,5} \sim 2l - e_1 - \dots - e_5$. Assume that $D_i \cdot \widetilde{C} = \gamma_1 + \dots + \gamma_5 = 0$. Hence we may write

$$\widetilde{C} \sim (2\beta + \alpha_{1,2} + \dots + \alpha_{4,5})l - (\beta + \alpha_{1,2} + \dots + \alpha_{1,5})e_1 - \dots - (\beta + \alpha_{4,5})e_5$$

Since $2\beta + \alpha_{1,2} + \dots + \alpha_{4,5} < \beta + \alpha_{1,2} + \dots + \alpha_{1,5} + \dots + \beta + \alpha_{4,5}$ we conclude that $\widetilde{C} \subset \widetilde{\Pi}$.

Then, a large enough multiple of D_i induces a birational morphism $f_{D_i} : X \to X_i$ contracting $\widetilde{\Pi}$ onto a $\mathbb{P}^1$ contained in X_i, and whose exceptional locus is exactly $\widetilde{\Pi}$, that is $\mathrm{Exc}(f_i) = \widetilde{\Pi}$. Indeed, $D_{i|\widetilde{\Pi}}$ yields the fibration $\widetilde{\Pi} \to \mathbb{P}^1$ induced by the linear system of conics in Π through $p_1, \dots, \widehat{p}_i, \dots, p_5$.

Let $f_{D_i} : X \to X_i$ and $f_{D_j} : X \to X_j$ be the morphisms induced respectively by D_i and D_j. From now on we will assume that $n \geqslant 4$ so that f_{D_i} is a small contraction. Then $\mathrm{Exc}(f_{D_i}) = \mathrm{Exc}(f_{D_j}) = \widetilde{\Pi}$. On the other hand, D_i and D_j give rise to two different flops

$$\begin{tikzcd}
X^- \arrow[dr] & & X \arrow[ll, dashed, "\chi^-"'] \arrow[rr, dashed, "\chi^+"] \arrow[dl, "f_{D_i}"] \arrow[dr, "f_{D_j}"'] & & X^+ \arrow[dl] \\
& X_i & & X_j &
\end{tikzcd}$$

Therefore, X^+ and X^- correspond to two different chambers $\mathcal{C}^+$ and $\mathcal{C}^-$ of the Mori chamber decomposition of $\mathrm{Mov}(X)$. On the other hand, by Nakamaye's theorem [Laz04, Theorem 10.3.5] the base locus of χ^+ is $\mathrm{Exc}(f_{D_j})$ and the base locus of χ^- is $\mathrm{Exc}(f_{D_i})$. These base loci are respectively the stable base loci of D_i and D_j.

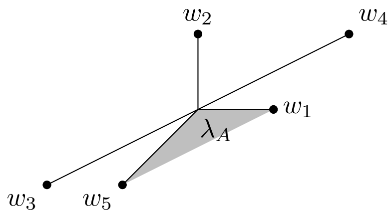
Finally, since $\mathrm{Exc}(f_{D_i}) = \mathrm{Exc}(f_{D_j}) = \widetilde{\Pi}$ we conclude that $\mathcal{C}^- \cup \mathcal{C}^+$ is a unique chamber of the stable base locus decomposition of $\mathrm{Mov}(X)$.

More generally, for $i = 1, \dots, 5$ we get five different chambers, all of them adjacent to $\mathrm{Nef}(X)$, of the Mori chamber decomposition of $\mathrm{Mov}(X)$ whose union gives a single chamber of its stable base locus decomposition. Furthermore, the stable base locus of a divisor in this chamber is exactly the surface $\widetilde{\Pi}$.

Example 3.2. (Cox rings viewpoint) Let X be the toric variety with Cox ring $\mathcal{R}(X) := K[T_1, \dots, T_5]$ whose grading matrix and irrelevant ideal are the following

$$Q = \begin{bmatrix} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & -1 & 1 & -1 \end{bmatrix} \qquad \mathcal{J}_{\mathrm{irr}}(X) = \langle T_1, T_4 \rangle \cap \langle T_1, T_5 \rangle \cap \langle T_3, T_5 \rangle$$

The degrees of the generators are displayed in the following picture together with the semi-ample cone λ_A which is the gray region



The matrix Q defines the following exact sequence where the right hand side $\mathbb{Z}^2$ is identified with the divisor class group of X

$$0 \to \mathbb{Z}^3 \to \mathbb{Z}^5 \xrightarrow{Q} \mathbb{Z}^2 \to 0$$

Denote by γ the positive quadrant of $\mathbb{Q}^5$. Since the Cox ring is a polynomial ring the $\mathfrak{F}$-faces are all the faces of γ and the orbit cones are all their projections in $\mathbb{Q}^2$. It follows that the maximal GIT chambers of X are all the 2-dimensional cones generated by pairs of consecutive rays. Recall that λ_A is the GIT chamber corresponding to the semi-ample cone, generated by w_1, w_5. The corresponding bunch of orbit cones is

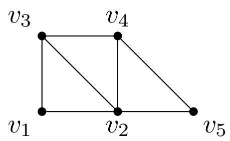
$$\Phi(\lambda_A) := \{\text{orbit cones } \omega \text{ such that } \omega^\circ \supseteq \lambda_A^\circ\}$$

In particular the only orbit cone of Φ which contains w_2 is the whole of $\mathbb{Q}^2$. Denote by $\lambda_{i,j}$ the cone determined by ω_i and ω_j, and observe that $\operatorname{cone}(\lambda_A \cup \lambda_{2,3}) = \operatorname{cone}(\lambda_A \cup \lambda_{2,4}) = \mathbb{Q}^2$. Thus, by Proposition 2.12, we conclude that $\lambda^{\mathrm{sbl}}(w_2) = \lambda_{2,3} \cup \lambda_{2,4}$.

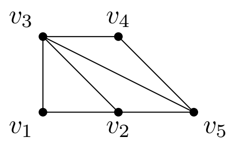
Example 3.3. (Fans viewpoint) Let $v_1 \dots, v_5 \in \mathbb{Z}^3$ be a set of vectors which is Gale dual to the set $w_1, \dots, w_5 \in \mathbb{Z}^2$ in Example 3.2. We can assume $v_1 \dots, v_5$ to be the five columns of the following matrix

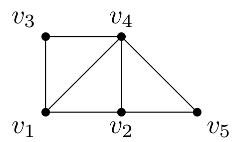
$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

These vectors generate the one dimensional cones of a fan Σ whose cones are displayed in the following picture



The toric variety $X = X(\Sigma)$ is the same as the one previously defined in Example 3.2. The displayed fan structure correspond to choosing the semi-ample chamber to be the GIT chamber λ_A of the previous picture. The following are the fan structures of the toric varieties whose semi-ample cone is respectively $\operatorname{cone}(w_1, w_4)$ and $\operatorname{cone}(w_3, w_5)$.





Example 3.4. (Compactification) Let Y be the toric variety with Cox ring $\mathcal{R} := K[T_1, \dots, T_6]$ whose grading matrix and irrelevant ideal are the following

$$Q = \begin{bmatrix} 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 2 & 1 \end{bmatrix} \qquad \begin{aligned} \mathcal{J}_{\mathrm{irr}}(Y) = & \langle T_1, T_4 \rangle \cap \langle T_1, T_5 \rangle \cap \langle T_3, T_5 \rangle \\ & \cap \langle T_2, T_3, T_6 \rangle \cap \langle T_2, T_4, T_6 \rangle \end{aligned}$$

The toric variety Y is a completion of the variety X in Example 3.2 obtained by adding the vector $(1, -4, -2)$ to the primitive generators of the one dimensional cones of X. The maximal cones of Y have the following indexes: $[1,2,3]$, $[2,3,4]$, $[2,4,5]$, $[1,2,6]$, $[2,5,6]$, $[1,3,6]$, $[3,4,6]$, $[4,5,6]$. The variety Y is $\mathbb{Q}$-factorial, non-Gorenstein, with $2K_Y$ Cartier. The effective cone of Y has 16 maximal GIT chambers, and the moving cone of

Y has 3 maximal chambers. The columns of each of the following matrices generate a maximal GIT chamber of the moving cone, where the first one corresponds to the semi-ample cone

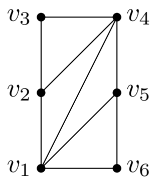
$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad \mathcal{M}_1 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 2 & 4 \end{bmatrix} \qquad \mathcal{M}_2 = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 4 & 8 \end{bmatrix}$$

The columns of each of the following matrices generate a maximal GIT chamber of the effective cone of X.

$$\mathcal{C}_1 = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 0 & 4 \end{bmatrix} \qquad \mathcal{C}_2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

The stable base locus of a divisor which lies in the interior of either $\mathcal{C}_1$ or $\mathcal{C}_2$ is $V(T_2)$. Thus the union of these two chambers is a unique stable base locus chamber. On the other hand, if we denote by X_i the projective toric variety whose semi-ample cone is given by $\mathcal{M}_i$, then from the point of view of X_i the GIT chambers coincide with the stable base locus chambers.

Example 3.5. Consider the toric variety $Z = Z(\Lambda)$ with fan $\Lambda \subset \mathbb{Q}^3$ given in the following picture



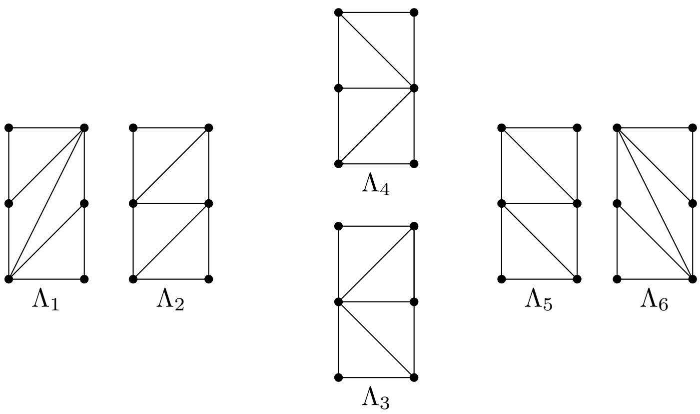
One can take for instance $v_1, \ldots, v_6$ to be the columns of the following matrix

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Then Z is a non-complete Mori dream space 3-fold with Picard number three and with Cox ring generated by six free variables with degrees given by the columns $w_1, \ldots, w_6$ of the following matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 2 & -2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

The effective cone of Z is the whole of $\mathbb{Q}^3$, it has 14 maximal GIT chambers and six of them are in the movable cone. Let us denote by $Z_j := Z(\lambda_j), j = 1, \ldots, 6$ the different models where $Z = Z_1$, and $\lambda_j, j = 1, \ldots, 6$ are the corresponding GIT chambers. The related fans are the following



and the possible flips are as follows

$$\begin{tikzcd}[column sep=small] & & & Z_4 \arrow[dr, dashed, "\eta_5"] & & & \\ Z_1 \arrow[r, dashed, "\eta_1"'] & Z_2 \arrow[urr, dashed, "\eta_3"] \arrow[drr, dashed, "\eta_2"'] & & & Z_5 \arrow[r, dashed, "\eta_6"'] & Z_6 \\ & & & Z_3 \arrow[ur, dashed, "\eta_4"'] & & & \end{tikzcd}$$

In Z we have that $\lambda_2, \lambda_3, \lambda_4$ are in distinct stable base locus chambers but λ_5 and λ_6 are inside the same stable base locus chamber. More precisely, if $w_j \in \lambda_j^\circ, j = 1, \dots, 6$ then the stable base loci are

$$\mathbf{B}(w_1) = \emptyset, \mathbf{B}(w_2) = l_{1,4}, \mathbf{B}(w_3) = l_{1,4} \cup l_{1,5}, \mathbf{B}(w_4) = l_{1,4} \cup l_{2,4}, \mathbf{B}(w_5) = \mathbf{B}(w_6) = l_{1,4} \cup l_{1,5} \cup l_{2,4}$$

where $l_{i,j}$ is the curve in the intersection of the toric divisors D_i, D_j corresponding to the vectors v_i, v_j in the fan Λ.

Geometrically we see that the flip η_6 has $l_{2,5}$ as flipping curve, and $l_{2,5}$ does not exist in Z_1. Therefore, the flipping curve in the last flip is not visible from the point of view of Z_1.

Note that we can produce an example of a complete 3-fold W with Picard number four such that $\mathrm{MCD}(W) \neq \mathrm{SBLD}(W)$ taking

$$v_8 = \frac{-v_1 - \cdots - v_7}{6} = (0, 0, -1)$$

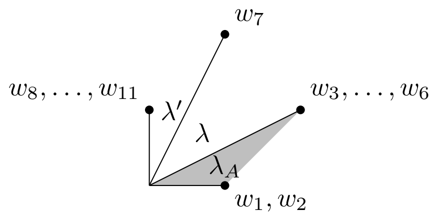
and including the relevant additional cones.

The following will be the leading example in the Section 5. Indeed, we will produce a complete Mori dream space X of Picard rank two such that $\mathrm{MCD}(X) \neq \mathrm{SBLD}(X)$.

Example 3.6. (Fundamental example) Let Z be the toric variety with Cox ring $K[T_1, \dots, T_{11}]$ whose grading matrix and irrelevant ideal are the following

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathcal{J}_{\mathrm{irr}}(Z) = \langle T_1, T_2 \rangle \cap \langle T_3, \dots, T_{11} \rangle$$

Denote by $w_i \in \mathbb{Z}^2$ the degree of T_i. The following picture displays the degrees of the generators of the Cox ring together with the three maximal GIT chambers of the moving cone, where the shaded one is the ample cone



Let $\overline{Z} = K^{11}$ be the spectrum of the Cox ring of Z and let $\overline{X}$ be the affine subvariety defined by

$$\overline{X} = \{F = G = 0\} \subset \overline{Z}$$

where F, G are general polynomial of degree $(2,2)$ in the T_i. that is general linear combinations of the following monomials

$$\left\{ \begin{array}{llllllll} T_1^2T_8^2 & T_1^2T_9^2 & T_1^2T_{10}^2 & T_1^2T_{11}^2 & T_2^2T_8^2 & T_2^2T_9^2 & T_2^2T_{10}^2 & T_2^2T_{11}^2; \\ T_3T_8 & T_3T_9 & T_3T_{10} & T_3T_{11} & T_4T_8 & T_4T_9 & T_4T_{10} & T_4T_{11}; \\ T_5T_8 & T_5T_9 & T_5T_{10} & T_5T_{11} & T_6T_8 & T_6T_9 & T_6T_{10} & T_6T_{11}; \\ T_1T_7 & T_2T_7. & & & & & & \end{array} \right. \tag{3.7}$$

It is immediate to check that for F, G general enough $\overline{X}$ is irreducible and $\mathrm{codim}_{\overline{X}}(\mathrm{Sing}(\overline{X})) \geqslant 2$. Since a local complete intersection is Cohen-Macaulay by Serre's criterion on normality we get that $\overline{X}$ is a normal variety. Let $p_Z \colon \widehat{Z} \to Z$ be the characteristic space morphism of Z and let $\widehat{X} := \overline{X} \cap \widehat{Z}$. The image of $\widehat{X}$ via p_Z is a subvariety X of Z. Since $\overline{X}$ is irreducible and normal, and X is a GIT quotient of $\overline{X}$ by a reductive group [Bri10, Theorem 1.24 (vi)] yields that X is irreducible and normal as well. We claim that

$$\widehat{X} = \text{ Zariski closure of } p_Z^{-1}(X \cap Z') \text{ in } \widehat{Z}$$

where Z' is the smooth locus of Z. Indeed Z' contains the open subset Z'' of Z obtained by removing the union of all the toric subvarieties of the form $p_Z(V(T_i,T_j))$, for any pair of indexes i,j. Since the Zariski closure of $p_Z^{-1}(X \cap Z'')$ in $\widehat{Z}$ equals $\widehat{X}$ the claim follows.

Observe that $\operatorname{codim}_{\overline{X}}(\overline{X}\backslash\widehat{X}) \geqslant 2$. Thus if one can show that the pull-back $\imath^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$, induced by the inclusion, is an isomorphism then by [ADHL15, Corollary 4.1.1.5], it follows that the Cox ring of X is

$$\mathcal{R}(X) = \frac{K[T_1,\ldots,T_{11}]}{I(\overline{X})}$$

Note that each of $w_{1..6}$, $w_{3..7}$ and $w_{7..11}$ is an orbit cone of $\overline{X}$. In particular, the three maximal chambers λ', λ, λ_A of the moving cone are GIT chambers. On the other hand, since $w_{1,2,7}$ is not an orbit cone, it follows that each orbit cone contains

$$\{\omega \,:\, \lambda_A \subseteq \omega \text{ and } \lambda' \subsetneq \omega\} = \{\omega \,:\, \lambda_A \subseteq \omega \text{ and } \lambda \subsetneq \omega\}$$

Then λ and λ' are contained in the same SBL chamber. It remains to show that

$$\imath^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$$

is an isomorphism. Note that the $K^* \times K^*$ action on $\overline{X}\backslash(V(T_2) \cup V(T_8))$ is trivial, so if we remove the images of $V(T_2) \cup V(T_8)$ from X, the resulting variety is isomorphic to an affine space. Therefore, $\mathrm{Cl}(X)$ is generated by the classes of the images of the two irreducible divisors $V(T_i) \cap \overline{X}$, with $i \in \{2,8\}$, and $\rho(X) \leqslant 2$.

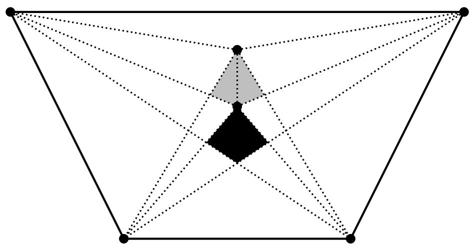
Note that crossing the wall corresponding to w_1, w_2 we get a morphism $f : Z \to \mathbb{P}^1$. Furthermore, $X \subset Z$ is not contained in any fiber of f, and hence f restricts to a surjective morphism $f_{|X} : X \to \mathbb{P}^1$. This forces $\rho(X) \geqslant 2$. Finally, we conclude that the images of $V(T_2), V(T_8)$ form a basis of $\mathrm{Cl}(X)$ and $\rho(X) = 2$.

We finish the present section with two examples of toric complete varieties in which the Mori chamber and stable base locus decomposition do not coincide even inside the movable cone.

Example 3.8. Let X be a toric variety with the following grading matrix

$$Q := \begin{bmatrix} 2 & 0 & 1 & 3 & 1 & 3 \\ 3 & 2 & 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 3 & 3 & 0 \end{bmatrix}$$

This is a toric 3-fold whose Mori chamber and stable base locus decomposition do not coincide even inside the movable cone. More precisely, its effective cone has 17 GIT chambers, of these 5 are inside the movable cone. The picture below shows a section of the Mori chamber decomposition of $\mathrm{Eff}(X)$. The 5 inner chambers, two triangles and three quadrilaterals, are the GIT chambers of the movable cone. There are five possible models, the picture shows in black a chosen semi-ample cone, and in the corresponding model the two gray GIT chambers $\mathcal{C}$ and $\mathcal{C}'$ share the stable base locus.

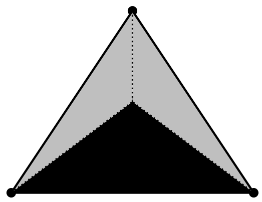


If one chooses any of the other four models the situation is similar, the two GIT chambers opposite to the chosen semi-ample cone inside the movable cone will have the same stable base locus.

Example 3.9. Let X be a toric variety with the following grading matrix

$$Q := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This is a toric 4-fold whose Mori chamber and stable base locus decomposition do not coincide inside the movable cone. More precisely, its effective cone has 3 GIT chambers, all of them inside the movable cone since $\mathrm{Eff}(X) = \mathrm{Mov}(X)$. The picture below shows a section of the Mori chamber decomposition of $\mathrm{Eff}(X)$. There are three possible models, the picture shows in black a chosen semi-ample cone, and in the corresponding model the two gray GIT chambers $\mathcal{C}$ and $\mathcal{C}'$ have the stable base locus. If one chooses any of the other two models the situation is the same.



Remark 3.10. All the computations of this section have been implemented in Magma [BCP97] and Maple [HK15] programs. For convenience of the reader we include, as ancillary files in the arXiv version of the paper, the following files:

- `Readme.txt`, a text on how to use the remaining files;
- `SBLib.m`, the Magma library containing all the functions needed to verify our examples;
- `Examples.txt`, the examples.

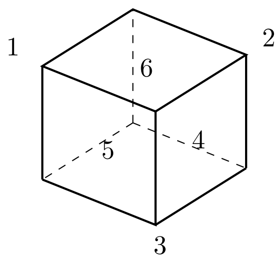
For an optimized version, implemented in Maple and Singular, of some of the algorithms presented in this library see [Kei12]. The library `SBLib.m` contains nine commands which we briefly describe here.

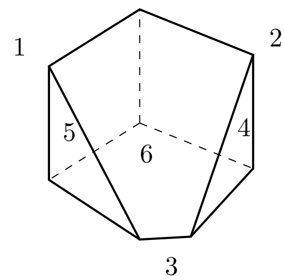
- `Ffaces`: computes the $\mathfrak{F}$-faces of an ideal I of a polynomial ring according to [ADHL15, Remark 3.1.1.11].
- `Eff`: computes the effective cone of a family of vectors in a rational vector space according to [ADHL15, Definition 2.2.2.5]. It takes as input the grading matrix whose columns are the relevant vectors.
- `Mov`: computes the moving cone of a family of vectors in a rational vector space according to [ADHL15, Definition 2.2.2.5]. It takes as input the grading matrix whose columns are the named vectors.
- `OrbitCones`: computes the orbit cones as projections of the $\mathfrak{F}$-faces according to [ADHL15, Proposition 3.1.1.10]. It takes as input a pair consisting of the set of $\mathfrak{F}$-faces together with the grading matrix.
- `GitChamber`: computes the GIT chamber defined by a class w according to [ADHL15, Definition 3.1.2.6]. It takes as input a pair consisting of the set of orbit cones together with a class w.
- `GitFan`: computes the GIT quasi-fan, that is the collection of all the git cones, of the set of orbit cones according to [Kei12, Algorithm 8]. It takes as input a pair consisting of the set of orbit cones together with a class w.
- `BunchCones`: computes a bunch of orbit cones defined by a GIT chamber λ according to [ADHL15, Example 3.1.3.6]. It takes as input a pair consisting of the set of orbit cones together with a class w in the relative interior of λ.
- `SameSbl`: decides whether two classes w_1, w_2 have the same stable base locus according to Proposition 2.8. It takes as input a triple consisting of a bunch of orbit cones together with the two classes w_1 and w_2.
- `FindTriples`: determines all the triples $(\lambda_A, \lambda_1, \lambda_2)$ of GIT chambers such that λ_1 and λ_2 are contained in the same stable base locus chamber of the variety whose semi-ample chamber is λ_A. It works according to Proposition 2.8, and takes as input a triple consisting of the grading matrix, the set of orbit cones and the GIT fan.

4. Smooth toric 3-folds of Picard rank three

In this section we prove Theorem 1.2. Let N be a finitely generated free abelian group, $M := \mathrm{Hom}(N, \mathbb{Z})$, $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $X := X(\Sigma)$ be a smooth projective toric variety defined by a fan $\Sigma \subseteq N_{\mathbb{Q}}$. Since X is projective Σ is the normal fan of the Riemann-Roch polytope $\Delta \subseteq M_{\mathbb{Q}}$ of any ample divisor of X, in particular each facet of Δ corresponds to a one dimensional cone of Σ. Since X is smooth each maximal cone of X is simplicial so that each vertex of Δ has valence $n := \dim(X)$.

When X is a threefold of Picard rank three the polytope Δ has six facets and all its vertexes have valency three. If we denote by (v, e, f) the vector consisting of the number of vertexes, edges and facets of Δ, then the conditions $f = 6$, $2e = 3v$ and $v - e + f = 2$ give $(v, e, f) = (8, 12, 6)$. According to the classification of hexahedra there are only two possible topological types for Δ, displayed in the following pictures.





Number the facets of each polytope from 1 to 6 and let C_i be the primitive generator of the inward normal vector to the i-th facet. Any vertex is labeled by a triple (i, j, k) in such a way that

$$\det(C_i, C_j, C_k) = 1. \tag{4.0}$$

For the type I polytope the vertexes labels are: $(1,2,3)$, $(2,4,3)$, $(1,6,2)$, $(1,3,5)$, $(4,2,6)$, $(1,5,6)$, $(3,4,5)$, $(4,6,5)$, while for the type II polytope the vertexes labels are: $(1,2,3)$, $(1,3,5)$, $(1,6,2)$, $(2,4,3)$, $(1,5,6)$, $(4,2,6)$, $(6,5,3)$, $(3,4,6)$. Without loss of generality we can assume $\{C_1, C_2, C_3\}$ to be the canonical basis of $N_{\mathbb{Q}} \simeq \mathbb{Q}^3$. Thus the matrix whose columns are the primitive generators of the fan Σ and its orthogonal have the form

$$P := \begin{bmatrix} 1 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 1 & c_1 & c_2 & c_3 \end{bmatrix} \qquad Q := \begin{bmatrix} -a_1 & -b_1 & -c_1 & 1 & 0 & 0 \\ -a_2 & -b_2 & -c_2 & 0 & 1 & 0 \\ -a_3 & -b_3 & -c_3 & 0 & 0 & 1 \end{bmatrix}$$

Applying conditions (4.0) to all type I triples gives the following equations $a_1 = b_2 = c_3 = -1$, $a_3c_1 = b_3c_2 = a_2b_1 = 0$, $a_2b_3c_1 + a_3b_1c_2 = 0$. These equations cut out the union of six three-dimensional affine spaces which are in a unique orbit of the S_3 action which permutes the coordinates. One of these spaces is given by $a_1 = b_2 = c_3 = -1$, $b_1 = c_1 = c_2 = 0$. The corresponding Q matrix is

$$Q_1 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -a_2 & 1 & 0 & 0 & 1 & 0 \\ -a_3 & -b_3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Any integer vector (a_2, a_3, b_3) gives an example of a smooth toric threefold whose defining fan is of type I.

Applying conditions (4.0) to all the type II triples gives the following equations $a_1 = b_2 = c_3 = -1$, $b_3c_2 = a_3c_1 = 0$, $a_2b_3 + a_3 + 1 = 0$, $b_3 + a_3b_1 + 1 = 0$. These equations cut out the union of three irreducible subvarieties of dimension two. Two of these are the affine spaces $a_1 = b_2 = b_3 = c_3 = -1$, $a_3 = c_2 = 0$, $a_2 = 1$ and $a_1 = b_2 = a_3 = c_3 = -1$, $b_3 = c_1 = 0$, $b_1 = 1$. The Q matrix for points on the first affine space is

$$Q_2 := \begin{bmatrix} 1 & -b_1 & -c_1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The second affine space gives a similar Q matrix. The third variety has equations $a_1 = b_2 = c_3 = -1$, $c_1 = c_2 = 0$, $a_2b_3 + a_3 + 1 = 0$, $a_3b_1 + b_3 + 1 = 0$. Considering only integer points this third variety is the union of a point and two one parameter families corresponding to the following grading matrices

$$Q_3 := \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, Q_4 := \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ -a_2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, Q_5 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -a_2 & 1 & 0 & 0 & 1 & 0 \\ -a_2+1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Note now that the case Q_3 is a sub-case of Q_2 and the case Q_5 is a sub-case of Q_1. Therefore it is enough to analyze Q_1, Q_2 and Q_4. We will call $A, B, \ldots, F$ the columns of Q. Note that in all cases $D = (1,0,0), E = (0,1,0)$ and $F = (0,0,1)$.

In the case of Q_1 we have $C = F$ and the point B moves in the half line from F to E. There are three possibilities $b_3 < 0, b_3 = 0, b_3 > 0$. If $b_3 = 0$ there is nine possibilities for the point A according to a_2 and a_3 being negative, zero or positive. If $b_3 < 0$ we now have four more possibilities depending if the point A is above or below the line DF, therefore 13 possibilities. For $b_3 > 0$ we have 13 possibilities as well. Summing up we have 35 possibilities. Choosing a particular value of b_2, a_2, a_3 for each of the 35 cases and computing the Mori chamber and the stable base locus decomposition one gets exactly two cases where the Mori chamber and stable base locus decomposition do not coincide, corresponding to the following matrices

$$G_1 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 1 & 0 \\ \beta & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \alpha,\beta > 0,\ G_2 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 1 & 0 \\ \beta & \gamma & 1 & 0 & 0 & 1 \end{bmatrix}, \alpha,\beta,\gamma < 0$$

Now, consider the case Q_2. In this case A is fixed, B moves in the line connecting $(0,1,1)$ to D and C moves in the line DF. If $c_1 = 0$ then $C = D$ and B has three possibilities according to sign of b_1. If $c_1 < 0$ there are five possibilities for b_1 depending on the sign and on which of the inequalities $b_1 \geqslant c_1$ and $b_1 < c_1$ holds. For $c_1 > 0$ there are five possibilities as well but now what matters is the sign of b_1 and if it is greater than $c_1 + 1$ or not. Therefore there are 13 possibilities to check. Computing explicitly the decompositions we get 4 new possible types of varieties whose Mori chamber and stable base locus decompositions do not coincide:

$$G_3 := \begin{bmatrix} 1 & 0 & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \gamma > 0, \qquad G_4 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \beta > 0 > \gamma$$

$$G_5 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \beta < 0 < \gamma, \quad G_6 := \begin{bmatrix} 1 & \beta & \gamma & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \beta < \gamma - 1 < -1$$

Finally, doing the same with Q_4 we have to consider four cases: $a_2 < 0$ (A inside the triangle DEF), $a_2 = 0$ (A in the midpoint of CF), $a_2 = 1$ (A in the line BF), and $a_2 > 2$ (A below the line BF). Doing this we get a single new type:

$$G_7 := \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This proves Theorem 1.2.

5. Mori dream spaces of Picard rank two

Let X be a Mori dream space with divisor class group $\mathrm{Cl}(X)$ of rank two. Since X is a projective variety its effective cone is pointed. Moreover $\mathrm{Cl}(X)$ has rank two so that we can fix a total order on the classes in the effective cone $w \leqslant w'$ if w is on the left of w'. Given two convex cones λ, λ' contained in the effective cone we will write

$$\lambda \leqslant \lambda' \quad \text{if } w \leqslant w' \text{ for any } w \in \lambda \text{ and } w' \in \lambda$$

Denote by $\{f_1, \dots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$ of X, and let $w_i = \deg(f_i)$ for any i.

Proposition 5.1. *Let X be a projective $\mathbb{Q}$-factorial toric variety with Picard rank two. Then the Mori chamber and the stable base locus decompositions of $\mathrm{Eff}(X)$ coincide.*

Proof. Let λ_A be the semi-ample cone of X and let λ', λ'' be two distinct maximal GIT chambers of X.

According to (2.10) it suffices to show that there exists an orbit cone of the bunch which contains one of $\lambda_A \cup \lambda'$, $\lambda_A \cup \lambda''$ but not the other. Since X is complete the effective cone is pointed, so that, since the Picard rank is two, one can order the GIT chambers of X.

Assuming $\lambda' \leqslant \lambda''$, we have three possibilities: either $\lambda' \leqslant \lambda_A \leqslant \lambda''$, or $\lambda_A \leqslant \lambda' \leqslant \lambda''$, or $\lambda' \leqslant \lambda'' \leqslant \lambda_A$. Since X is toric each pair of degrees of generators of the Cox ring span an orbit cone. Thus in the first two cases we can find an orbit cone which contains $\lambda_A \cup \lambda'$ but not $\lambda_A \cup \lambda''$, while in the last case we can find an orbit cone which contains $\lambda_A \cup \lambda''$ but not $\lambda_A \cup \lambda'$. □

The following is our first simple criterion implying the equality of the two chamber decompositions for a $\mathbb{Q}$-factorial Mori dream space of Picard rank two.

Proposition 5.2. *Let X be a projective $\mathbb{Q}$-factorial Mori dream space with Picard rank two. If all the generators of $\mathcal{R}(X)$ appear in the walls of the stable base locus decomposition of $\mathrm{Eff}(X)$ then the Mori chamber and the stable base locus decompositions of $\mathrm{Eff}(X)$ coincide.*

Proof. By (2.7) the Mori chamber decomposition is a subdivision of $\mathrm{Eff}(X)$ whose walls are given by some of the generators of $\mathcal{R}(X)$, and it is a refinement of the stable base locus decomposition. Since, by hypothesis all the generators appear as walls of the stable base locus decomposition such a refinement must be trivial. $\square$

Now, we develop some technical results in order to describe the semi-stable loci corresponding to the GIT chambers of the Mori chamber decomposition of a Mori dream space of Picard rank two.

Lemma 5.3. *Let X be a Mori dream space with Picard rank two, $\lambda \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ a maximal GIT chamber of X, and $\overline{X}^{\mathrm{ss}}(\lambda)$ the corresponding subset of semi-stable points of $\overline{X}$. Then the following holds*

$$\overline{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda) = V(f_i : w_i \leqslant \lambda) \cup V(f_i : \lambda \leqslant w_i)$$

Proof. By [ADHL15, Theorem 3.1.2.8] we have $\overline{X}^{\mathrm{ss}}(\lambda) = \overline{X}^{\mathrm{ss}}(w)$ for any $w \in \lambda^\circ$, where the second semi-stable locus is the complement of the zero set of all the homogeneous sections of the Cox ring whose degree is a positive multiple of w. If we choose such a class $w \in \lambda^\circ$ so that $w < w_i$ for any $w_i \in \lambda^\circ$ then each monomial in $f_1, \dots, f_r$ of degree nw must contain at least one f_i with $w_i \leqslant \lambda$. Thus the inclusion

$$V(f_i : w_i \leqslant \lambda) \subseteq \overline{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda)$$

follows. The analogous inclusion for $V(f_i : \lambda \leqslant w_i)$ can be proved in a similar way.

To prove the opposite inclusion observe that if $\overline{x} \in \overline{X}$ is a point which does not belong to the union $V(f_i : w_i \leqslant \lambda) \cup V(f_i : \lambda \leqslant w_i)$, then there exist two sections f_i, with $w_i \leqslant \lambda$, and f_j, with $\lambda \leqslant w_j$, each of which does not vanish on $\overline{x}$.

Take non negative $a, b \in \mathbb{Z}$ such that $aw_i + bw_j \in \lambda^\circ$. Since $f_i^a f_j^b$ is a homogeneous element of the Cox ring of degree $aw_i + bw_j \in \lambda^\circ$ which does not vanish on $\overline{x}$, the point $\overline{x}$ is in $\overline{X}^{\mathrm{ss}}(\lambda)$. $\square$

Lemma 5.4. *Let X be a Mori dream space with Picard rank two and let $\lambda, \lambda' \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ be two maximal distinct GIT chambers of X with $\lambda \leqslant \lambda'$. Then the following inclusion is strict*

$$V(f_i : w_i \leqslant \lambda') \subsetneqq V(f_i : w_i \leqslant \lambda)$$

Proof. Assume that the equality $V(f_i : w_i \leqslant \lambda') = V(f_i : w_i \leqslant \lambda)$ holds. By hypothesis the inclusion $V(f_i : \lambda \leqslant w_i) \subseteq V(f_i : \lambda' \leqslant w_i)$ holds. Thus by Lemma 5.3 there would be an inclusion $\overline{X}^{\mathrm{ss}}(\lambda') \subseteq \overline{X}^{\mathrm{ss}}(\lambda)$. By [ADHL15, Theorem 3.1.2.8] the latter inclusion would imply $\lambda \subseteq \lambda'$, a contradiction. $\square$

Recall that a Mori dream space X is a good quotient of its characteristic space $\widehat{X} = \overline{X}^{\mathrm{ss}}(\lambda_A)$, and denote by $p_{\lambda_A} : \widehat{X} \to X$ the good quotient map. The following simple characterization of stable base loci will be fundamental for the rest of the paper.

Lemma 5.5. *If $\lambda \leqslant \lambda_A$ then the stable base locus of a class $w \in \lambda$ is*

$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda)) \tag{5.6}$$

If $\lambda \geqslant \lambda_A$ then the stable base locus of a class $w \in \lambda$ is

$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda)) \tag{5.7}$$

Proof. In order to prove (5.6) just note that the first equality holds by definition while the second equality is due to Lemma 5.3 and the fact that $\lambda \leqslant \lambda_A$. Clearly (5.7) can be proved using a completely analogous argument. $\square$

The following is the main technical tool of the paper.

Theorem 5.8. *Let $X = X(\lambda_A)$ be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two corresponding to the maximal chamber λ_A of the Mori chamber decomposition of* $\mathrm{Eff}(X)$. *If for any $\lambda' \leqslant \lambda \leqslant \lambda_A$ we have*

$$V(f_i : w_i \leqslant \lambda)\backslash V(f_i : w_i \geqslant \lambda_A) \subsetneqq V(f_i : w_i \leqslant \lambda')\backslash V(f_i : w_i \geqslant \lambda_A) \tag{5.9}$$

and for any $\lambda_A \leqslant \lambda \leqslant \lambda'$ we have

$$V(f_i : w_i \geqslant \lambda)\backslash V(f_i : w_i \leqslant \lambda_A) \subsetneqq V(f_i : w_i \geqslant \lambda')\backslash V(f_i : w_i \leqslant \lambda_A) \tag{5.10}$$

then the Mori chamber and the stable base locus decompositions of $\mathrm{Eff}(X)$ *coincide.*

Furthermore, if for any $\lambda \leqslant \lambda_A \leqslant \lambda'$ we have that

$$V(f_i : w_i \leqslant \lambda)\backslash(V(f_i : w_i \leqslant \lambda_A) \cup V(f_i : w_i \geqslant \lambda_A))$$

is different from
$$V(f_i : w_i \geqslant \lambda')\backslash(V(f_i : w_i \leqslant \lambda_A) \cup V(f_i : w_i \geqslant \lambda_A))$$
then the stable base locus chambers of $\mathrm{Eff}(X)$ *are convex.*

Proof. Let λ be a maximal non-ample GIT chamber of X. Assume that $\lambda \leqslant \lambda_A$, where λ_A is the ample cone of X. By (5.6) in Lemma 5.5 the stable base locus of a class $w \in \lambda$ is
$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda))$$
Now let $\lambda' \leqslant \lambda_A$ be any maximal GIT chamber distinct from λ and λ_A. Without loss of generality we can assume that $\lambda' \leqslant \lambda$ then by Lemma 5.4 we deduce the following
$$\begin{gathered} V(f_i : w_i \leqslant \lambda_A) \subsetneqq V(f_i : w_i \leqslant \lambda) \subsetneqq V(f_i : w_i \leqslant \lambda') \\ V(f_i : w_i \geqslant \lambda_A) \supsetneqq V(f_i : w_i \geqslant \lambda) \supsetneqq V(f_i : w_i \geqslant \lambda') \end{gathered}$$
where all the inclusions are strict. Taking the intersection with $\widehat{X}$ is equivalent to remove from $V(f_i : w_i \leqslant \lambda) \subsetneq V(f_i : w_i \leqslant \lambda')$ the common subset $V(f_i : w_i \leqslant \lambda_A)$ and their intersection with $V(f_i : w_i \geqslant \lambda_A)$. So hypothesis (5.9) yields that
$$\widehat{X} \cap V(f_i : w_i \leqslant \lambda) \neq \widehat{X} \cap V(f_i : w_i \leqslant \lambda')$$
Since X is $\mathbb{Q}$-factorial, the good quotient $p_{\lambda_A}\colon \widehat{X} \to X$ is geometric [ADHL15, Corollary 1.6.2.7]. It follows that the images of the above sets via p_{λ_A} remain distinct in X and thus that $\mathbf{B}(w) \neq \mathbf{B}(w')$ for any $w \in \lambda^\circ$ and $w' \in \lambda'^\circ$.

Now, assume that $\lambda_A \leqslant \lambda$. Then (5.7) in Lemma 5.5 yields that the stable base locus of a class $w \in \lambda$ is
$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X}\backslash\overline{X}^{\mathrm{ss}}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda))$$
In this case if λ' is a maximal chamber distinct from λ such that $\lambda_A \leqslant \lambda \leqslant \lambda'$ Lemma 5.3 yields the following strict inclusions
$$\begin{gathered} V(f_i : w_i \geqslant \lambda_A) \subsetneqq V(f_i : w_i \geqslant \lambda) \subsetneqq V(f_i : w_i \geqslant \lambda') \\ V(f_i : w_i \leqslant \lambda_A) \supsetneqq V(f_i : w_i \leqslant \lambda) \supsetneqq V(f_i : w_i \leqslant \lambda') \end{gathered}$$
To conclude it is enough to argue as in the previous case using (5.10) instead of (5.9).

Summing up we showed that any pair of distinct GIT chambers lying on the same side of λ_A gives two different stable base locus chambers. Therefore, the Mori chamber decomposition of $\mathrm{Eff}(X)$ coincide with its stable base locus decomposition.

Finally, an analogous argument shows that if $\lambda \leqslant \lambda_A \leqslant \lambda'$ and our last hypothesis holds then for any $w \in \lambda$ and $w' \in \lambda'$ we have $\mathbf{B}(w) \neq \mathbf{B}(w)$, and hence the stable base locus chambers of $\mathrm{Eff}(X)$ are convex. □

Remark 5.11. Let us consider the Mori dream space X in Example 3.6. Note that (3.7) yields
$$V(f_i : w_i \geqslant \lambda_A) = \{T_1 = T_2 = \widetilde{F} = \widetilde{G} = 0\}$$
where $\widetilde{F}, \widetilde{G}$ are general linear combinations of the following monomials
$$\begin{cases} T_3T_8 \quad T_3T_9 \quad T_3T_{10} \quad T_3T_{11} \quad T_4T_8 \quad T_4T_9 \quad T_4T_{10} \quad T_4T_{11}; \\ T_5T_8 \quad T_5T_9 \quad T_5T_{10} \quad T_5T_{11} \quad T_6T_8 \quad T_6T_9 \quad T_6T_{10} \quad T_6T_{11}; \end{cases}$$
$$\begin{aligned} &V(f_i : w_i \leqslant \lambda) = \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\} \\ &V(f_i : w_i \leqslant \lambda') = \{T_1 = T_2 = T_8 = T_9 = T_{10} = T_{11} = 0\} \cup \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\} \end{aligned}$$
and $V(f_i : w_i \geqslant \lambda_A) \supset \{T_1 = T_2 = T_8 = T_9 = T_{10} = T_{11} = 0\}$ contains a component of the set $V(f_i : w_i \leqslant \lambda')$.

In what follows we work out some interesting consequences of Theorem 5.8.

Corollary 5.13. *Let* $X = X(\lambda_A)$ *be a* $\mathbb{Q}$*-factorial Mori dream space with Picard rank two corresponding to the maximal chamber* λ_A *of the Mori chamber decomposition of* $\mathrm{Eff}(X)$*. If for any maximal chamber* λ *we have that* $V(f_i : w_i \leqslant \lambda)$ *and* $V(f_i : w_i \geqslant \lambda)$ *are irreducible then the Mori chamber and the stable base locus decompositions of* $\mathrm{Eff}(X)$ *coincide.*

Proof. Without loss of generality we may assume that $\lambda' \leqslant \lambda \leqslant \lambda_A$. Since $V(f_i : w_i \leqslant \lambda')$ is irreducible either $V(f_i : w_i \geqslant \lambda_A) \supseteq V(f_i : w_i \leqslant \lambda')$ or $V(f_i : w_i \geqslant \lambda_A) \cap V(f_i : w_i \leqslant \lambda')$ is a closed subset of $V(f_i : w_i \leqslant \lambda')$.

Assume that $V(f_i : w_i \geqslant \lambda_A) \supseteq V(f_i : w_i \leqslant \lambda')$. Then since $V(f_i : w_i \geqslant \lambda') \subseteq V(f_i : w_i \geqslant \lambda_A)$ we get that $X^{\mathrm{ss}}(\lambda_A) \subseteq X^{\mathrm{ss}}(\lambda')$, and [ADHL15, Theorem 3.1.2.8] yields that $\lambda' \subseteq \lambda_A$, a contradiction. Therefore, $V(f_i : w_i \geqslant \lambda_A)$ intersects $V(f_i : w_i \leqslant \lambda')$ in a closed subset, and to conclude it is enough to apply Theorem 5.8. ☐

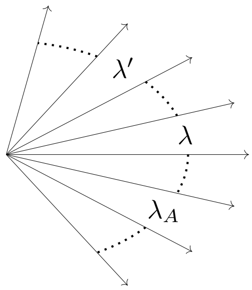
Now, we are ready to prove the main result of the paper.

Theorem 5.14. *Let X be a $\mathbb{Q}$-factorial Mori dream space with Picard rank two, $\{f_1, \dots, f_r\}$ a minimal set of homogeneous generators for the Cox ring $\mathcal{R}(X)$, $w_i := \deg(f_i)$, and λ_A be the ample chamber of X. Denote by c the codimension of X into its canonical toric embedding* [ADHL15, Section 3.2.5]. *Define*

$$h^+ := \#\{f_i : w_i \geqslant \lambda_A\} \quad \text{and} \quad h^- := \#\{f_i : w_i \leqslant \lambda_A\}$$

If $h^- > c$ and $h^+ > c$, then the Mori chamber and the stable base locus decomposition of $\mathrm{Eff}(X)$ *coincide.*

Proof. Consider $p_{\lambda_A} : \widehat{X} \to X$, let $\overline{X}$ be the total space of X, and $\overline{Z} \cong \mathbb{A}^r$ be the affine space with coordinates given by the f_i. Let λ', λ be two chambers of the Mori chamber decomposition of $\mathrm{Eff}(X)$ as in the following picture



Recall that by (5.6) in Lemma 5.5 the stable base loci of classes $w \in \lambda$, $w' \in \lambda'$ are given respectively by

$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \backslash \overline{X}^{\mathrm{ss}}(\lambda)) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda))$$
$$\mathbf{B}(w') = p_{\lambda_A}(\widehat{X} \backslash \overline{X}^{\mathrm{ss}}(\lambda')) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda'))$$

and the non semi-stable locus of λ_A is

$$V(f_i : w_i \leqslant \lambda_A) \cup V(f_i : w_i \geqslant \lambda_A)$$

Now, $\overline{X} \subset \mathbb{A}^r$ has dimension $\dim(X) + 2$, and hence any irreducible component of the intersection $\overline{X} \cap V(f_i : w_i \leqslant \lambda')$ has dimension greater than or equal to $\dim(X) + 2 - h'$, where $h' = \#\{f_i : w_i \leqslant \lambda'\}$. Assume that an irreducible component of $\overline{X} \cap V(f_i : w_i \leqslant \lambda')$ is contained in $\overline{X} \cap V(f_i : w_i \geqslant \lambda_A)$. Then such component must be contained in

$$V(f_i, f_j : w_i \leqslant \lambda', w_j \geqslant \lambda_A)$$

which has dimension $r - h' - h^+$. This forces $h^+ \leqslant c$, a contradiction with our hypothesis. Now, to conclude that λ, λ' are two different stable base locus chambers it is enough to recall that Lemma 5.4 yields $V(f_i : w_i \leqslant \lambda) \subsetneqq V(f_i : w_i \leqslant \lambda')$.

When $\lambda_A \leqslant \lambda \leqslant \lambda'$ we argue in a completely analogous way, and then to conclude it is enough to apply Theorem 5.8. ☐

The following is the first immediate consequence of Theorem 5.14.

Corollary 5.15. *Let Z be a projective normal $\mathbb{Q}$-factorial toric variety with $\mathrm{rk}(\mathrm{Cl}(Z)) = 2$, and $X \subseteq Z$ a projective normal $\mathbb{Q}$-factorial Mori dream hypersurface such that $\imath^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism. Then the Mori chamber and the stable base locus decompositions of both* $\mathrm{Eff}(Z)$ *and* $\mathrm{Eff}(X)$ *coincide.*

Proof. For a toric variety the claim follows from Proposition 5.1 and it is also an immediate consequence of Theorem 5.14 with $c = 0$. In general, following the notation in the proof of Theorem 5.14, there are always at least two generators in the sets $\{f_i : w_i \geqslant \lambda_A\}$, $\{f_i : w_i \leqslant \lambda_A\}$ otherwise λ_A would be a chamber of $\mathrm{Eff}(X) \backslash \mathrm{Mov}(X)$. Since $c = \mathrm{codim}_Z(X) = 1$ we conclude by Theorem 5.14. ☐

Remark 5.16. Theorem 5.14 is sharp. Indeed, the Mori dream space in Example 3.6 has three Mori chamber but just two stable base locus chambers. In this example $h^+ = c = 2$.

Remark 5.17. An intrinsic quadric is a normal $\mathbb{Q}$-factorial projective Mori dream space with Cox ring defined by a single quadratic relation. Smooth intrinsic quadrics with small Picard rank have been studied recently in [FH18]. By Corollary 5.15 the Mori chamber and the stable base locus decomposition of the effective cone of an intrinsic quadric of Picard rank two coincide.

5.17. **Picard rank two varieties with a torus action of complexity one.** Recall that a variety with a torus action of complexity one is a normal complete algebraic variety X with an effective action of a torus T such that the biggest T-orbits are of codimension one in X.

Proposition 5.18. *Let X be smooth rational projective variety of Picard rank two that admits a torus action of complexity one. Then the Mori chamber and the stable base locus decomposition of* $\operatorname{Eff}(X)$ *coincide.*

Proof. By [FHN16, Theorem 1.1] any smooth rational projective variety of Picard rank two with a torus action of complexity one, with just one exception, is a Mori dream hypersurface in its canonical toric embedding. Therefore, with the exception of the variety No. 13 in the statement of [FHN16, Theorem 1.1] the claim follows directly from Corollary 5.15.

On the hand, the Cox ring of the exceptional variety X has eight generators $T_1, \dots, T_8$, with $\deg(T_1) = \deg(T_3) = \deg(T_5) = \deg(T_7)$, and $\deg(T_2) = \deg(T_4) = \deg(T_6) = \deg(T_8)$. Therefore, both the Mori chamber and the stable base locus decomposition of $\operatorname{Eff}(X)$ consist of a single chamber which is indeed the nef cone of X. □

In what follows we apply the techniques developed in this section to compute the stable base locus decomposition which by Proposition 5.18 coincide with the Mori chamber decomposition, of the effective cones of the varieties in [FHN16, Theorem 1.1].

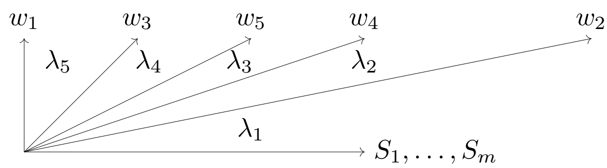
Example 5.19. (No. 6 in [FHN16, Theorem 1.1]) In this case X is a variety of dimension $m+3$ with Cox ring given by

$$\mathcal{R}(X) \cong \frac{k[T_1, \dots, T_6, S_1, \dots, S_m]}{(T_1T_2 + T_3T_4 + T_5^2T_6)}$$

with $m \geqslant 1$, and grading matrix

$$\begin{pmatrix} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

with $a, b, c \geqslant 0$, $a < b$ and $a + b = 2c + 1$. Here we develop the case $0 < a < c$, when $a = 0$ or $a = c$ a similar argument will work. Therefore, $\operatorname{MCD}(X)$ is a possibly trivial coarsening of the following decomposition



where $\lambda_1 = \lambda_A$ is the ample cone of X. Note that (5.6) in Lemma 5.5 yields

$$\begin{array}{ll} \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda_2)) = p_{\lambda_A}(\widehat{X} \cap \{T_4 = T_5 = T_3 = T_1 = 0\}) & \text{if } w \in \lambda_2; \\ \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda_3)) = p_{\lambda_A}(\widehat{X} \cap \{T_5 = T_3 = T_1 = 0\}) & \text{if } w \in \lambda_3; \\ \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda_4)) = p_{\lambda_A}(\widehat{X} \cap \{T_3 = T_1 = T_5^2T_6 = 0\}) & \text{if } w \in \lambda_4; \\ \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \leqslant \lambda_5)) = p_{\lambda_A}(\widehat{X} \cap \{T_1 = T_3T_4 + T_5^2T_6 = 0\}) & \text{if } w \in \lambda_5. \end{array}$$

Therefore, $\operatorname{MCD}(X) = \operatorname{SBLD}(X) = \{\lambda_A, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$.

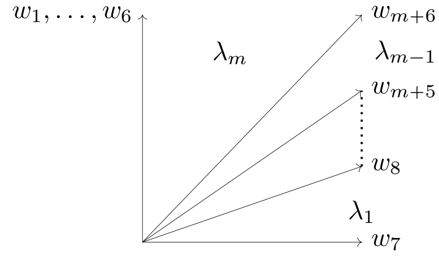
Example 5.20. (No. 8 in [FHN16, Theorem 1.1]) In this case X is a variety of dimension $m+3$ with Cox ring given by

$$\mathcal{R}(X) \cong \frac{k[T_1, \dots, T_6, S_1, \dots, S_m]}{(T_1T_2 + T_3T_4 + T_5T_6)}$$

with $m \geqslant 2$, and grading matrix

$$\left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right)$$

with $0 \leqslant a_2 \leqslant \cdots \leqslant a_m$ and $a_m > 0$. We develop the case $0 < a_2 < \cdots < a_m$, the same argument will work in the remaining cases as well. Therefore, $\mathrm{MCD}(X)$ is a possibly trivial coarsening of the following decomposition



where $\lambda_m = \lambda_A$ is the ample cone of X. Note that (5.7) in Lemma 5.5 yields

$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_j)) = p_{\lambda_A}(\widehat{X} \cap \{S_j = \cdots = S_1 = T_1T_2 + T_3T_4 + T_5T_6 = 0\})$$

if $w \in \lambda_j$, for $j = 1, \dots, m-1$. Therefore, $\mathrm{MCD}(X) = \mathrm{SBLD}(X) = \{\lambda_A, \lambda_{m-1}, \dots, \lambda_1\}$.

For all the other varieties listed in [FHN16, Theorem 1.1], with the exception of the varieties No. 3 and No. 12 for generic parameters, arguing similarly we get that the Mori chamber decomposition of the variety coincide with the one of the ambient toric variety which is given by the corresponding grading matrix in [FHN16, Theorem 1.1]. In the following we study the two exceptional cases.

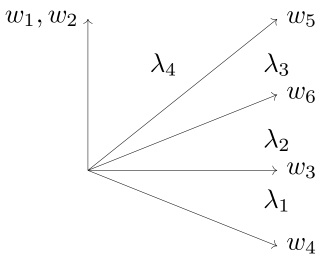
Example 5.21. (No. 3 in [FHN16, Theorem 1.1]) In this case X is a 3-fold with Cox ring given by

$$\mathcal{R}(X) \cong \frac{k[T_1, \dots, T_6]}{(T_1T_2T_3^2 + T_4T_5 + T_6^2)}$$

with $m \geqslant 2$, and grading matrix

$$\left(\begin{array}{cccccc} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right)$$

with $a \geqslant 1$. Therefore, in the case $a \geqslant 3$ the $\mathrm{MCD}(X)$ is a possibly trivial coarsening of the following decomposition

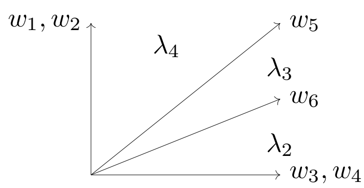


where $\lambda_4 = \lambda_A$ is the ample cone of X. Note that (5.7) in Lemma 5.5 yields

$$\begin{array}{ll} \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_3)) = p_{\lambda_A}(\widehat{X} \cap \{T_6 = T_3 = T_4 = 0\}) & \text{if } w \in \lambda_3; \\ \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_2)) = p_{\lambda_A}(\widehat{X} \cap \{T_6 = T_3 = T_4 = 0\}) & \text{if } w \in \lambda_2; \\ \mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_1)) = p_{\lambda_A}(\widehat{X} \cap \{T_4 = 0\}) & \text{if } w \in \lambda_1. \end{array}$$

Therefore, $\mathrm{MCD}(X) = \mathrm{SBLD}(X) = \{\lambda_A, \lambda_2 \cup \lambda_3, \lambda_1\}$.

In the case $a = 2$, $\mathrm{MCD}(X)$ is a possibly trivial coarsening of the following decomposition

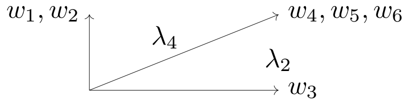


where $\lambda_4 = \lambda_A$ is the ample cone of X. Note that (5.7) in Lemma 5.5 yields

$$\begin{aligned}
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_2)) = p_{\lambda_A}(\widehat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_2;\\
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_3)) = p_{\lambda_A}(\widehat{X} \cap \{T_6 = T_3 = T_4 = 0\}) \quad \text{if } w \in \lambda_3.
\end{aligned}$$

Therefore, $\mathrm{MCD}(X) = \mathrm{SBLD}(X) = \{\lambda_A, \lambda_2 \cup \lambda_3\}$.

In the case $a = 1$, $\mathrm{MCD}(X)$ is a possibly trivial coarsening of the following decomposition



where $\lambda_4 = \lambda_A$ is the ample cone of X. Now (5.7) in Lemma 5.5 yields

$$\mathbf{B}(w) = p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_2)) = p_{\lambda_A}(\widehat{X} \cap \{T_3 = 0\}) \quad \text{if } w \in \lambda_2.$$

Therefore, $\mathrm{MCD}(X) = \mathrm{SBLD}(X) = \{\lambda_A, \lambda_2\}$.

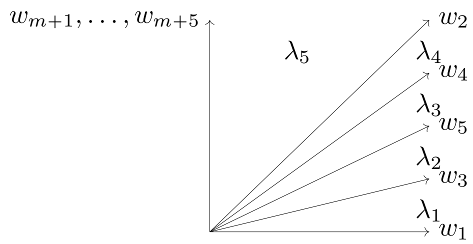
Example 5.22. (No. 12 in [FHN16, Theorem 1.1]) In this case X is a variety of dimension $m+2$ with Cox ring given by

$$\mathcal{R}(X) \cong \frac{k[T_1, \dots, T_5, S_1, \dots, S_m]}{(T_1T_2 + T_3T_4 + T_5^2)}$$

with $m \geqslant 2$, and grading matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & \dots & 1 \end{pmatrix}$$

with $0 \leqslant a \leqslant c \leqslant b$ and $a + b = 2c$. Therefore, in the case $0 < a < c < b$, $\mathrm{MCD}(X)$ is a possibly trivial coarsening of the following decomposition



where $\lambda_5 = \lambda_A$ is the ample cone of X. Note that (5.7) in Lemma 5.5 yields

$$\begin{aligned}
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_4)) = p_{\lambda_A}(\widehat{X} \cap \{T_4 = T_5 = T_3 = T_1 = 0\}) \quad \text{if } w \in \lambda_4;\\
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_3)) = p_{\lambda_A}(\widehat{X} \cap \{T_5 = T_3 = T_1 = 0\}) \quad \text{if } w \in \lambda_3;\\
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_2)) = p_{\lambda_A}(\widehat{X} \cap \{T_3 = T_1 = T_5 = 0\}) \quad \text{if } w \in \lambda_2.\\
\mathbf{B}(w) &= p_{\lambda_A}(\widehat{X} \cap V(f_i : w_i \geqslant \lambda_1)) = p_{\lambda_A}(\widehat{X} \cap \{T_1 = T_3T_4 + T_5^2 = 0\}) \quad \text{if } w \in \lambda_1.
\end{aligned}$$

Therefore, $\mathrm{MCD}(X) = \mathrm{SBLD}(X) = \{\lambda_A, \lambda_4, \lambda_2 \cup \lambda_3, \lambda_1\}$.

If there is an equality in any of the inequalities $0 \leqslant a \leqslant c \leqslant b$, then some w_j coincide and the corresponding chambers collapse as in Example 5.21. For instance, if $a = 0$ then $w_1 = w_3$ and the chamber λ_1 does not exist.

6. Grassmannians blow-ups

Let $\mathbb{G}(r,n)$ be the Grassmannian parametrizing r-planes in $\mathbb{P}^n$, and $\mathbb{G}(r,n)_k$ the blow-up of $\mathbb{P}^n$ at k general points. These blow-ups have been studied in [MR18]. In particular the stable base locus decomposition of $\mathrm{Eff}(\mathbb{G}(r,n)_1)$ has been computed in [MR18, Theorem 1.3].

In this section we will compute the Cox ring of $\mathbb{G}(r,n)_1$ by exploiting its spherical nature, and as a consequence of Proposition 5.2 we will answer positively to [MR18, Question 6.9] which ask whether the decomposition given in [MR18, Theorem 1.3] is the Mori chamber decomposition of $\mathrm{Eff}(\mathbb{G}(r,n)_1)$.

Definition 6.1. A *spherical variety* is a normal variety X together with an action of a connected reductive affine algebraic group $\mathscr{G}$, a Borel subgroup $\mathscr{B} \subseteq \mathscr{G}$, and a base point $x_0 \in X$ such that the $\mathscr{B}$-orbit of x_0 in X is a dense open subset of X.

Let $(X, \mathscr{G}, \mathscr{B}, x_0)$ be a spherical variety. We distinguish two types of $\mathscr{B}$-invariant prime divisors: a *boundary divisor* of X is a $\mathscr{G}$-invariant prime divisor on X, a *color* of X is a $\mathscr{B}$-invariant prime divisor that is not $\mathscr{G}$-invariant.

For instance, any toric variety is a spherical variety with $\mathscr{B} = \mathscr{G}$ equal to the torus. For a toric variety there are no colors, and the boundary divisors are the usual toric invariant divisors.

Set $\Lambda := \{I \subset \{0, \dots, n\}, |I| = r+1\}$ and $N := |\Lambda| + 1$. Define the *Hamming distance* on Λ as

$$d(I,J) = |I| - |I \cap J| = |J| - |I \cap J|$$

for each $I, J \in \Lambda$. Note that, with respect to this distance, the diameter of Λ is $r+1$. We consider the Grassmannian $\mathbb{G}(r,n)$ in the usual Plücker embedding $\mathbb{G}(r,n) \subset \mathbb{P}^N$.

For each pair $I = \{i_0 < \cdots < i_{r-1}\}, J = \{j_0 < \cdots < j_{r+1}\} \subset \{0, \dots, n\}$ with $|I| = r, |J| = r+2$, define a quadratic polynomial

$$F_{IJ} = \sum_{t=0}^{r+1} (-1)^t p_{i_0 \dots i_{r-1} j_t} p_{j_1 \dots \hat{j_t} \dots j_{r+1}} \tag{6.2}$$

Then the ideal of $K[p_I, I \in \Lambda]$ generated by the F_{IJ} is the ideal defining $\mathbb{G}(r,n) \subset \mathbb{P}^N$ [Sha13, Section I.4]. We denote by $\mathbb{G}(r,n)_1$ the blow-up of $\mathbb{G}(r,n)$ at $p = \langle e_0, \dots, e_r \rangle$, where $\{e_0, \dots, e_n\}$ is the canonical basis of K^{n+1}, by H the pull-back to $\mathbb{G}(r,n)_1$ of the hyperplane section of $\mathbb{P}^N$, and by E the exceptional divisor of the blow-up.

Proposition 6.3. *In the polynomial ring $K[S, T_I, I \in \Lambda]$ consider the ideal $\mathfrak{J}$ generated by the Plücker relations (6.2) in the coordinates T_I. Then*

$$\mathcal{R}(\mathbb{G}(r,n)_1) \cong \frac{K[S, T_I, I \in \Lambda]}{\mathfrak{J}}$$

and the degree of the variable T_I in $\mathrm{Cl}(\mathbb{G}(r,n)_1) = \mathbb{Z}[H] + \mathbb{Z}[E]$ is $(1, -d(I, \{0, \dots, r\}))$, where $\deg(S) = (0,1)$.

Proof. By [MR18, Proposition 4.1] under the action of the reductive group

$$\mathscr{G} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in GL_{r+1}, B \in GL_{n-r} \mid \det(A)\det(B) = 1 \right\} \subset SL_{n+1}$$

the blow-up $\mathbb{G}(r,n)_1$ is a spherical variety. We consider the Borel subgroup $\mathscr{B} \subset \mathscr{G}$ of matrices with upper triangular blocks. Consider the divisors $D_0, \dots, D_{r+1}$ in $\mathbb{G}(r,n)$ defined as $D_j := \{[\Sigma] \in \mathbb{G}(r,n) : \Sigma \cap \Gamma_j \neq \varnothing\}$, where

$$\begin{cases} \Gamma_0 = \langle e_{r+1}, \dots, e_n \rangle; \\ \Gamma_1 = \langle e_0, e_{r+1}, \dots, e_{n-1} \rangle; \\ \vdots \\ \Gamma_{r+1} = \langle e_0, \dots, e_r, e_{r+1}, \dots, e_{n-r-1} \rangle; \\ \Gamma' = \langle e_0, \dots, e_r \rangle. \end{cases}$$

Pulling-back these divisors via the blow-up map we obtain divisors in $\mathbb{G}(r,n)_1$. For sake of simplicity we will use the same notation for divisors in $\mathbb{G}(r,n)$ and their pull-backs in $\mathbb{G}(r,n)_1$.

Now, note that $\mathscr{G}$ preserves the dimension of the intersection of a given subspace of $\mathbb{P}^n$ with Γ_0 and with Γ'. Therefore $\mathscr{G} \cdot D_0 = D_0$ and $\mathscr{G} \cdot E = E$ that is, D_0 and E are boundary divisors. Note also that each D_j is a $\mathscr{B}$-invariant but not a $\mathscr{G}$-invariant prime divisor, and therefore $D_1, \dots, D_{r+1}$ are colors.

In order to determine the $\mathscr{G}$-orbit of $D_1, \dots, D_{r+1}$ we have to describe these divisors explicitly as zeros of polynomials in the Plücker coordinates.

In $\mathbb{G}(r,n)$ the divisor D_0 is given by $D_0 = V(p_{0,\dots,r})$. Indeed, if $q \in D_0$ then $q = [\Sigma]$ with $\Sigma \cap \Gamma_0 \neq \varnothing$ and therefore it can be represented with a matrix whose first row is of the form $(0, \dots, 0, a_{r+1}, \dots, a_n)$.

This implies that $p_{0\dots r}(q) = 0$. Conversely, if $p_{0\dots r}(q) = 0$ then the most left $(r+1) \times (r+1)$ sub-matrix of any matrix representing q has zero determinant, therefore there is another representation of q such that the first row has the following form $(0, \dots, 0, a_{r+1}, \dots, a_n)$, and we conclude that $q \in D_0$.

Similarly, setting $I_j = \{e_0, \dots, e_{j-1}, e_{r+1}, \dots, e_{n-j}\}$ for $j = 0, \dots, r+1$, we have $D_j = V(p_{I_j})$. Note that $d(I_j, I_0) = j$ for any j. More generally, one can consider the prime divisor $D_I := V(p_I)$ for any $I \in \Lambda$. We claim that the linear span of the orbit $\mathscr{G} \cdot D_j$ is given by

$$\operatorname{lin}(\mathscr{G} \cdot D_j) = \langle \{D_J; d(J, I_0) = j\} \rangle, j = 0, \dots, r+1 \tag{6.4}$$

Note that given $I, I' \in \Lambda$ with distance one and such that the non shared indexes, say $i \in I \backslash I', i' \in I' \backslash I$ are in $\{0, \dots, r\}$ we can find a $g \in \mathscr{G}$ such that $g(e_i) = e_{i'}$ and $g(e_j) = e_j$ for $j \neq i, i'$. The same holds if the non shared indexes are in $\{r+1, \dots, n\}$, and we get that

$$\{D_J; d(J, I_0) = j\} \subset \mathscr{G} \cdot D_j, j = 0, \dots, r+1$$

Since the D_J such that $d(J, I_0) = j, j = 0, \dots, r+1$, give a generating set of $H^0(\mathbb{G}(r,n), \mathcal{O}_{\mathbb{G}(r,n)}(1))$ we get (6.4). Now, let S and T_I be the canonical sections associated respectively to E and D_I. By [ADHL15, Theorem 4.5.4.6] $S, T_I, I \in \Lambda$ are generators of $\mathcal{R}(\mathbb{G}(r,n)_1)$. Furthermore, by [Ris17, Lemma 7.2.1] for any $I \in \Lambda$ we have that

$$\operatorname{mult}_{\langle e_0, \dots, e_r \rangle} D_I = 1 + \dim(\langle e_i, i \notin I \rangle \cap \langle e_0, \dots, e_r \rangle) = |(\{0, \dots, n\} \backslash I) \cap \{0, \dots, r\}| = d(I, I_0)$$

Therefore, if $\deg(S) = (0,1)$ the degree of the other generators of $\mathcal{R}(\mathbb{G}(r,n)_1)$ in $\operatorname{Pic}(\mathbb{G}(r,n)_1) = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$ is given by $\deg(T_I) = (1, -d(I, I_0))$.

The matrix representing this grading has size $2 \times (N+1)$ and is of the following form

$$A = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & 0 & -1 & \dots & -1 & -2 & \dots & -(r+1) \end{pmatrix}$$

Our next aim is to find relations among the generators of $\mathcal{R}(\mathbb{G}(r,n)_1)$. Note that for each pair $I = \{i_0 < \dots < i_{r-1}\}, J = \{j_0 < \dots < j_{r+1}\} \subset \{0, \dots, n\}$ with $|I| = r, |J| = r+2$, the polynomial

$$G_{IJ} = \sum_{t=0}^{r+1} (-1)^t T_{i_0 \dots i_{r-1} j_t} T_{j_1 \dots \widehat{j_t} \dots j_{r+1}}$$

is homogeneous of degree $(2, -|I \backslash I_0| - |J \backslash I_0|)$. Let $\mathfrak{J} \subset K[T_I, I \in \Lambda]$ be the ideal generated by the G_{IJ}. Since $\frac{K[T_I, I \in \Lambda]}{\mathfrak{J}}$ is the homogeneous coordinate ring of $\mathbb{G}(r,n)$, then

$$\dim(K[T_I, I \in \Lambda]/\mathfrak{J}) = (r+1)(n-r)+1$$

and Remark 2.4 yields

$$\dim(K[S, T_I, I \in \Lambda]/\mathfrak{J}) = (r+1)(n-r)+2 = \dim(\mathbb{G}(r,n)_1) + \operatorname{rank}(\operatorname{Pic}(\mathbb{G}(r,n)_1))) = \dim(\mathcal{R}(\mathbb{G}(r,n)_1))$$

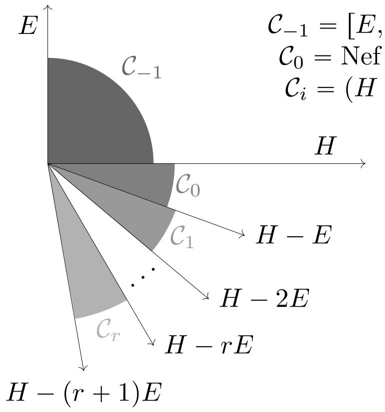
where we denote by $\mathfrak{J}$ the ideal generated by the polynomials G_{IJ} in $K[S, T_I, I \in \Lambda]$. We conclude that there are no further relations in $\mathcal{R}(\mathbb{G}(r,n)_1)$ and hence $\mathcal{R}(\mathbb{G}(r,n)_1) = \frac{K[S, T_I, I \in \Lambda]}{\mathfrak{J}}$ as claimed. □

Now, we are ready to compute the Mori chamber decomposition of $\operatorname{Eff}(\mathbb{G}(r,n)_1)$.

Proposition 6.5. *Let $\mathbb{G}(r,n)_1$ be the blow-up of the Grassmannian $\mathbb{G}(r,n)$ at a point. Then we have that* $\operatorname{Eff}(\mathbb{G}(r,n)_1) = \langle E, H-(r+1)E \rangle$, $\operatorname{Nef}(\mathbb{G}(r,n)_1) = \langle H, H-E \rangle$ *and*

$$\operatorname{Mov}(\mathbb{G}(r,n)_1) = \begin{cases} \langle H, H-rE \rangle & \text{if } n = 2r+1; \\ \langle H, H-(r+1)E \rangle & \text{if } n > 2r+1. \end{cases}$$

Furthermore, $\operatorname{MCD}(\mathbb{G}(r,n)_1)$ *and* $\operatorname{SBLD}(\mathbb{G}(r,n)_1)$ *coincide and their walls are given by the divisors $E, H, H-E, \dots, H-(r+1)E$ as represented in the following picture*



$$\mathcal{C}_{-1} = [E, H),$$
$$\mathcal{C}_0 = \mathrm{Nef}(\mathbb{G}(r,n)_1) = \langle H, H-E\rangle,$$
$$\mathcal{C}_i = (H-iE, H-(i+1)E] \textit{ for } i = 1,\dots,r,$$

where with the notation $\mathcal{C}_i = (H-iE, H-(i+1)E]$ we mean that the ray spanned by $H-(i+1)E$ belongs to $\mathcal{C}_i$ but the ray spanned by $H-iE$ does not, and similarly with the notation $\mathcal{C}_{-1} = [E,H)$ we mean that the ray spanned by E belongs to $\mathcal{C}_{-1}$ but the ray spanned by H does not.

Proof. The claims on the effective, nef and movable cones follow from [MR18, Theorem 1.3]. Furthermore, by [MR18, Theorem 1.3] the decomposition displayed in the statement is the stable base locus decomposition of $\mathrm{Eff}(\mathbb{G}(r,n)_1)$. Now, by Proposition 6.3 all the generators of $\mathcal{R}(\mathbb{G}(r,n)_1)$ appear in the walls of the stable base locus decomposition of $\mathrm{Eff}(\mathbb{G}(r,n)_1)$, and then Proposition 5.2 yields that the Mori chamber and the stable base locus decomposition of $\mathrm{Eff}(\mathbb{G}(r,n)_1)$ coincide. □

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