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A DIRECT AND ELEMENTARY DERIVATION OF THE MULTIVARIATE FAÀ DI BRUNO FORMULA

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Abstract. By applying Taylor expansion and the Leibniz rule for partial derivatives, we provide a straightforward deduction of the multivariate Faà di Bruno formula, yielding a concise and simple expression for the derivatives of the composite function $g \circ \boldsymbol{f} \colon \mathbb{R}^n \to \mathbb{R}$, with $\boldsymbol{f} \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^m \to \mathbb{R}$. Our formulation extends the original Faà di Bruno formula to the multidimensional case in a natural way.

1. Introduction and results

The well-known formula for the derivatives of composite functions, published by Faà di Bruno [4] in 1855 (see also [9] for a modern formulation and a simple proof) and dating back to [1], has been widely extended to the case of functions of several variables. In literature there are several versions and proofs of the multivariate Faà di Bruno formula for $\boldsymbol{f} \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^m \to \mathbb{R}$. See, among others, [3,5–8]. But, typically, the proofs are either excessively long or involve unnecessarily sophisticated tools.

Here we just apply the Taylor expansion for g and Leibniz rule for partial derivatives of the powers of $\boldsymbol{f}$ to give a straightforward proof of the multivariate Faà di Bruno formula. Furthermore, by these elementary methods, we obtain a simple formulation, (1.3) below, which is the natural extension to the multidimensional case of the original formula of Faà di Bruno.

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Notation. We use boldface for vectors and multi-indices: we write $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ and, for $\mathbb{N} = \{0, 1, 2, \dots\}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\boldsymbol{\nu} = (\nu_1, \dots \nu_m) \in \mathbb{N}^m$ with $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_n$ and $|\boldsymbol{\nu}| = \nu_1 + \dots + \nu_m$; for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$, $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ means $\beta_i \leq \alpha_i$ for all $1 \leq i \leq n$; $\boldsymbol{\beta}_p, \boldsymbol{\omega}_q \in \mathbb{N}^n$ (for suitable indices $p, q \in \mathbb{N}$) will denote n-dimensional multi-indices that will be used in the calculations.

Moreover, $\boldsymbol{y}^{\boldsymbol{\nu}} = \prod_{i=1}^m y_i^{\nu_i}$ and, for $\boldsymbol{f} = (f_1, \dots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{f}^{\boldsymbol{\nu}} = \prod_{i=1}^m f_i^{\nu_i}\colon \mathbb{R}^n \to \mathbb{R}$, with the convention that if h is a given function, then $h^0 \equiv 1$.

Also for partial derivatives we use standard notation: for $g\colon \mathbb{R}^m \to \mathbb{R}$ we set

$$\partial^{\boldsymbol{\nu}} g = g^{(\boldsymbol{\nu})} = \frac{\partial^{|\boldsymbol{\nu}|} g}{\partial y_1^{\nu_1} \cdots \partial y_m^{\nu_m}}\,.$$

Similar notation will be used for any function $h\colon \mathbb{R}^n \to \mathbb{R}$. We shall apply the following Leibniz rule for the partial derivatives of a product of functions $h_1, \dots, h_k\colon \mathbb{R}^n \to \mathbb{R}$:

$$\partial^{\boldsymbol{\alpha}} \big(h_1 \cdots h_k\big) = \sum_{\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_k = \boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\omega}_1! \cdots \boldsymbol{\omega}_k!}\, h_1^{(\boldsymbol{\omega}_1)} \cdots h_k^{(\boldsymbol{\omega}_k)}, \tag{1.1}$$

where $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!$, and $\boldsymbol{\omega}_1!$, …, $\boldsymbol{\omega}_k!$ are defined similarly. As usual, let $0! = 1$.

Finally, we shall always mean that the value of a given sum over a given set of indices is zero when that set is empty.

THEOREM 1.1. *Let $\boldsymbol{f} = (f_1, \dots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^m \to \mathbb{R}$, be smooth functions. Then, for $\boldsymbol{\alpha} \in \mathbb{N}^n$, $|\boldsymbol{\alpha}| \geq 1$, $\partial^{\boldsymbol{\alpha}}(g \circ \boldsymbol{f})$ has the following equivalent expressions*:

$$\begin{aligned} \partial^{\boldsymbol{\alpha}}(g \circ \boldsymbol{f}) &= \boldsymbol{\alpha}! \sum_{1 \leq |\boldsymbol{\nu}| \leq |\boldsymbol{\alpha}|} \frac{g^{(\boldsymbol{\nu})}(\boldsymbol{f})}{\boldsymbol{\nu}!} \\ &\times \sum_{\substack{\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_m = \boldsymbol{\alpha} \\ \nu_i \leq |\boldsymbol{\omega}_i| \leq |\boldsymbol{\alpha}|\nu_i}} \prod_{\substack{1 \leq i \leq m \\ \nu_i \geq 1}} \Bigg(\sum_{\substack{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_{\nu_i} = \boldsymbol{\omega}_i \\ |\boldsymbol{\beta}_j| \geq 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\boldsymbol{\beta}_j)}}{\boldsymbol{\beta}_j!} \Bigg), \end{aligned} \tag{1.2}$$

where $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{N}^m$ and $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_m, \boldsymbol{\beta}_1, \dots \boldsymbol{\beta}_{\nu_i} \in \mathbb{N}^n$;

$$\partial^{\boldsymbol{\alpha}}(g\circ \boldsymbol{f}) = \boldsymbol{\alpha}! \sum_{1\le|\boldsymbol{\nu}|\le|\boldsymbol{\alpha}|} g^{(\boldsymbol{\nu})}(\boldsymbol{f}) \tag{1.3}$$
$$\times \sum_{\boldsymbol{\omega}_1+\cdots+\boldsymbol{\omega}_m=\boldsymbol{\alpha}} \prod_{1\le i\le m} \Bigg(\sum_{\substack{\sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}=\nu_i \\ \sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}\boldsymbol{\beta}=\boldsymbol{\omega}_i}} \prod_{\boldsymbol{\beta}\in J} \frac{1}{k_{\boldsymbol{\beta}}!}\left(\frac{f_i^{(\boldsymbol{\beta})}}{\boldsymbol{\beta}!}\right)^{k_{\boldsymbol{\beta}}}\Bigg),$$

where $J = J(\boldsymbol{\alpha}) = \{\boldsymbol{\beta}\in\mathbb{N}^n \mid |\boldsymbol{\beta}|\ge 1,\ \boldsymbol{\beta}\le\boldsymbol{\alpha}\},\ k_{\boldsymbol{\beta}}\in\mathbb{N}$ *for all* $\boldsymbol{\beta}\in J$.

REMARK 1.2. Formula (1.3) is the natural extension to the multidimensional case of the original Faà di Bruno's formula. Note that in (1.3) it is not necessary to impose the conditions $\nu_i\le|\boldsymbol{\omega}_i|\le|\boldsymbol{\alpha}|\nu_i$ and $\nu_i\ge 1$ that we have in (1.2) for the summation over $\boldsymbol{\omega}_i$ and the product over $1\le i\le m$ respectively, because of the different expression in the parentheses of (1.3) with respect to that in (1.2). See the last part of the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. To begin with, we write the Taylor expansion of order $k\ge1$ of g at a given point $\boldsymbol{b}\in\mathbb{R}^m$. For $\boldsymbol{y}\in\mathbb{R}^m$, we have (see [2], [10]):

$$g(\boldsymbol{y}) = g(\boldsymbol{b}) + \sum_{1\le|\boldsymbol{\nu}|\le k} \frac{g^{(\boldsymbol{\nu})}(\boldsymbol{b})}{\boldsymbol{\nu}!}(\boldsymbol{y}-\boldsymbol{b})^{\boldsymbol{\nu}} + R_k(\boldsymbol{y},\boldsymbol{b}), \tag{1.4}$$

with the integral remainder

$$R_k(\boldsymbol{y},\boldsymbol{b}) = (k+1)\int_0^1 \Bigg[\sum_{|\boldsymbol{\nu}|=k+1} \frac{g^{(\boldsymbol{\nu})}(\boldsymbol{b}+t(\boldsymbol{y}-\boldsymbol{b}))}{\boldsymbol{\nu}!}(\boldsymbol{y}-\boldsymbol{b})^{\boldsymbol{\nu}}\Bigg](1-t)^k\,dt. \tag{1.5}$$

Hence, setting $\boldsymbol{b}=\boldsymbol{f}(\boldsymbol{a})$, with $\boldsymbol{a}$ a given point of $\mathbb{R}^n$, we find

$$g(\boldsymbol{f}(\boldsymbol{x})) = g(\boldsymbol{f}(\boldsymbol{a})) + \sum_{1\le|\boldsymbol{\nu}|\le k} \frac{g^{(\boldsymbol{\nu})}(\boldsymbol{f}(\boldsymbol{a}))}{\boldsymbol{\nu}!}\big(\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{a})\big)^{\boldsymbol{\nu}} + R_k(\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{f}(\boldsymbol{a})). \tag{1.6}$$

We shall now obtain $\partial^{\boldsymbol{\alpha}}(g\circ\boldsymbol{f})(\boldsymbol{a})$ by deriving the right-hand side of the expression (1.6) with respect to $\boldsymbol{x}$ and then setting $\boldsymbol{x}=\boldsymbol{a}$. Given $\boldsymbol{\nu}=(\nu_1,\dots,\nu_m)\in\mathbb{N}^m$, such that $|\boldsymbol{\nu}|\ge1$, from Leibniz rule (1.1), we have

$$\partial^{\boldsymbol{\alpha}}\Big(\big(\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{a})\big)^{\boldsymbol{\nu}}\Big) = \partial^{\boldsymbol{\alpha}}\Bigg(\prod_{i=1}^m \big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\Bigg) \tag{1.7}$$
$$= \sum_{\boldsymbol{\omega}_1+\cdots+\boldsymbol{\omega}_m=\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\omega}_1!\cdots\boldsymbol{\omega}_m!}\Bigg(\prod_{i=1}^m \partial^{\boldsymbol{\omega}_i}\Big(\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\Big)\Bigg)$$

$$= \sum_{\substack{\boldsymbol{\omega}_1+\cdots+\boldsymbol{\omega}_m=\boldsymbol{\alpha} \\ |\boldsymbol{\omega}_i|\le|\boldsymbol{\alpha}|\nu_i}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\omega}_1!\dots\boldsymbol{\omega}_m!}\Bigg(\prod_{\substack{1\le i\le m \\ \nu_i\ge 1}} \partial^{\boldsymbol{\omega}_i}\Big(\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\Big)\Bigg),$$

where in the last expression we took into account that $\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\equiv 1$ if $\nu_i=0$, thus

$$\partial^{\boldsymbol{\omega}_i}\Big(\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\Big)=0 \quad \text{if } \nu_i=0 \text{ and } |\boldsymbol{\omega}_i|>0\,; \tag{1.8}$$

since $\boldsymbol{\omega}_1+\cdots+\boldsymbol{\omega}_m=\boldsymbol{\alpha}$, the condition $|\boldsymbol{\omega}_i|\le|\boldsymbol{\alpha}|\nu_i$, added in the last expression of (1.7), simply means that we take $|\boldsymbol{\omega}_i|=0$ when $\nu_i=0$.

Furthermore, for $\nu_i\ge 1$, Leibniz rule (1.1) for $k=\nu_i$ and $h_1=\cdots=h_{\nu_i}=f_i(\boldsymbol{x})-f_i(\boldsymbol{a})$ gives that

$$\begin{aligned} &\partial^{\boldsymbol{\omega}_i}\Big(\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)^{\nu_i}\Big) \\ &= \sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i}=\boldsymbol{\omega}_i} \frac{\boldsymbol{\omega}_i!}{\boldsymbol{\beta}_1!\dots\boldsymbol{\beta}_{\nu_i}!}\Bigg(\prod_{j=1}^{\nu_i}\partial^{\boldsymbol{\beta}_j}\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)\Bigg) \\ &= \boldsymbol{\omega}_i! \sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i}=\boldsymbol{\omega}_i} \Bigg(\prod_{j=1}^{\nu_i}\frac{1}{\boldsymbol{\beta}_j!}\,\partial^{\boldsymbol{\beta}_j}\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)\Bigg). \end{aligned} \tag{1.9}$$

For $\boldsymbol{x}=\boldsymbol{a}$ we have

$$\partial^{\boldsymbol{\beta}_j}\big(f_i(\boldsymbol{x})-f_i(\boldsymbol{a})\big)\Big|_{\boldsymbol{x}=\boldsymbol{a}} = \begin{cases} 0 & \text{for } |\boldsymbol{\beta}_j|=0, \\ f_i^{(\boldsymbol{\beta}_j)}(\boldsymbol{a}) & \text{for } |\boldsymbol{\beta}_j|\ge 1, \end{cases} \tag{1.10}$$

and also

$$\partial^{\boldsymbol{\alpha}}R_k(\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{f}(\boldsymbol{a}))\big|_{\boldsymbol{x}=\boldsymbol{a}}=0 \quad \text{for } |\boldsymbol{\alpha}|\le k, \tag{1.11}$$

since, by (1.5), the remainder $R_k(\boldsymbol{y},\boldsymbol{b})$ in (1.4) satisfies the condition

$$\partial^{\boldsymbol{\nu}}R_k(\boldsymbol{y},\boldsymbol{b})\big|_{\boldsymbol{y}=\boldsymbol{b}}=0 \quad \text{for } \boldsymbol{\nu}\in\mathbb{N}^m,\ |\boldsymbol{\nu}|\le k.$$

Therefore, applying (1.7), (1.9), (1.10) and (1.11) to (1.6) for $\boldsymbol{x}=\boldsymbol{a}$ and $|\boldsymbol{\alpha}|=k\ge 1$, we get the expression (1.2) of $\partial^{\boldsymbol{\alpha}}(g\circ\boldsymbol{f})$:

$$\begin{aligned} &\partial^{\boldsymbol{\alpha}}(g\circ\boldsymbol{f})(\boldsymbol{a}) \\ &= \sum_{1\le|\boldsymbol{\nu}|\le|\boldsymbol{\alpha}|}\frac{g^{(\boldsymbol{\nu})}(\boldsymbol{f}(\boldsymbol{a}))}{\boldsymbol{\nu}!}\,\partial^{\boldsymbol{\alpha}}\big((\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{a}))^{\boldsymbol{\nu}}\big)\big|_{\boldsymbol{x}=\boldsymbol{a}} + \partial^{\boldsymbol{\alpha}}R_{|\boldsymbol{\alpha}|}(\boldsymbol{f}(x),\boldsymbol{f}(\boldsymbol{a}))\big|_{\boldsymbol{x}=\boldsymbol{a}} \end{aligned} \tag{1.12}$$

$$= \boldsymbol{\alpha}! \sum_{1\le|\boldsymbol{\nu}|\le|\boldsymbol{\alpha}|} \frac{g^{(\boldsymbol{\nu})}(f(\boldsymbol{a}))}{\boldsymbol{\nu}!} \sum_{\substack{\boldsymbol{\omega}_1+\cdots+\boldsymbol{\omega}_m=\boldsymbol{\alpha}\\ \nu_i\le|\boldsymbol{\omega}_i|\le|\boldsymbol{\alpha}|\nu_i}} \prod_{\substack{1\le i\le m\\ \nu_i\ge 1}} \Bigg(\sum_{\substack{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i}=\boldsymbol{\omega}_i\\ |\boldsymbol{\beta}_j|\ge 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\boldsymbol{\beta}_j)}(\boldsymbol{a})}{\boldsymbol{\beta}_j!} \Bigg),$$

where we have added the conditions $|\boldsymbol{\beta}_j| \ge 1$ in the third sum (because of (1.10)) and, consequently, $\nu_i \le |\boldsymbol{\omega}_i|$ in the second sum by $\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i}=\boldsymbol{\omega}_i$.

Finally, to prove that $\partial^{\boldsymbol{\alpha}}(g\circ \boldsymbol{f})$ satisfies (1.3), we consider the terms inside the parentheses in the second line of (1.12). We claim that, for $i \in \{1,\dots,m\}$ such that $\nu_i \ge 1$,

(1.13)
$$\sum_{\substack{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i}=\boldsymbol{\omega}_i\\ |\boldsymbol{\beta}_j|\ge 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\boldsymbol{\beta}_j)}(\boldsymbol{a})}{\boldsymbol{\beta}_j!} = \sum_{\substack{\sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}=\nu_i\\ \sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}\boldsymbol{\beta}=\boldsymbol{\omega}_i}} \frac{\nu_i!}{\prod_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}!} \prod_{\boldsymbol{\beta}\in J} \left(\frac{f_i^{(\boldsymbol{\beta})}(\boldsymbol{a})}{\boldsymbol{\beta}!}\right)^{k_{\boldsymbol{\beta}}},$$

where $J = J(\boldsymbol{\alpha}) = \{\boldsymbol{\beta}\in\mathbb{N}^n \mid 1\le|\boldsymbol{\beta}|,\ \boldsymbol{\beta}\le\boldsymbol{\alpha}\}$ and $k_{\boldsymbol{\beta}}\in\mathbb{N}$ for $\boldsymbol{\beta}\in J$. Indeed, we have that

$$\left(\sum_{\boldsymbol{\beta}\in J} \frac{f_i^{(\boldsymbol{\beta})}(\boldsymbol{a})}{\boldsymbol{\beta}!}\right)^{\nu_i} \tag{1.14}$$

contains all addends of the sum inside the left-hand side of (1.13) and can be written, by the multinomial theorem, as

$$\left(\sum_{\boldsymbol{\beta}\in J} \frac{f_i^{(\boldsymbol{\beta})}(\boldsymbol{a})}{\boldsymbol{\beta}!}\right)^{\nu_i} = \sum_{\sum\limits_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}=\nu_i} \frac{\nu_i!}{\prod\limits_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}!} \prod_{\boldsymbol{\beta}\in J} \left(\frac{f_i^{(\boldsymbol{\beta})}(\boldsymbol{a})}{\boldsymbol{\beta}!}\right)^{k_{\boldsymbol{\beta}}}. \tag{1.15}$$

Therefore, in order to select in (1.14) only the addends that appear in the left-hand side of (1.13), it is necessary and sufficient to add the condition

$$\sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}\boldsymbol{\beta} = \boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_{\nu_i} = \boldsymbol{\omega}_i \tag{1.16}$$

in the right-hand side of (1.15). Then, substituting (1.13) into (1.12) we obtain (1.3), but with the conditions $\nu_i \le |\boldsymbol{\omega}_i| \le |\boldsymbol{\alpha}|\nu_i$ in the second sum and $\nu_i \ge 1$ in the first product.

Now, due to the fact that we use the convention that $h^0 \equiv 1$, it is clear that the right-hand side of (1.13) is equal to 1 when $\nu_i = 0$ (and hence $|\boldsymbol{\omega}_i| = 0$, because of $|\boldsymbol{\omega}_i| \le |\boldsymbol{\alpha}|\nu_i$). Thus, we can avoid the condition $\nu_i \ge 1$ in the first product of (1.3). Furthermore, when in the second sum of (1.3) the condition $\nu_i \le |\boldsymbol{\omega}_i| \le |\boldsymbol{\alpha}|\nu_i$ does not hold for some $i\in\{1,\dots,m\}$, the right-hand side of (1.13) is 0. In fact, in this case the conditions $\sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}} = \nu_i$ and

$\sum_{\boldsymbol{\beta}\in J} k_{\boldsymbol{\beta}}\boldsymbol{\beta} = \boldsymbol{\omega}_i$ cannot both be satisfied, so the sum is empty. This means that in the second sum of (1.3) we can avoid the condition $\nu_i \le |\boldsymbol{\omega}_i| \le |\boldsymbol{\alpha}|\nu_i$. $\square$

In the following example we see an application of Theorem 1.1 in the case $n = 1, m = 2$.

Example 1.3. Compute the (ordinary) derivative $D^h g(f_1(x), f_2(x))$ for $x \in \mathbb{R}$, $h \in \mathbb{N}$, $h \ge 1$. From (1.3) we have that

$$D^h g(f_1, f_2) = h! \sum_{1\le \nu_1+\nu_2\le h} g^{(\nu_1,\nu_2)}(f_1, f_2)$$

$$\times \sum_{r_1+r_2=h} \Bigg(\sum_{\substack{k_1+\cdots+k_h=\nu_1\\ 1\cdot k_1+2\cdot k_2+\cdots+h\cdot k_h=r_1}} \frac{1}{k_1!\cdots k_h!}\Big(\frac{Df_1}{1!}\Big)^{k_1}\cdots\Big(\frac{D^h f_1}{h!}\Big)^{k_h}\Bigg)$$

$$\times \Bigg(\sum_{\substack{k_1+\cdots+k_h=\nu_2\\ 1\cdot k_1+2\cdot k_2+\cdots+h\cdot k_h=r_2}} \frac{1}{k_1!\cdots k_h!}\Big(\frac{Df_2}{1!}\Big)^{k_1}\cdots\Big(\frac{D^h f_2}{h!}\Big)^{k_h}\Bigg),$$

since $J = J(h) = \{1, 2, \ldots, h\}$.

In the particular case of $g(y_1, y_2) = \frac{1}{y_1 y_2}$, assuming $f_1(x), f_2(x) \ne 0$, we get

$$D^h \frac{1}{f_1 f_2} = \frac{h!}{f_1 f_2} \tag{1.17}$$

$$\times \sum_{\substack{1\le\nu_1+\nu_2\le h\\ r_1+r_2=h}} \sum_{\substack{k_1+\cdots+k_h=\nu_1\\ 1\cdot k_1+2\cdot k_2+\cdots+h\cdot k_h=r_1}} \frac{(-1)^{\nu_1}\nu_1!}{k_1!\cdots k_h!}\Big(\frac{Df_1}{1!\,f_1}\Big)^{k_1}\cdots\Big(\frac{D^h f_1}{h!\,f_1}\Big)^{k_h}$$

$$\times \sum_{\substack{k_1+\cdots+k_h=\nu_2\\ 1\cdot k_1+2\cdot k_2+\cdots+h\cdot k_h=r_2}} \frac{(-1)^{\nu_2}\nu_2!}{k_1!\cdots k_h!}\Big(\frac{Df_2}{1!\,f_2}\Big)^{k_1}\cdots\Big(\frac{D^h f_2}{h!\,f_2}\Big)^{k_h},$$

since

$$g^{(\boldsymbol{\nu})}(y_1, y_2) = (-1)^{\nu_1+\nu_2}\frac{\nu_1!}{y_1^{\nu_1+1}}\cdot\frac{\nu_2!}{y_2^{\nu_2+1}}, \quad \boldsymbol{\nu} = (\nu_1, \nu_2).$$

Note that formula (1.17) may be useful for estimating $\left|D^h \frac{1}{f_1 f_2}\right|$ since

$$\sum_{\substack{k_1+\cdots+k_h=\nu_i\\ 1\cdot k_1+2\cdot k_2+\cdots+h\cdot k_h=r_i}} \frac{\nu_i!}{k_1!\cdots k_h!} \le 2^{\nu_i}, \quad i = 1, 2.$$

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References

[1] L. F. A. Arbogast, *Du calcul des dérivations*, Levrault (Strasbourg, 1800).

[2] S. Campanato, *Lezioni di Analisi Matematica*, seconda parte, Libreria Scientifica G. Pellegrini (Pisa, 1977).

[3] G. M. Constantine and T. H. Savits, A multivariate Faa di Bruno formula with applications, *Trans. Amer. Math. Soc.*, **348** (1996), 503–520.

[4] F. Faà di Bruno, Sullo sviluppo delle funzioni, *Annali di Scienze Matematiche e Fisiche*, **6** (1855), 479–480.

[5] L. E. Fraenkel, Formulae for high derivatives of composite functions, *Mat. Proc. Cambridge Philos. Soc.*, **83** (1978), 159–165.

[6] H. Gzyl, Multidimensional extension of Faa di Bruno's formula, *J. Math. Anal. Appl.*, **116** (1986), 450–455.

[7] L. Hernández Encinas and J. Muñoz Masqué, A short proof of the generalized Faà di Bruno's formula, *Appl. Math. Lett.*, **16** (2003), 975–979.

[8] R. B. Leipnik and C. E. M. Pearce, The multivariate Faà di Bruno formula and multivariate Taylor expansions with explicit integral remainder term, *ANZIAM J.*, **48** (2007), 327–341.

[9] K. Spindler, A short proof of the formula of Faà di Bruno, *Elem. Math.*, **60** (2005), 33–35.

[10] M. Taylor, *Introduction to Analysis in Several Variables–Advanced Calculus*, Pure Appl. Undergrad. Texts, 46, Amer. Math. Soc. (Providence, RI, 2020).

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