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Cremona equivalence and log Kodaira dimension

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ABSTRACT

Two projective varieties are said to be Cremona equivalent if there is a Cremona modification sending one onto the other. In the last decade, Cremona equivalence has been investigated widely, and we now have a complete theory for non-divisorial reduced schemes. The case of irreducible divisors is completely different, and not much is known besides the case of plane curves and a few classes of surfaces. In particular, for plane curves it is a classical result that an irreducible plane curve is Cremona equivalent to a line if and only if its log-Kodaira dimension is negative. This can be interpreted as the log version of Castelnuovo's rationality criterion for surfaces. One expects that a similar result for surfaces in projective space should not be true, as it is false, the generalization in higher dimensions of Castelnuovo's Rationality Theorem. In this paper, the first example of such behavior is provided, exhibiting a rational surface in the projective space with negative log-Kodaira dimension, which is not Cremona equivalent to a plane. This can be thought of as a sort of log Iskovkikh-Manin, Clemens-Griffith, Artin-Mumford example. Using this example, it is then possible to show that Cremona equivalence to a plane is neither open nor closed among log pairs with negative Kodaira dimension.

RÉSUMÉ

Deux variétés projectives sont dites Cremona équivalentes s'il existe une modification de Cremona envoyant l'une sur l'autre. Au cours de la dernière décennie, l'équivalence de Cremona a été largement étudiée, et nous disposons maintenant d'une théorie complète pour les schémas réduits non divisoriels. Le cas des diviseurs irréductibles est complètement différent, et peu de choses sont connues en dehors du cas des courbes planes et de quelques classes de surfaces. En particulier, pour les courbes planes, c'est un résultat classique qu'une courbe plane irréductible est Cremona équivalente à une droite si et seulement si sa dimension de Kodaira logarithmique est négative. Cela peut être interprété comme la version logarithmique du critère de rationalité de Castelnuovo pour les surfaces. On s'attend à ce qu'un résultat similaire pour les surfaces dans l'espace projectif ne soit pas vrai, car il est faux, la généralisation en dimensions supérieures du théorème de rationalitè de Castelnuovo.

Dans cet article, le premier exemple d'un tel comportement est fourni, exhibant une surface rationnelle dans l'espace projectif avec une dimension de Kodaira

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logarithmique négative, qui n'est pas Cremona équivalente à un plan. Cela peut être considéré comme une sorte d'exemple logarithmique de type Iskovkikh-Manin, Clemens-Griffith, Artin-Mumford. En utilisant cet exemple, il est alors possible de montrer que l'équivalence de Cremona à un plan n'est ni ouverte ni fermée parmi les paires logarithmiques avec une dimension de Kodaira négative.

1. Introduction

Let $X, Y \subset \mathbb{P}^N$ be irreducible birational subvarieties. It is quite natural to ask if there is a birational self-map of the projective space $\omega : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ such that $\omega(X) = Y$. If this is the case, X is said to be Cremona equivalent to Y.

The notion of Cremona equivalence is old and already at the end of the XIX$^{\text{th}}$ century, both the Italian and the English school of algebraic geometry approached the problem, with special regard to the plane curves.

In this context, a lot of attention was devoted to rational curves Cremona equivalent to a line. It is easy to give examples of rational curves that are not Cremona equivalent to a line, for instance, a nodal sextic. It is less immediate to characterize rational curves that are Cremona equivalent to a line. I am not going to explain the long and intricate story of this theorem and its proofs. Let me simply say that the key idea is that the vanishing of all the adjoints, in modern language, the negativity of the log Kodaira dimension, is enough to provide singularities that force the Cremona equivalence of a rational curve to a line.

Theorem 1.1. *[9, vol III pg 188] [6, pg 406] A rational curve $C \subset \mathbb{P}^2$ is Cremona equivalent to a line if and only if all adjoint linear systems to C vanish, i.e. $\overline{\kappa}(\mathbb{P}^2, C) < 0$.*

Where $\overline{\kappa}(\mathbb{P}^2, C)$ is the log Kodaira dimension of the pair $(\mathbb{P}^2, C)$, see the next section for a precise definition. This result can be seen as a log version of Castelnuovo rationality criterion, saying that to detect the Cremona equivalence to a line it is enough to check the vanishing of some prescribed cohomological groups. Note that [20] improved Coolidge result and proved that it is enough to check the vanishing of the second adjoint, increasing the similarity to the Castelnuovo criterion.

My aim is to investigate the higher-dimensional version of Theorem 1.1. First, one should ask for the Cremona equivalence of arbitrary rational subvarieties of $\mathbb{P}^n$. It is amazing, but as soon as the codimension is at least 2, any birational equivalence of reduced schemes can be obtained via a Cremona modification. This is the content of a series of papers I dedicated to the subject, [21] [4] [18]. For divisors, the situation is much more intricate and only very special examples of Cremona equivalence are known, [21] [16][2][3] [17] [19].

Like the failure of a higher-dimensional version of Castelnuovo rationality criterion, proved in a series of seminal papers [13] [5] [1], the existence of rational surfaces with negative log Kodaira dimension that are not Cremona equivalent to a plane in $\mathbb{P}^3$ was largely expected. On the other hand, since no known birational invariant can distinguish those pairs from a plane in $\mathbb{P}^3$, no examples were known.

The aim of this paper is to provide the first example of such a behavior. Let $W \subset \mathbb{P}^7$ be the minimal degree embedding of $\mathbb{F}_0$, that is the embedding given by the linear system $\mathcal{O}_{\mathbb{F}_0}(1, 3)$.

Theorem 1.2. *Let $S \subset \mathbb{P}^3$ be a general linear projection of W. Then $\overline{\kappa}(S, \mathbb{P}^3) < 0$ and S is not Cremona equivalent to a plane.*

Let me spend some words on the reason I focused on this special surface. Any rational surface of degree at most 4 is Cremona equivalent to a plane, [17]. All rational quintic surfaces I was able to test are Cremona equivalent to a plane. Unfortunately, the classification of rational quintic surfaces is incomplete, so I do not have a full statement in this degree. All but S, sextic rational surfaces I studied are Cremona equivalent to a plane.

The surface S can be detected with a 2-ray game inherited by the Sarkisov program. My heuristic approach is based on the following pattern. Produce Sarkisov links out of the singularities of the surface and then perform a 2-ray game. In the context of Cremona equivalence, the linear system that provides the map is hidden, the starting surface is usually an irreducible component of a very special divisor in the linear system that realizes the equivalence. For this reason, this approach alone can only provide positive answers in very special situations. Nonetheless, the surface S is the first case of my personal zoo, where this method failed to produce a non-terminal 3-fold as output. That is, the 2-ray game led me to a non-terminal variety. I interpreted this failure as a suggestion to go deeper into the geometry of the pair.

Once the attention is focused on this example, a brute force analysis of the intersection theoretic behavior of a possible Cremona equivalence between S and a plane allows us to prove Theorem 1.2. This, again, reminds me of the first proof of the non-rationality of quartic 3-folds, [13]. Unfortunately, at the moment, a more conceptual approach is not at hand, and this lack prevents a more general treatment of Cremona equivalence of surfaces with negative Kodaira dimension.

The example also allows us to prove that being Cremona equivalent is neither open nor closed, even for families of pairs with negative log Kodaira dimension, see Theorems 4.1 and 4.4 for the precise statements.

The paper is constructed as follows. In Section 2, general results about Cremona equivalence are proposed; in particular, Lemma 2.5 provides a very useful log resolution of the pair $(\mathbb{P}^3, S)$. The section ends with an application of the 2-ray game to a different projected surface Cremona equivalent to a plane. This construction of Cremona equivalence, even if not strictly necessary, helps to understand the geometric reason that makes this special example work. In Section 3, Theorem 1.2 is proved. In the final section, special families of log varieties are studied to prove that Cremona equivalence to a plane is neither open nor closed among log pairs of negative log-Kodaira dimension.

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2. Cremona equivalence: definition and first results

We work over the complex field. Let us start introducing the main relation we are going to analyze.

Definition 2.1. Let $X, Y \subset \mathbb{P}^N$ be two birational subvarieties. We say that X is Cremona equivalent to Y if there is a birational modification $\omega : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$, defined on the generic point of X, such that $\omega(X) = Y$.

For a log pair $(\mathbb{P}^n, X)$ we define $\overline{\kappa}(\mathbb{P}^n, X)$ as follows.

Definition 2.2. Let $X \subset \mathbb{P}^n$ be a divisor and let $\mu : Z \to \mathbb{P}^n$ be a log resolution of the pair, that is, Z is smooth and $\mu^*X = X_Z + \Delta$, for some effective μ-exceptional divisor Δ, is a normal crossing divisor. We define $\overline{\kappa}(\mathbb{P}^n, X)$ to be the log Kodaira dimension of the pair (Z, X_Z), that is

$$\overline{\kappa}(\mathbb{P}^n, X) := \kappa(Z, X_Z).$$

Remark 2.3. It is well known that Definition 2.2 does not depend on the log resolution chosen and therefore $\overline{\kappa}(\mathbb{P}^n, X)$ is well defined.

When $\overline{\kappa}(\mathbb{P}^n, X) < 0$ all linear systems $m(K_Z + X_Z)$ are without sections, for $m \geq 1$. Note that since X_Z is effective, this is equivalent to having

$$H^0(Z, aK_Z + bX_Z) = 0,$$

for all $a \geq b \geq 1$. This is exactly the vanishing of all adjoint linear systems mentioned in Theorem 1.1.

Definition 2.4. For a linear system $\mathcal{L}$ on a variety X let $\varphi_{\mathcal{L}} : X \dashrightarrow \mathcal{L}^*$ be the map induced by divisors in $\mathcal{L}$.

In this paper, we are focused on rational surfaces $S \subset \mathbb{P}^3$ and in particularly on the generic projection of smooth surfaces to $\mathbb{P}^3$.

2.1. Cremona equivalence for projected surfaces in $\mathbb{P}^3$

Let us start by recalling some known results on projected surfaces, see for instance [23].

Let $f : W \to S \subset \mathbb{P}^3$ be a generic projection of a smooth surface $W \subset \mathbb{P}^n$. The singular locus of S is a curve Γ whose only singularities are t ordinary triple points with transverse tangent directions. The surface S has ordinary triple points at the singular points of Γ. The curve Γ is a curve of ordinary double points for S except for a bunch of pinch points. Except in the case of generic projections of the Veronese surface, the curve Γ is irreducible.

The next Lemma is the key to understanding the blow-up of the singular curve $\Gamma \subset S \subset \mathbb{P}^3$.

Lemma 2.5. *Let $\Gamma \subset X$ be an irreducible reduced curve in a smooth 3-fold X. Assume that the only singularities of Γ are t ordinary triple points. Let $\nu : T \to X$ be the blow up of Γ. Then T is a variety with terminal singularities and the only singularities of T are t points of type $1/2(1, -1, 1)$ over the singularities of Γ.*

Assume that $S := \pi(W) \subset \mathbb{P}^3$ is a general projection of a smooth surface $W \subset \mathbb{P}^n$, different from the Veronese surface, and Γ its curve of singularities. Let $\nu : T \to \mathbb{P}^3$ be the blow up of Γ and S_T the strict transform of S. Then $S_T \cong W$, $\nu_{|S_T} = \pi_{|W}$ is the projection and S_T is on the smooth locus of T.

Proof. Let $p \in \Gamma$ be a triple point and F_p the corresponding fiber in T. It is clear that it is enough to prove that T is terminal with the required singularities in a neighborhood of F_p.

We can therefore assume, without loss of generality, that the curve Γ has a unique triple point, say p. Consider the blow up of the point p, say $\nu_p : X_p \to X$, with exceptional divisor E_p and then the blow up of the curve Γ, $\nu_\Gamma : X_\Gamma \to X_p$, with exceptional divisor E_Γ.

By hypothesis ν_Γ restrict to E_p as the blow up of 3 not aligned points. Let r_i be the lines joining the 3 points. Then we have

$$K_{X_\Gamma} \cdot r_i = 0 \quad \text{and} \quad N_{r_i/X_\Gamma} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Therefore, we may flop these lines with a map $\psi : X_\Gamma \dashrightarrow Y$.

By construction, $\psi_{|E_p}$ is the contraction of the three lines. In particular, the composition $(\psi \circ \nu_\Gamma)_{|E_p}$ is a standard Cremona transformation for the plane E_p. Let $E \subset Y$ be the strict transform of E_p then

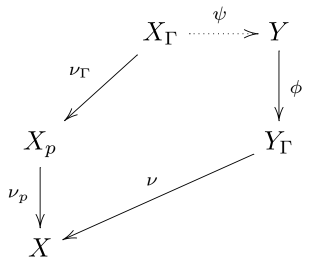
$$E_{|E} \sim \mathcal{O}(-2).$$

This shows that we may blow down E to a singular point of type $1/2(1, -1, 1)$. Let $\phi : Y \to Y_\Gamma$ be the blow down of E. Note that the construction is relative over X. Therefore, there is a canonical morphism $g : Y_\Gamma \to X$. The irreducibility of Γ yields

$$\operatorname{Rank} Pic(Y_\Gamma/X) = 1.$$

Therefore, Y_Γ is the terminal elementary extraction of X and $g = \nu : Y_\Gamma \to X$ is the unique blow-up of the curve Γ. This concludes the first part of the proof.

Let us summarize all the maps involved in the following diagram



Let $\pi : W \to S$ be the linear projection. By hypothesis, W is smooth and π is a finite birational map. In particular, W is the normalization of S. To prove the latter statement in the Lemma, let us follow the birational modifications on the strict transforms of S along the blow-up diagram.

The surface S has triple ordinary points on the singularities of the curve Γ and double points on the smooth points of Γ. The latter are ordinary double points except for finitely many cusps.

Then $S_p \subset X_p$ is singular along Γ and

$$S_{p|E_p} = r_1 + r_2 + r_3,$$

being a cubic singular in the three points of intersection $\{r_i \cap r_j\}_{i \neq j}$.

The surface $S_\Gamma \subset X_\Gamma$ is smooth along E_p and r_i is a local complete intersection of E_Γ and S_Γ. In particular, the self-intersection of r_i in S_Γ is (-1). This shows that $\psi_{|S_\Gamma} : S_\Gamma \to S_Y$ is the blow down of three (-1)-curves and $S_Y \cap E_p = \emptyset$. Let $S_{Y_\Gamma} = \phi(S_Y)$ be the image of S_Y. Then the surface S_{Y_Γ} is smooth, and it is on the smooth locus of Y_Γ. This shows that $\nu_{|S_{Y_\Gamma}} : S_{Y_\Gamma} \to S$ is a finite birational morphism. In particular $\nu_{|S_{Y_\Gamma}}$ is the normalization of S and we conclude

$$W \cong S$$

and $\nu_{|W} = \pi$. $\square$

Remark 2.6. The main feature of Lemma 2.5 we are going to use is the fact that the log variety (T, S_T) has Rank $Pic(T) = 2$ and S_T is smooth. This drastically simplifies the computations needed to study Cremona Equivalence.

It is time to shed light on the surface we are looking for.

Notation 2.7. Let $W \subset \mathbb{P}^7$ be the Segre-Veronese embedding of $\mathbb{F}_0$ with the linear system $\mathcal{O}_{\mathbb{F}_0}(1,3)$ and $\pi : W \to S \subset \mathbb{P}^3$ a general linear projection of W. Then S is a surface of degree 6 singular along an irreducible curve Γ.

Remark 2.8. Note that the class of rational ruled surfaces in $\mathbb{P}^3$ has been extensively studied by Edge, [10, Chapter IV], and many of the results we are going to use were known to Zeuthen. In particular, the one we are considering is a "general" sextic ruled surface. Besides the results in [10], one may find much information and all the properties we need in Dolgachev's book [8].

The degree of Γ can be computed via the sectional genus of W. Indeed, a general plane section of S is a plane sextic curve of geometric genus $g(W) = 0$ with $\deg \Gamma$ ordinary double points. That is

$$\deg \Gamma = \binom{5}{2} = 10. \qquad (1)$$

For what follows, it is not crucial to know the number of triple points and cusps, but they are respectively 4 and 8, see either [23] or [8, Theorem 10.4.9]. Let $\Gamma_W = \pi^{-1}(\Gamma)$ be the double point locus of the projection π. Then $\Gamma_W \subset \mathbb{F}_0$ is a curve of degree 20. Set $\nu : T \to \mathbb{P}^3$ the blow up of Γ with exceptional divisor E_Γ. Then by Lemma 2.5 S_T is a smooth quadric.

Firstly, we determine the class of the divisor Γ_W, see also the double point class formula in [8, Equation 10.52].

Lemma 2.9. *Let $l \subset S$ be a general line then*

$$\Gamma \cap l = 4$$

and $\Gamma_W \sim \mathcal{O}_{\mathbb{F}_0}(4, 8)$.

Proof. Let $r := \pi^{-1}(l) \subset W$ be the preimage and let $H \subset \mathbb{P}^7$ be a general hyperplane containing r. Then $r^2 = 0$ and $H_{|W} = r + R$ for some residual curve R. In particular, we have

$$1 = r \cdot H = r^2 + r \cdot R.$$

This shows that

$$r \cdot R = 1.$$

Let $P \subset \mathbb{P}^3$ be a general plane containing l. Then $P_{|S} = l + D$, for D a plane curve of degree 5. That is $D \cdot l = 5$.

Combining the two equations, we get

$$r \cdot \Gamma_W = D \cdot l - r \cdot R = 4.$$

Since the projection is general

$$r \cdot \Gamma_W = l \cap \Gamma = 4.$$

In particular $\Gamma_W \sim \mathcal{O}(4, a)$ and since $\deg \Gamma_W = 20$ we conclude

$$20 = \mathcal{O}(4, a) \cdot \mathcal{O}(1, 3) = 12 + a.$$

That is $\Gamma_W \sim \mathcal{O}(4, 8)$. □

Next, we compute the log Kodaira dimension of $(\mathbb{P}^3, S)$

Lemma 2.10. *In the above notation we have $\overline{\kappa}(\mathbb{P}^3, S) < 0$.*

Proof. Let $\nu : T \to \mathbb{P}^3$ be the blow up of Γ, with exceptional divisor E_Γ and S_T the strict transform of S. By Lemma 2.5 $\overline{\kappa}(\mathbb{P}^3, S) = \kappa(T, S_T)$. We have

$$K_T + S_T = \nu^* \mathcal{O}_{\mathbb{P}^3}(2) - E_\Gamma,$$

therefore to conclude, we have to prove that there is no surface in $\mathbb{P}^3$ of degree d having multiplicity at least $\frac{d}{2}$ along Γ. Assume that such a surface exists and call it D. Since the multiplicity of S along Γ is less than $\frac{6}{2}$, we may assume that $D \not\supset S$. By construction $D \cdot S$ contains Γ of multiplicity at least $\deg D$, hence we obtain the contradiction

$$\deg(D \cdot S) = 6 \deg D \geq \deg D \deg \Gamma = 10 \deg D. \quad \square$$

Let us now apply Lemma 2.5 and Lemma 2.9 to study the 2-ray game originated by the blow up of Γ.

Let $\nu : T \to \mathbb{P}^3$ be the blow up of the singular curve $\Gamma \subset S$, with exceptional divisor E_Γ. Then $\nu^* S = S_T + 2E_\Gamma$, $Pic(T)$ has rank 2 and the strict transform S_T is a smooth quadric, by Lemma 2.5. Let f_1 and f_2 be the two rulings, then, by Lemma 2.9,

$$S_T \cdot f_1 = 18 - 16 = 2, \quad S_T \cdot f_2 = 6 - 8 = -2.$$

This shows that the cone of curves $NE(T)$ is spanned by: fibers of the blow up and the class of a curve contained in S. On the other hand, S is a smooth quadric and $S \cdot f_2 < 0$ therefore $[f_2]$ spans the second ray. The 2-ray game will force us to contract f_2, with a morphism $\eta : T \to X$. Since $S_T \cdot f_2 = -2$ we have $K_T \cdot f_2 = 0$. Hence X is a Fano 3-fold with a curve of canonical singularities. As already stated, this is not enough to conclude our theorem, see for instance Example 2.11. Indeed, there is no guarantee that the blow-up of Γ is the first step of a Sarkisov factorization based on the linear system, say $\mathcal{H}$, that produces the Cremona equivalence. This is due to the fact that we could have $S + R \in \mathcal{H}$ for some effective non trivial divisor R and the canonical threshold of the pair $(\mathbb{P}^3, \mathcal{H})$ could be smaller than $\frac{1}{2}$. On the other hand, it is an indication to study deeper the geometry of the log pair $(\mathbb{P}^3, S)$.

Let me add a different example of the 2-ray game, where, even if the Sarkisov factorization does not work; the surface is Cremona equivalent to a plane.

Example 2.11. Let $W \subset \mathbb{P}^6$ be a del Pezzo surface of degree 6 and $\pi : W \to S \subset \mathbb{P}^3$ a general projection. This time W has sectional genus 1 and, as before or via the double point formula [8, Equation 10.52], we prove that the singular curve Γ has degree 9. Let $\nu : T \to \mathbb{P}^3$ be the blow up of Γ, with exceptional divisor E and strict transform S_T. Then S_T is a del Pezzo surface of degree 6 and, with a computation similar to that in Lemma 2.9, we see that $S_T \cdot m_i = 0$ for any (-1)-curve m_i in S_T. Note that lines, i.e. (-1)-curves, generate $Pic(S_T)$; therefore, S_T is a fiber of a pencil of del Pezzo surfaces of degree 6. Incidentally, note that $36 = 6^2 = 9 \cdot 4$. The del Pezzo fibration is the second ray of the two-ray game. Therefore, also in this case, we cannot continue the Sarkisov factorization. On the other hand, this time the fibration can be used to produce the required Cremona equivalence.

Let $\pi : T \to \mathbb{P}^1$ be the dP_6 fibration and S_ξ the generic fiber over $k = \mathbb{C}(t)$. It is well known, [15] see also [14, Chap. IV Theorem 6.8], that S_ξ is rational over k, since the Brauer group of a C_1-field is trivial. Therefore the pair $(\mathbb{P}^1 \times \mathbb{P}^2, \mathbb{P}^2)$ is a birational model of $(\mathbb{P}^3, S)$. This is enough to conclude that $(\mathbb{P}^3, S)$ is Cremona equivalent to a plane.

In general, I am not trying to factor the Cremona equivalence in Sarkisov links. I only use this technique to improve the log pairs and understand their geometry better.

The last ingredient we need is the following, probably well-known Lemma, of independent interest.

Lemma 2.12. *Let $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and $H \subset \mathbb{P}^3$ is a plane. Then there is a birational map $\Omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that*

$$\Omega_{|H} = \omega.$$

In particular, $\Omega(H)$ is a plane embedded linearly in $\mathbb{P}^3$.

Remark 2.13. Note that Lemma 2.12 works in arbitrary dimension for linear spaces. I thank Jérémy Blanc for the nice remark.

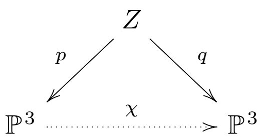
Proof. By Noether–Castelnuovo Theorem, ω can be factored into a sequence of linear automorphisms and standard Cremona transformations. Hence, to conclude, it is enough to prove that those are extendible to $\mathbb{P}^3$. For linear automorphisms, it is clear. For the latter note that the quadro quadric map of $\mathbb{P}^3$, associated to the linear system of quadrics containing a point, say p, and a conic, C, restricts to the standard Cremona modification on a general plane through the base point p. To conclude, observe that linear automorphisms preserve linear spaces and the quadro-quadric map, centered in a point p, sends planes through p to planes. □

We are ready to prove the Theorem stated in the introduction.

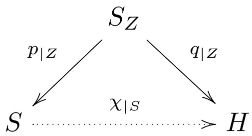
3. The surface S is not Cremona equivalent to a plane

We prove the statement by contradiction. Assume that there is a birational map $\chi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\chi(S) = H$ is a plane.

Set $\chi = \varphi_{\mathcal{H}}$ and $\chi^{-1} = \varphi_{\mathcal{H}'}$, for linear systems $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(h)|$ and $\mathcal{H}' \subset |\mathcal{O}_{\mathbb{P}^3}(h')|$. Consider a resolution of the map χ



and let S_Z be the strict transform of S, that coincides with strict transform of H, in Z. I'm assuming that both Z and S_Z are smooth. This induces the following restricted diagram



We aim to study the restricted diagram next.

Recall that the surface S is the projection of : $W \subset \mathbb{P}^7$, where W is the embedding of $\mathbb{F}_0$ via the linear system $|\mathcal{O}_{\mathbb{F}_0}(1,3)|$. Let $\pi : W \to S$ be the projection. For our purposes, it is also useful to describe W in a different way.

Let $\mathcal{L} = |\mathcal{I}_{q_1^3 \cup q_2}(4)| \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the linear system of quartics with a 3-ple point in q_1 and passing through q_2, then

$$W = \varphi_{\mathcal{L}}(\mathbb{P}^2).$$

Note that $\varphi_{\mathcal{L}}^{-1} : W \dashrightarrow \mathbb{P}^2$ is induced by the linear system $|\mathcal{I}_{p_1}(1,1)| \subset |\mathcal{O}_{\mathbb{F}_0}(1,1)|$ of rational quartics, in the embedding of $W \subset \mathbb{P}^7$, with a simple base point p_1. Then there is a base-point-free linear system of quartics $\Lambda' \subset \mathcal{L}$ such that

$$\psi := \varphi_{\Lambda'} : \mathbb{P}^2 \dashrightarrow S.$$

Set

$$\omega := \psi^{-1} \circ (\chi^{-1})_{|H} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

and

$$\Omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

its extensions to $\mathbb{P}^3$, as in Lemma 2.12. Then by construction $\Omega \circ \chi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is well defined on the generic point of S and

$$(\Omega \circ \chi)_{|S} = \psi^{-1}.$$

Then, up to replacement χ with $\Omega \circ \chi$, we may assume that $(\chi^{-1})_{|H} = \psi$.

Remark 3.1. Here we are reviving Cremona's classifying method for Cremona transformations of $\mathbb{P}^3$. That is, reduce the study of birational modifications of $\mathbb{P}^3$ to the one of plane transformations, [7].

In other words, we may assume that

$$\mathcal{H}'_{|H} = \Lambda' + F',$$

for some fixed divisor F' and

$$\mathcal{H}_{|S} = \Lambda + F,$$

for some fixed divisor F and $\pi_*^{-1}\Lambda \in |\mathcal{O}_{\mathbb{F}_0}(1,1)|$.

Let me stress some interesting consequences:

- S_Z is the blow up of $\mathbb{P}^2$ in the points, q_1, q_2, equivalently the blow up of S along Γ and subsequently in a point p_1,
- $p_{|Z}$ is the blowing down of the line M spanned by q_1 and q_2 to the point p_1, followed by a finite morphism
- $q_{|Z}$ is the blowing down of the two rulings, say F_1 and F_2, of S passing through p_1,
- the rulings of S are mapped to the pencils of lines through q_1 and q_2,
- $Pic(S_Z) = \langle F_1, F_2, M\rangle$, and all the three generators are (-1)-curves with $F_1 \cdot F_2 = 0$ and $F_i \cdot M = 1$.

A divisor $aF_1 + bF_2 + cM \in Pic(S_Z)$ will be denoted by (a, b, c). In this notation, the strict transform of the ruling of lines in S is $(0, 1, 1)$ while the strict transform of the ruling of twisted cubics corresponds to $(1, 0, 1)$. A divisor $D = (a, b, c)$ is effective if and only if $a \geq 0$, $b \geq 0$ and $c \geq 0$ and

i) it is the pull back of divisor in S_T if and only if $a + b = c$
ii) it is the pull back of a divisor in H if and only if $a = b = c$.

Let us set some further notation:

- $\nu : T \to \mathbb{P}^3$ the blow up of Γ, with exceptional divisor E_Γ and strict transform S_T; recall that by construction S_T is a smooth quadric with $\nu^*\mathcal{O}_S(1) \sim \mathcal{O}_{\mathbb{F}_0}(1,3)$ and, by Lemma 2.9 and Lemma 2.5, $E_{\Gamma|S_T} \sim (4,8)$,
- $\nu_1 : T_1 \to T$ the blow up of the fixed component F; the surface S is smooth, then $F \subset S$ is a Cartier divisor and $S_{T_1} \cong S_T \cong \mathbb{F}_0$, by the universal property of blow up, [12, Corollary 7.15].
- $\mu : X \to \mathbb{P}^3$ the blow up the fixed component F'; the surface H is smooth, then $F' \subset H$ is a Cartier divisor and $H_X \cong H \cong \mathbb{P}^2$, by the universal property of blow up, [12, Corollary 7.15].

After these blow ups we have $\mathrm{Bs}\mathcal{H}_{T_1} \cap S_{T_1} \subset p_1$ and $\mathrm{Bs}\mathcal{H}'_X \cap H_X \subset \{q_1, q_2\}$. Then we have

$$p = \nu_2 \circ \nu_1 \circ \nu$$

and

$$q = \mu_1 \circ \mu,$$

for some birational morphisms $\nu_2 : Z \to T_1$ and $\mu_1 : Z \to X$ such that the restricted morphisms $\nu_{2|S_Z}$ is the blow up of the point p_1 and $\mu_{1|S_Z}$ is the blow up of the points q_1, q_2.

By our construction and Lemma 2.9, we have

$$\begin{array}{l} ((\nu_2 \circ \nu_1)^* E_\Gamma)_{|S_Z} \sim (4, 8, 12), \\ p^* \mathcal{O}_{\mathbb{P}^3}(1)_{|S_Z} \sim (1, 3, 4), \\ q^* \mathcal{O}_{\mathbb{P}^3}(1)_{|S_Z} \sim (1, 1, 1) \end{array} . \tag{2}$$

And we may write

$$p^* S_{|S_Z} = S_{Z|S_Z} + (8, 16, 24) + E_S + (0, 0, a+1) \sim (6, 18, 24) \tag{3}$$

$$q^* H_{|S_Z} = S_{Z|S_Z} + E_H + (b_1 + 1, b_2 + 1, 0) \sim (1, 1, 1) \tag{4}$$

where:

- E_S is the total transform of curves in S_T, blown up along the map ν_1,
- E_H is the total transform of curves in H, blown up along the map μ,
- a is a non negative integer related to the map ν_2,
- b_1 and b_2 are non negative integers related to the map μ_1.

In particular E_S and E_H are the pull-back of curves in S_T and H respectively, then by items i) and ii), for non-negative integers s_1, s_2 and e, we have

$$E_S \sim (s_1, s_2, s_1 + s_2), \quad E_H \sim (e, e, e). \tag{5}$$

From Equations (3) (4) we get

$$E_H \sim E_S + (2 - b_1, -2 - b_2, 2 + a),$$

and finally, plugging in Equation (5)

$$(e, e, e) \sim (s_1 + 2 - b_1, s_2 - 2 - b_2, s_1 + s_2 + 2 + a). \tag{6}$$

Then we have

$$\begin{cases} e &= s_1 + 2 - b_1 \\ e &= s_2 - 2 - b_2 \\ e &= s_1 + s_2 + 2 + a \end{cases} \tag{7}$$

therefore

$$\begin{cases} e+b_1 &= s_1+2 \\ e &= s_2-2-b_2 \\ 0 &= b_1+s_2+a \end{cases} . \tag{8}$$

Since all integers are non-negative, the third equation yields $a = b_1 = s_2 = 0$ and from the second equation we derive the impossible $e = -2 - b_2$. This contradiction shows that the map χ cannot exist and proves that S is not Cremona equivalent to a plane.

Remark 3.2. Let me stress that, in the notation of [22], thanks to [19, Corollary 1.7], this shows that any good model (X, S_X) birational to $(\mathbb{P}^3, S)$ is such that $\rho(X, S_X) = 0$. In other words, for any Mori fiber space (X, S_X), with S_X smooth, birational to $(\mathbb{P}^3, S)$, the surface S_X is never transverse to the MfS fibration. This answers to a question in [22, Remark 4.8].

4. Cremona equivalence to a plane is neither open nor closed

It is easy and not surprising that being Cremona equivalent to a plane, like rationality for 3-fold hypersurfaces, is neither closed nor open among log pairs, without further hypothesis. For instance, one can consider a family of smooth cubic surfaces degenerating to a smooth cubic cone or a family of smooth quartic surfaces degenerating to a rational quartic.

In this section, we use the result in Theorem 1.2 to prove that restricting to pairs with a negative log Kodaira dimension is not enough to gain neither openness nor closedness.

Theorem 4.1. *There exist families of projective surfaces in $\mathbb{P}^3$, $\phi : X \to B$ over connected varieties B, such that for every $b \in B$ the fiber $X_b = \phi^{-1}(b)$ satisfies $\overline{\kappa}(\mathbb{P}^3, X_b) < 0$ and for every $b \neq 0$ X_b is not Cremona equivalent to a plane, while X_0 is Cremona equivalent to a plane.*

Proof. Let $W \subset \mathbb{P}^7$ be the Segre-Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ we considered in Notation 2.7. Fix 5 general points $\{x_0, \dots, x_4\} \subset W$ and let $\Lambda = \langle x_0, \dots, x_4\rangle$. Set $\Pi_0 \subset \Lambda$ a general linear space of dimension 3 and $\Pi_1 \subset \mathbb{P}^7$ is a general linear space of dimension 3. Let $B \subset \mathbf{G}(3, 7)$ be a rational curve, parameterized by t, through $[\Pi_0]$ and $[\Pi_1]$ in the Grassmannian variety. Let X_t be the projection of W from Π_t, the linear space associated with the parameter t. Then, without loss of generality, we may assume that, for $t \neq 0$, X_t is a general projection. Then, by Theorem 1.2 for $t \neq 0$, X_t is not Cremona equivalent to a plane. On the other hand, by construction X_0 is a sextic with a 5-tuple point, and it is therefore Cremona equivalent to a plane, like any monoid, see for instance [18]. □

To study closedness, we start by studying the Cremona equivalence of a different quartic surface, known as the Bordiga surface.

Lemma 4.2. *Fix 10 general points $\{x_1, \dots, x_{10}\} \subset \mathbb{P}^2$. Consider the linear system $\mathcal{L} := |\mathcal{I}_{x_1 \cup \ldots \cup x_{10}}(4)|$ and let $X := \varphi_{\mathcal{L}}(\mathbb{P}^2) \subset \mathbb{P}^4$ be the image in $\mathbb{P}^4$. Let S' be a general linear projection of X to $\mathbb{P}^3$, then $\overline{\kappa}(\mathbb{P}^3, S') < 0$ and S' is Cremona equivalent to a plane.*

Proof. Let $\nu' : T' \to \mathbb{P}^3$ be the blow up of Γ', the singular curve of S'. Then, by Lemma 2.5, the strict transform $S'_{T'} \cong X$ is the blow up of $\mathbb{P}^2$ in $Z := \{p_1, \dots, p_{10}\}$. The sectional genus of X is 3 and therefore we have $\deg \Gamma' = 7$, recall the computation in Notation 2.7.

Claim. $\overline{\kappa}(\mathbb{P}^3, S') < 0$

Proof of the Claim. As in the proof of Lemma 2.10, the claim is equivalent to proving that there are no effective irreducible divisors $D \subset \mathbb{P}^3$ such that

$$\operatorname{mult}_\Gamma D \geq \frac{\deg D}{2}.$$

Assume that such a divisor exists, then $D \cdot S$ has to contain the curve Γ' of multiplicity $\geq \deg D$. Hence we derive the contradiction

$$6 \deg D = \deg(D \cdot S) \geq \deg \Gamma' \deg D = 7 \deg D. \quad \square$$

Let $m_i \subset X$ be the exceptional divisors of the blow-up. Then $m_i \subset X$ is a line and arguing as in Lemma 2.9 we have, with ν the blow up of the curve Γ,

$$\sharp(\Gamma \cap \nu(m_i)) = 3 \tag{9}$$

This gives

$$S'_{T'} \cdot m_i = 6 - 6 = 0, \quad K_{T'} \cdot m_i = -4 + 3 = -1. \tag{10}$$

The Picard group of X is generated by the exceptional divisors $\{m_i\}$ and the strict transform of a line through two points in Z, say $c \subset X$. Then c is a conic and $c^2 = -1$. Let $H \subset \mathbb{P}^4$ be a hyperplane containing c and R the residual curve

$$H \cap X = c + R.$$

Then we have

$$2 = (c + R) \cdot c = -1 + R \cdot c. \tag{11}$$

Looking at this configuration after the projection π to $\mathbb{P}^3$ gives:

- $\pi(c)$ is a conic,
- $\pi(R)$ is a plane curve of degree 4.

This yields

$$\sharp(\pi(R) \cap \pi(c)) = 8$$

and together with Equation (11)

$$\sharp(\Gamma \cap \pi(c)) = 5.$$

Hence we conclude

$$S_{T'} \cdot c = 12 - 10 = 2, \quad K_{T'} \cdot c = -8 + 5 = -3. \tag{12}$$

Let $C \subset S_T$ be an irreducible effective curve, then

$$C \equiv \alpha c + \sum_1^{10} \beta_i m_i,$$

and either $\alpha = 0$, and $\sum \beta_i = 1$ or $\alpha > 0$. In particular if $C \neq m_i$

$$S_{T'} \cdot C > 0. \tag{13}$$

Equations (13) and (10) show that $S_{T'}$ is nef.

Claim. T' is a Fano 3-fold.

Proof of the Claim. The cone of effective curves has dimension 2, $S_{T'}$ is nef and $S_{T'} \cdot m_i = 0$. Therefore, the cone is closed and spanned by the curves contracted by ν' and the curves numerically proportional to m_i. The map ν' is a terminal blow up, hence $-K_{T'}$ is relatively ample and by Equation (10) we have $-K_{T'} \cdot m_i = 1$. Then T' is Fano. □

Let $\mu : T' \to \tilde{T}$ be the contraction of the second extremal ray $[m_i]$. We have $S_{T'} \cdot m_i = 0$ and by Equation (13) we conclude that μ is a birational contraction. In particular, there is an effective divisor $D \subset T'$ with $D \cdot m_i < 0$. Then $S_{T'}$ is in the interior of the effective cone and the effective threshold

$$\rho(T', S_{T'}) := \inf\{t \in \mathbb{Q} | tK_{T'} + S_{T'} \text{ is effective}\}$$

is strictly positive. In the notation of [17], by Lemma 2.5, $(T', S_{T'})$ is a good model of the pair $(\mathbb{P}^3, S')$. Therefore, by [17, Corollary 1.7] S is Cremona equivalent to a plane. □

Next, we recall a degeneration argument to study point collisions. For this, we use notations and results in [11].

Construction 4.3 *(Specialization with 6 collapsing simple points).* Set $V = \mathbb{P}^2 \times \Delta$, for $\Delta \ni 0$ a complex disk, and

- $\pi : V \to \Delta$, $\tau : V \to \mathbb{P}^2$ the canonical projections,
- $|\mathcal{O}_V(d)| := |\tau^*(\mathcal{O}_{\mathbb{P}^2}(d)|$.
- $\mathcal{L}_t := \mathcal{L}_{|\pi^{-1}(t)}$ the restriction to the fiber of a linear system $\mathcal{L}$ on V.

Fix 6 general sections $\{\sigma_1, \ldots, \sigma_6\}$ such that $\sigma_i(0) = p_1$, and set $Z := \cup_i \sigma_i(\Delta)$. Let $Y \to V$ be the blow up of V at the point p_1, with exceptional divisor E. Then we have natural morphisms $\tau_Y : Y \to \mathbb{P}^2$, a degeneration $\pi_Y : Y \to \Delta$, and sections $\sigma_{Y,i} : \Delta \to Y$. The special fiber W_0 is given by $E \cup V_0$, where V_0 is $\mathbb{P}^2$ blown up in one point and $E \cong \mathbb{P}^2$. Keep in mind that since the sections σ_i's are general $\{\sigma_{Y,i}(0)\}$ are general points of E.

We are interested in the flat limit scheme Z_0. By [11, Lemma 20] Z_0 is a scheme of length 6 and multiplicity 3. Then Z_0 is the triple point in p_1.

We are ready to prove that Cremona equivalence is not a closed condition among pairs with negative log Kodaira dimension.

Theorem 4.4. *There exist a family of projective surfaces in $\mathbb{P}^3$, $\phi : X \to B$ over a connected variety B, such that for every $b \in B$ the fiber $X_b = \phi^{-1}(b)$ satisfies $\overline{\kappa}(\mathbb{P}^3, X_b) < 0$ and for every $b \neq 0$ X_b is Cremona equivalent to a plane, while X_0 is not Cremona equivalent to a plane.*

Proof. To prove the theorem, we produce a linear system $\mathcal{H}$ on $V := \mathbb{P}^2 \times \Delta$, such that, with the notation of Lemma 4.2 and Theorem 1.2, for $t \neq 0$ $\varphi_{\mathcal{H}_t}(\mathbb{P}^2) \cong S'$, while $\varphi_{\mathcal{H}_0}(\mathbb{P}^2) \cong S$.

We already observed that a general sublinear system of dimension 3, $\Lambda_0 \subset |\mathcal{I}_{p_1^3 \cup p_2}(4)|$ is such that

$$S = \varphi_{\Lambda_0}(\mathbb{P}^2) \subset \mathbb{P}^3.$$

First, we introduce a fixed component in the linear systems Λ_0. Let $L \subset \mathbb{P}^2$ be a general line and $y_1, \ldots, y_n \in L$ general points. Note that the linear system

$$\mathcal{H}_0 := \Lambda_0 + L \subset |\mathcal{O}_{\mathbb{P}^2}(5)|$$

is such that $\varphi_{\mathcal{H}_0}(\mathbb{P}^2) = \varphi_{\Lambda_0}(\mathbb{P}^2) = S$, and $\mathcal{H}_0 = |\mathcal{I}_{p_1^3 \cup p_2 \cup y_1 \cup \ldots \cup y_h}(5)|$ for any $h \geq 6$.

We aim to obtain $\mathcal{H}_0$ as a specialization of linear systems in $\mathbb{P}^2$ providing birational embedding of the surfaces described in Lemma 4.2.

For this, keeping in mind the Construction 4.3, consider a degeneration $\pi : V \to \Delta$ together with the following sections:

- $\{\sigma_1, \ldots, \sigma_6\}$ such that $\sigma_i(0) = p_1$,
- $\{s_1, s_2, s_3\}$ with $s_i(0) \in L$,
- $\{y_1, \ldots, y_6\}$ with $y_i(t) \in L$, for any $t \in \Delta$,
- $\{p_2\}$ with $p_2(t) = p_2$, for any $t \in \Delta$.

Let $Z = \cup \sigma_i \cup s_j \cup y_k \cup p_2$ and $\mathcal{L} = |\mathcal{I}_Z(5)| \subset |\mathcal{O}_V(5)|$ be the linear system containing Z. Let

$$\mathcal{H} \subset \mathcal{L}$$

be a general sublinear system of relative dimension 3 then

$$\mathcal{H}_t = \Lambda_t + L.$$

For $t \neq 0$ the linear system Λ_t is contained in $|\mathcal{I}_{x_1 \cup \ldots \cup x_{10}}(4)|$, with x_i general points in $\mathbb{P}^2$ and, with the notation of Lemma 4.2, $\varphi_{\mathcal{H}_t}(\mathbb{P}^2) = S'$. For $t = 0$, keeping in mind Construction 4.3, we have

$$\mathcal{H}_0 = \Lambda_0 + L \subset |\mathcal{I}_{p_1^3 \cup p_2}(4)| + L.$$

To conclude the proof, we need to show that, in the notation of Theorem 1.2, the special fiber is isomorphic to S. In other words we have to prove that Λ_0 is a general linear space in $|\mathcal{I}_{p_1^3 \cup p_2}(4)|$. Since $\dim |\mathcal{I}_{p_1^3 \cup p_2}(4)| = 7$, we have to produce a dominant map from the space parameterizing our degenerations to $\mathbb{G}(3, 7)$.

In the notation of Construction 4.3, the restriction $\mathcal{H}_{Y|E}$ is given by cubics passing through 6 general points. The linear system $\mathcal{H}_{Y|E}$ is complete by [11, Lemma 24] and it is therefore the dimension 3 linear system of cubics through the six points $\{\sigma_{Y,1}(0), \ldots, \sigma_{Y,6}(0)\}$. In particular different 6-tuples of points $\{\sigma_{Y,1}(0), \ldots, \sigma_{Y,6}(0)\}$ produce different degenerations. The linear system $\mathcal{L}_t$ has dimension 4, therefore its flat limit $\mathcal{L}_0$ has dimension 4 and to any point $q \in \mathcal{L}_0^* \cong \mathbb{P}^4$ we may associate a sublinear system $\mathcal{H}_q \subset \mathcal{L}$ such that $[\mathcal{H}_{q0}] = q$. Again, changing the point produces different limit linear systems $\mathcal{H}_0$. Hence the choice of the points $(\sigma_{Y,1}(0), \ldots, \sigma_{Y,6}(0)) \in (\mathbb{P}^2)^6$ and the point $q \in \mathbb{P}^4$ yields a generically finite map

$$\tau : (\mathbb{P}^2)^6 \times \mathbb{P}^4 \dashrightarrow \mathbb{G}(3, 7),$$

mapping the points to the limit linear system $\mathcal{H}_0$. Both varieties have dimension 16, then τ is dominant and, for a general degeneration $\mathcal{H}$, we have $\varphi_{\mathcal{H}_0}(\mathbb{P}^2) = S$, concluding the proof. □

Remark 4.5. It is interesting to stress that it is not possible to provide the same degeneration with linear systems of relative degree 4, indeed, there is no flat limit of 10 general points to the scheme of length 7 given by a triple point and a simple point. The trick is to force a fixed component that absorbs the excess base points of the general fiber without changing the birational map.

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