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KdV-type equations in projective Gevrey spaces

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A B S T R A C T

We prove well-posedness of the Cauchy problem for a class of third order quasilinear evolution equations with variable coefficients in projective Gevrey spaces. The class considered is connected with several equations in Mathematical Physics as the KdV and KdVB equation and some of their many generalizations.

R É S U M É

Nous prouvons que le problème de Cauchy est bien posé pour une classe d'équations d'évolution quasi linéaires du troisième ordre à coefficients variables dans des espaces de Gevrey projectifs. La classe considérée est liée à plusieurs équations en Physique Mathématique comme les équations KdV et KdVB et certaines de leurs nombreuses généralisations.

1. Introduction and main result

The Korteweg-de Vries equation

$$\partial_t u + \frac{1}{2}\sqrt{\frac{g}{h}}\sigma\partial_x^3 u + \sqrt{\frac{g}{h}}\left(\alpha + \frac{3}{2}u\right)\partial_x u = 0, \qquad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.1}$$

has been introduced in [27] to describe the wave motion in shallow waters; $u(t,x)$ represents the wave elevation, h is the (constant) water level, g the gravity, α a (small) constant, $\sigma = \frac{h^3}{3} - \frac{Th}{\rho g}$, T describes the

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surface tension and ρ the water density. It is the most famous example of dispersive third order evolution equation with (real) constant coefficients. Denoting $D = -i\partial$, the equation (1.1) can be written in the form $P(u, D_t, D_x)u = 0$, where

$$P(u, D_t, D_x) = D_t - \frac{1}{2}\sqrt{\frac{g}{h}}\sigma D_x^3 + \sqrt{\frac{g}{h}}\left(\alpha + \frac{3}{2}u\right) D_x. \tag{1.2}$$

Notice that the principal symbol of P (in the sense of Petrowski) is given by

$$\sigma_{principal}(\tau, \xi) := \tau - \frac{1}{2}\sqrt{\frac{g}{h}}\sigma\xi^3$$

and admits the *real* characteristic root $\tau = \frac{1}{2}\sqrt{\frac{g}{h}}\sigma\xi^3$. An operator of the form (1.2) can be referred to as a quasilinear 3-evolution operator, cf. [3,30]. A huge number of variants of the equation (1.1) has been introduced and studied along the years to model different phenomena connected with the wave propagation, see for instance [26,29,33] and the references therein. One of these variants is the so-called KdV-Burgers (KdVB) equation, see [21,22], which appears for instance in the analysis of the flow of liquids containing gas bubbles and of the propagation of waves in an elastic tube containing a viscous fluid. The KdVB equation reads as follows

$$\partial_t u + 2au\partial_x u + 5b\partial_x^2 u + c\partial_x^3 u = 0, \tag{1.3}$$

cf. [21], where a, b, c are real constants. The associated operator

$$P(u, D_t, D_x) = D_t - cD_x^3 + 5ibD_x^2 + 2auD_x \tag{1.4}$$

is again a semilinear 3-evolution operator with constant coefficients. With respect to (1.2), the operator (1.4) admits complex-valued coefficients in the lower order terms. We recall that complex-valued coefficients naturally arise in the study of other evolution equations of physical interest (think for instance to the Euler-Bernoulli vibrating beam operator studied in [8]). We also observe that assuming the coefficients of the equations (1.1), (1.3) to be constant is just a simplification; in principle some of the coefficients may depend on t and/or x.

Starting from these considerations our aim is to consider a class of quasilinear 3-evolution equations with variable coefficients connected with the previous physical models. Namely we shall consider the Cauchy problem for equations of the form $P(t, x, u, D_t, D_x)u(t, x) = f(t, x)$ where

$$P(t, x, u, D_t, D_x) = D_t + a_3(t)D_x^3 + \sum_{j=0}^{2} a_j(t, x, u)D_x^j, \qquad (t, x) \in [0, T] \times \mathbb{R}, \tag{1.5}$$

and f is an assigned function. Before addressing the general problem, let us spend some words about the linear case, that is the case when the coefficients $a_j, j = 0, 1, 2$, do not depend on u. In this situation, we are led to consider the initial value problem

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) \\ u(0, x) = g(x) \end{cases}, \qquad (t, x) \in [0, T] \times \mathbb{R}, \tag{1.6}$$

where

$$P(t, x, D_t, D_x) = D_t + a_3(t)D_x^3 + \sum_{j=0}^{2} a_j(t, x)D_x^j. \tag{1.7}$$

When the coefficients $a_j, j = 0, 1, 2, 3$, are all smooth and real-valued, the related Cauchy problem is well posed in L^2 and in Sobolev spaces H^m for every $m \in \mathbb{R}$, whereas when $a_2(t, x)$ is complex-valued, in [5] it was proved that if the Cauchy problem is well-posed in $H^\infty(\mathbb{R}) = \cap_{m\in\mathbb{R}} H^m(\mathbb{R})$, then there exist $M, N > 0$ such that $\forall \varrho > 0$

$$\sup_{x\in\mathbb{R}} \min_{0\le\tau\le t\le T} \int_{-\varrho}^{\varrho} \operatorname{Im} a_2(t, x + 3a_3(\tau)\theta)d\theta \le M \log(1 + \varrho) + N. \tag{1.8}$$

On the other hand, by [4] we know that if there exists $C > 0$ such that for every $(t, x) \in [0, T] \times \mathbb{R}$

$$|\operatorname{Im} a_2(t, x)| \le \frac{C}{\langle x\rangle} \quad \text{and} \quad |\operatorname{Im} a_1(t, x)| + |\partial_x \operatorname{Re} a_2(t, x)| \le \frac{C}{\langle x\rangle^{1/2}}, \tag{1.9}$$

with $\langle x\rangle = (1 + |x|^2)^{1/2}$, then the Cauchy problem is well-posed in $H^\infty(\mathbb{R})$ with a loss of derivatives. Namely, given $f(t), g \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, there exists a unique solution with values in $H^{s-\delta}(\mathbb{R})$ for some suitable $\delta > 0$. This type of results has been also extended to general linear p-evolution operators of the form

$$P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t, x)D_x^j, \qquad (t, x) \in [0, T] \times \mathbb{R}, \tag{1.10}$$

where p is a positive integer, see [4,5]. In the recent paper [2], we considered the Cauchy problem (1.6) for the operator (1.7) under weaker decay conditions (compared to (1.9)) on the second order terms. Namely, we replaced the decay of $|\operatorname{Im} a_2|$ in (1.9) by a decay of type $\langle x\rangle^{-\sigma}$ for some $\sigma \in (1/2, 1)$. In this case, H^∞ well-posedness is lost due to the violation of (1.8). However, in analogy with the case $p = 2$ treated in [11,23], under suitable assumptions on the regularity of the coefficients, it is natural to study the Cauchy problem in the Gevrey-Sobolev spaces

$$H^m_{\rho;\theta}(\mathbb{R}) = \{u \in \mathscr{S}'(\mathbb{R}) : \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} u \in L^2(\mathbb{R})\}, \quad \theta \ge 1,\ m, \rho \in \mathbb{R},$$

where $\langle D\rangle^m$ and $e^{\rho\langle D\rangle^{\frac{1}{\theta}}}$ are the Fourier multipliers with symbols $\langle\xi\rangle^m$ and $e^{\rho\langle\xi\rangle^{\frac{1}{\theta}}}$ respectively. These spaces are Hilbert spaces with the following inner product

$$\langle u, v\rangle_{H^m_{\rho;\theta}} = \langle \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} u, \langle D\rangle^m e^{\rho\langle D\rangle^{\frac{1}{\theta}}} v\rangle_{L^2}, \quad u, v \in H^m_{\rho;\theta}(\mathbb{R}).$$

The spaces

$$\mathcal{H}^\infty_\theta(\mathbb{R}) := \bigcup_{\rho>0} H^m_{\rho;\theta}(\mathbb{R}), \quad H^\infty_\theta(\mathbb{R}) := \bigcap_{\rho>0} H^m_{\rho;\theta}(\mathbb{R})$$

are related to Gevrey classes in the following sense:

$$G^\theta_0(\mathbb{R}) \subset \mathcal{H}^\infty_\theta(\mathbb{R}) \subset G^\theta(\mathbb{R}), \quad \gamma^\theta_0(\mathbb{R}) \subset H^\infty_\theta(\mathbb{R}) \subset \gamma^\theta(\mathbb{R}),$$

where $G^\theta(\mathbb{R})$ (respectively, $\gamma^\theta(\mathbb{R})$) denotes the space of all smooth functions f on $\mathbb{R}$ such that

$$\sup_{\alpha\in\mathbb{N}^n} \sup_{x\in\mathbb{R}} h^{-|\alpha|} \alpha!^{-\theta} |\partial^\alpha f(x)| < +\infty \tag{1.11}$$

for some $h > 0$ (resp., for every $h > 0$), and $G^\theta_0(\mathbb{R})$ (resp. $\gamma^\theta_0(\mathbb{R})$) is the space of all compactly supported functions contained in $G^\theta(\mathbb{R})$ (resp. $\gamma^\theta(\mathbb{R})$).
In [2], we proved that if

(i) $a_3 \in C([0,T];\mathbb{R})$ and there exists $C_{a_3} > 0$ such that $|a_3(t)| \geq C_{a_3}$ $\forall t \in [0,T]$,
(ii) $a_j \in C([0,T]; G^{\theta_0}(\mathbb{R}))$, $\theta_0 > 1$, for $j = 0, 1, 2$,
(iii) $\exists \sigma \in (\frac{1}{2}, 1)$ with $\theta_0 < \frac{1}{2(1-\sigma)}$ and $C_{a_2} > 0$ such that $|\partial_x^\beta a_2(t,x)| \leq C_{a_2}^{\beta+1} \beta!^{\theta_0} \langle x \rangle^{-\sigma}$ for every $t \in [0,T]$, $x \in \mathbb{R}, \beta \in \mathbb{N}_0$,
(iv) $\exists C_{a_1}$ such that $|\operatorname{Im} a_1(t,x)| \leq C_{a_1} \langle x \rangle^{-\frac{\sigma}{2}}$ for every $t \in [0,T]$, $x \in \mathbb{R}$,

then the Cauchy problem (1.6) for the operator (1.7) is well-posed in $\mathcal{H}^\infty_\theta(\mathbb{R})$ for every $\theta \in [\theta_0, \frac{1}{2(1-\sigma)})$. Moreover, the solution satisfies the energy estimate

$$\|u(t,\cdot)\|^2_{H^m_{\rho-\delta;\theta}} \leq C \left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau,\cdot)\|^2_{H^m_{\rho;\theta}} \, d\tau \right), \quad t \in [0,T], \tag{1.12}$$

for a suitable $\delta \in (0,\rho)$. More recently, we realized that also well-posedness in $H^\infty_\theta(\mathbb{R})$ can be obtained with minor modifications in the proof of the latter result, and assuming the coefficients a_j to satisfy suitable *projective* Gevrey estimates; in this case, we can prove an energy estimate of the form (1.12) *for every* $\delta \in (0,\rho)$, and by this estimate well-posedness in $H^\infty_\theta(\mathbb{R})$ follows. The proof of this result is a particular case of Theorem 3.1 here below when the coefficients a_j are independent of u, cf. Corollary 3.10.

Going back to quasilinear equations, in this paper we shall consider the Cauchy problem

$$\begin{cases} P(t,x,u(t,x),D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{1.13}$$

for the operator (1.5) in the Gevrey setting described above. As far as we know, there are only a few results concerning KdV-type equations with constant coefficients in Gevrey spaces, see [15,17,18,20]. Due to the loss of regularity appearing in the linear case, it is not possible in general to deduce local well-posedness for the problem (1.13) from the above mentioned results for linear equations via a standard fixed point argument but we need more sophisticated techniques. To prove our main result we use an approach inspired by the method proposed in [13,14] for hyperbolic equations and in [3] for p-evolution equations in the H^∞ setting. Here we adapt this method to the Gevrey setting. The proof relies on the application of Nash-Moser inversion theorem and gives the existence of a unique solution u of (1.13) in $C^1([0,T^*], H^\infty_\theta(\mathbb{R}))$ for some $T^* \in [0,T]$ by solving the equivalent integral equation $Ju \equiv 0$ in $[0,T^*]$, where the map $J : C^1([0,T], H^\infty_\theta(\mathbb{R})) \to C^1([0,T], H^\infty_\theta(\mathbb{R}))$ is defined by

$$J(u) = u - g + i \int_0^t (Pu)(s)ds - i \int_0^t f(s)ds. \tag{1.14}$$

This can be achieved by proving that J is a locally invertible map. The main reason to work in $H^\infty_\theta(\mathbb{R})$ instead than in $\mathcal{H}^\infty_\theta(\mathbb{R})$ is the following: Nash-Moser theorem applies in the category of tame Fréchet spaces and $H^\infty_\theta(\mathbb{R})$, equipped with its natural topology, is such a space, whereas this is not the case for $\mathcal{H}^\infty_\theta(\mathbb{R})$.
In order to give a precise assumption on the regularity and decay of the coefficients, we need the following definition.

Definition 1.1. For $\theta_0 > 1$ and $\tau \geq 0$ we denote by $\Gamma^{\theta_0,\tau}(\mathbb{R}\times\mathbb{C})$ the space of all functions $f(x,w)$ defined on $\mathbb{R}\times\mathbb{C}$ which are smooth in x and holomorphic in w and satisfy the following condition: for every $A > 0$ and every compact set $K \subset \mathbb{C}$ there exists a constant $C_K > 0$ such that

$$\sup_{\beta,\gamma\in\mathbb{N}_0}\sup_{x\in\mathbb{R},w\in K}|\partial_x^\beta\partial_w^\gamma f(x,w)|C_K^{-\gamma}\gamma!^{-1}A^{-\beta}\beta!^{-\theta_0}\langle x\rangle^\tau < +\infty,$$

where ∂_x stands for a real derivative and ∂_w stands for a complex derivative.

We recall the notion of convergence in $\Gamma^{\theta_0,\tau}(\mathbb{R}\times\mathbb{C})$. For $\{f_j\}_{j\in\mathbb{N}_0}\subset\Gamma^{\theta_0,\tau}(\mathbb{R}\times\mathbb{C})$ and $f\in\Gamma^{\theta_0,\tau}(\mathbb{R}\times\mathbb{C})$ we have $f_j\to f$ in $\Gamma^{\theta_0,\tau}(\mathbb{R}\times\mathbb{C})$ as $j\to\infty$, whenever for every $A>0$ and every compact K there exists $C_K>0$ such that

$$\sup_{\beta,\gamma\in\mathbb{N}_0}\sup_{x\in\mathbb{R},w\in K}|\partial_x^\beta\partial_w^\gamma\{f_j(x,w)-f(x,w)\}|C_K^{-\gamma}\gamma!^{-1}A^{-\beta}\beta!^{-\theta_0}\langle x\rangle^\tau\to 0,\quad \text{as}\, j\to\infty.$$

Now we are ready to state the main result of this manuscript.

Theorem 1.2. *Let $a_3\in C([0,T];\mathbb{R})$ such that $|a_3(t)|\geq C_{a_3}>0$ for some constant C_{a_3} and for every $t\in[0,T]$. Let moreover $\sigma\in\left(\frac{1}{2},1\right)$ and $\theta_0<\frac{1}{2(1-\sigma)}$ and assume that for $j=0,1,2$ the coefficients $a_j\in C([0,T],\Gamma^{\theta_0,\frac{j\sigma}{2}}(\mathbb{R}\times\mathbb{C}))$. Then the Cauchy problem (1.13) is locally in time well-posed in $H^\infty_\theta(\mathbb{R})$ for every $\theta\in\left[\theta_0,\frac{1}{2(1-\sigma)}\right)$: namely for all $f\in C([0,T];H^\infty_\theta(\mathbb{R}))$ and $g\in H^\infty_\theta(\mathbb{R})$, there exists $T^*=T^*(g,f)\in(0,T]$ and a unique solution $u\in C^1([0,T^*];H^\infty_\theta(\mathbb{R}))$ of (1.13). Moreover, $T^*(g,f)$ is lower semi-continuous with respect to the data g and f (in the $H^\infty_\theta(\mathbb{R})\times C([0,T^*];H^\infty_\theta(\mathbb{R}))$ topology).*

Example 1.3. Simple examples of coefficients a_j satisfying the assumptions of Theorem 1.2 are given by $a_j(t,x,w)=a(t,x)\langle x\rangle^{-\frac{\sigma j}{2}}b(w)$ with $a\in C([0,T];\gamma^{\theta_0}(\mathbb{R}))$ and $b(w)=w^r$, $r\in\mathbb{N}$, or $b(w)=e^w$ or some other entire function. Indeed, given an entire function h, for every compact $K\subset\mathbb{C}$ there exists a positive constant C_K such that for every $w\in K$ we have $|\partial_w^\alpha h(w)|\leq C_K^{\alpha+1}\alpha!$.

Remark 1.4. The result obtained in this paper concerns 3-evolution equations in one space dimension as (1.1), (1.3). The extension of this result to higher space dimension requires a major technical effort in the definition of the change of variable needed to study the linearized problem associated to (1.13). We will treat this extension in a future paper.

The paper is organized as follows. In Section 2 we recall some basic definitions and properties of tame Fréchet spaces and the statement of Nash-Moser theorem. Moreover, we prove that $H^\infty_\theta(\mathbb{R})$ is a tame Fréchet space. Then, we introduce pseudodifferential operators of infinite order which are employed in the next sections to study the linearized Cauchy problem associated to (1.13). Section 3 is devoted to the study of this linear problem which is done using similar techniques as the ones used in [2] adapted to the projective Gevrey setting. Finally, in Section 4 we apply Nash-Moser theorem to obtain local in time well-posedness of (1.13).

2. Preliminaries

2.1. Function spaces

In this subsection we recall some basic facts concerning tame Fréchet spaces and prove that $H^\infty_\theta(\mathbb{R})$ is such a space. Moreover, we recall the statement of Nash-Moser inversion theorem, see [19]. A *graded* Fréchet space X is a Fréchet space endowed with a *grading*, i.e. an increasing sequence of semi-norms:

$$|x|_n \le |x|_{n+1}, \qquad \forall n \in \mathbb{N}_0,\, x \in X.$$

Example 2.1. Given a Banach space B, consider the space $\Sigma(B)$ of all sequences $\{v_k\}_{k\in\mathbb{N}_0} \subset B$ such that

$$|\{v_k\}|_n := \left(\sum_{k=0}^{\infty} e^{2nk}\|v_k\|_B^2\right)^{1/2} < +\infty, \qquad \forall n \in \mathbb{N}_0.$$

We have that $\Sigma(B)$ is a graded Fréchet space with the topology induced by the family of seminorms $|\cdot|_n$ (which is in fact a grading on $\Sigma(B)$).

We say that a linear map $L:\ X \to Y$ between two graded Fréchet spaces is a *tame linear map* if there exist $r, n_0 \in \mathbb{N}$ such that for every integer $n \ge n_0$ there exists a constant $C_n > 0$, depending only on n, such that

$$|Lx|_n \le C_n |x|_{n+r}, \qquad \forall x \in X. \tag{2.1}$$

The numbers n_0 and r are called respectively *base* and *degree* of the *tame estimate* (2.1).

Definition 2.2. A graded Fréchet space X is said to be *tame* if there exist a Banach space B and two tame linear maps $L_1 : X \to \Sigma(B)$ and $L_2 : \Sigma(B) \to X$ such that $L_2 \circ L_1$ is the identity on X.

Obviously, given a graded Fréchet space X and a tame space Y, if there exist two linear tame maps $L_1 : X \to Y$ and $L_2 : Y \to X$ such that $L_2 \circ L_1$ is the identity on X, then also X is a tame space.

Theorem 2.3. *The space $H_\theta^\infty(\mathbb{R}^n)$ is a tame Fréchet space.*

Proof. As standard, we shall denote here and throughout the paper the Fourier transform of a function (or a distribution) u by $\hat{u}$ or by $\mathcal{F}(u)$. First of all, it is easy to verify that $H_\theta^\infty(\mathbb{R}^n)$ is a graded Fréchet space with the increasing family of seminorms

$$|f|_k := \|f\|_{H^0_{k,\theta}} = \|e^{k\langle\cdot\rangle^{1/\theta}} \hat{f}(\cdot)\|_{L^2}, \qquad k = 1, 2, 3, \ldots.$$

Consider now the space $\Sigma(L^2(\mathbb{R}^n))$ and the map $L_1 : H_\theta^\infty(\mathbb{R}^n) \to \Sigma(L^2(\mathbb{R}^n))$ defined as $L_1(f) = \{f_j\}, j = 1, 2, 3, \ldots$, where $f_j = \mathcal{F}^{-1}(\chi_j \hat{f})$ and the functions χ_j are such that $\chi_j(\xi) = 1$ if $j^\theta \le \langle\xi\rangle < (j+1)^\theta$ and $\chi_j(\xi) = 0$ otherwise. Then we have

$$\begin{aligned}
|\{f_j\}|_k^2 &= \sum_{j=1}^{\infty} e^{2jk}\|\hat{f}_j\|_{L^2}^2 = \sum_{j=1}^{\infty} e^{2jk}\|\chi_j\hat{f}\|_{L^2}^2 \\
&= \sum_{j=1}^{\infty} e^{2jk} \int_{\mathbb{R}^n} |\chi_j(\xi)e^{-\rho\langle\xi\rangle^{1/\theta}} e^{\rho\langle\xi\rangle^{1/\theta}} \hat{f}(\xi)|^2\, d\xi \\
&\le \sum_{j=1}^{\infty} e^{2j(k-\rho)} \|e^{\rho\langle\cdot\rangle^{1/\theta}} \hat{f}(\cdot)\|_{L^2}^2 \le C_{k,\rho}\|f\|^2_{H^0_{\theta,\rho}}
\end{aligned}$$

for every $\rho > k$. In particular, for $\rho = k+1$ we obtain that $|\{f_j\}|_k \le C'_k |f|_{k+1}$, hence L_1 is a tame linear map. Similarly, we define the map $L_2 : \Sigma(L^2(\mathbb{R}^n)) \to H_\theta^\infty(\mathbb{R}^n)$ as

$$L_2(\{f_j\}) = \mathcal{F}^{-1}\left(\sum_{j=1}^{\infty} \chi_j \hat{f}_j\right).$$

We have

$$|L_2\{f_j\}|_k^2 = \left\| e^{k\langle\cdot\rangle^{1/\theta}} \sum_{j=1}^{\infty} \chi_j(\cdot)\hat{f}_j(\cdot) \right\|_{L^2}^2 = \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} e^{2k\langle\xi\rangle^{1/\theta}} |\chi_j(\xi)\hat{f}_j(\xi)|^2 \, d\xi$$

$$\leq \sum_{j=1}^{\infty} e^{2k(j+1)} \int_{\mathbb{R}^n} |\chi_j(\xi)\hat{f}_j(\xi)|^2 \, d\xi \leq \sum_{j=1}^{\infty} e^{2k(j+1)} \|\hat{f}_j\|_{L^2}^2 = e^{2k}|\{f_j\}|_k^2.$$

Hence, also L_2 is a tame linear map. Moreover, it is easy to verify that $L_2 \circ L_1$ is the identity map on $H_\theta^\infty(\mathbb{R}^n)$. □

Definition 2.4. Let X, Y be two graded spaces, U be an open subset of X. A map $T : U \to Y$ is said to be tame if for every $u \in U$ there exist a neighborhood U' of u, $r \geq 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a constant $C_n > 0$ such that

$$|T(u)|_n \leq C_n(1 + |u|_{n+r})$$

for all $u \in U'$. The map T is said to be smooth tame if T is $\mathcal{C}^\infty$ and its derivatives $D^nT : U \times X^n \to Y$ are tame for every $n \in \mathbb{N}$.

Finally, we recall the statement of Nash-Moser inversion theorem, cf. [19].

Theorem 2.5. *(Nash-Moser) Let X, Y be tame Fréchet spaces, U be an open subset of X and let $T : U \to Y$ be a smooth tame map. If for every fixed $u \in U, h \in Y$ the equation $DT(u)v = h$ has a unique solution $v = S(u, h)$ and if the map $S : U \times Y \to X$ is smooth tame, then T is locally invertible at any point and each local inverse is smooth tame.*

2.2. Pseudodifferential operators

In this subsection we introduce the pseudodifferential operators of infinite order which will be used to prove the well-posedness for the linearized Cauchy problem associated to (1.13). Although the arguments in the next sections concern one space dimensional problems, it is convenient to introduce these operators in arbitrary dimension for future applications.

Fixed $\mu \geq 1$, $A > 0$ and $m, m_1, m_2 \in \mathbb{R}$ we will consider the following Banach spaces:

$$p(x,\xi) \in S_\mu^m(\mathbb{R}^{2n}; A) \iff \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| A^{-|\alpha+\beta|} (\alpha!\beta!)^{-\mu} \langle\xi\rangle^{-m+|\alpha|} < +\infty,$$

$$p(x,\xi) \in \tilde{S}_\mu^m(\mathbb{R}^{2n}; A) \iff |p|_A := \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| A^{-|\alpha+\beta|} (\alpha!\beta!)^{-\mu} \langle\xi\rangle^{-m} < +\infty,$$

$$p(x,\xi) \in \mathbf{SG}_\mu^{m_1,m_2}(\mathbb{R}^{2n}; A) \Leftrightarrow \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ x,\xi\in\mathbb{R}^n}} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| A^{-|\alpha+\beta|} (\alpha!\beta!)^{-\mu} \langle\xi\rangle^{-m_1+|\alpha|} \langle x\rangle^{-m_2+|\beta|} < +\infty.$$

We set

$$S_\mu^m(\mathbb{R}^{2n}) := \bigcup_{A>0} S_\mu^m(\mathbb{R}^{2n}; A), \qquad \tilde{S}_\mu^m(\mathbb{R}^{2n}) := \bigcup_{A>0} \tilde{S}_\mu^m(\mathbb{R}^{2n}; A), \tag{2.2}$$

$$\mathbf{SG}_\mu^{m_1,m_2}(\mathbb{R}^{2n}) := \bigcup_{A>0} \mathbf{SG}_\mu^{m_1,m_2}(\mathbb{R}^{2n}; A) \tag{2.3}$$

endowed with the inductive limit topology and

$$\Gamma^m_\mu(\mathbb{R}^{2n}) := \bigcap_{A>0} S^m_\mu(\mathbb{R}^{2n}; A), \qquad \tilde{\Gamma}^m_\mu(\mathbb{R}^{2n}) := \bigcap_{A>0} \tilde{S}^m_\mu(\mathbb{R}^{2n}; A), \tag{2.4}$$

$$\Gamma G^{m_1,m_2}_\mu(\mathbb{R}^{2n}) := \bigcap_{A>0} \mathbf{SG}^{m_1,m_2}_\mu(\mathbb{R}^{2n}; A) \tag{2.5}$$

endowed with the projective limit topology.

Remark 2.6. We observe that if $\mu < \theta$, then for every $A > 0$ we have $S^m_\mu(\mathbb{R}^{2n}; A) \subset \Gamma^m_\theta(\mathbb{R}^{2n})$, $\tilde{S}^m_\mu(\mathbb{R}^{2n}; A) \subset \tilde{\Gamma}^m_\theta(\mathbb{R}^{2n})$ and $\mathbf{SG}^{m_1,m_2}_\mu(\mathbb{R}^{2n}; A) \subset \Gamma G^{m_1,m_2}_\theta(\mathbb{R}^{2n})$.

Taking into account the latter remark, in the sequel we shall consider symbols satisfying the estimates above for a fixed constant $A > 0$ as subsets of some projective symbol classes with a weaker Gevrey regularity as in (2.4), (2.5). For this reason we shall state the next results only for this type of classes.
For a given symbol $p \in \tilde{\Gamma}^m_\theta(\mathbb{R}^{2n})$ we denote by $p(x, D)$ or by $\mathrm{op}(p)$ the pseudodifferential operator defined by

$$p(x, D)u(x) = \int e^{i\xi x} p(x, \xi)\hat{u}(\xi) đ\xi, \quad u \in \gamma^\theta_0(\mathbb{R}^n), \tag{2.6}$$

where $đ\xi = (2\pi)^{-n} d\xi$. Arguing as in [32, Theorem 3.2.3] or [34, Theorem 2.4] it is easy to verify that operators of the form (2.6) with symbols from $\tilde{\Gamma}_\theta(\mathbb{R}^{2n})$ map continuously $\gamma^\theta_0(\mathbb{R}^n)$ into $\gamma^\theta(\mathbb{R}^n)$. Moreover, from the classical theory of pseudodifferential operators, they extend to linear and continuous operators from $H^{m'}(\mathbb{R}^n)$ to $H^{m'-m}(\mathbb{R}^n)$. For our purposes, it is also important to state the action of these operators on the Gevrey-Sobolev spaces defined in the Introduction. The following result is a direct consequence of [24, Proposition 6.3] applied to symbols from $\tilde{\Gamma}^m_\theta(\mathbb{R}^{2n})$.

Proposition 2.7. *Let $p \in \tilde{\Gamma}^m_\theta(\mathbb{R}^{2n})$. Then the operator $p(x, D)$ maps continuously $H^{m'}_{\rho;\theta}(\mathbb{R}^n)$ into $H^{m'-m}_{\rho;\theta}(\mathbb{R}^n)$ for every $m', \rho \in \mathbb{R}$.*

By [24, Proposition 6.4], given $p \in \Gamma^m_\theta(\mathbb{R}^{2n})$ and $q \in \Gamma^{m'}_\theta(\mathbb{R}^{2n})$, the operator $p(x, D)q(x, D)$ is a pseudodifferential operator with symbol s given for every $N \geq 1$ by

$$s(x, \xi) = \sum_{|\alpha|<N} (\alpha!)^{-1} \partial^\alpha_\xi p(x, \xi) D^\alpha_x q(x, \xi) + r_N(x, \xi),$$

where $r_N \in \Gamma^{m+m'-N}_\theta(\mathbb{R}^{2n})$.
In the following we shall consider also particular symbols of infinite order, that is growing exponentially at infinity. Such operators are frequently used in the analysis of evolution equations in the Gevrey setting, see for instance [1,2,6,7,10,11,23,24,34]. In particular, in this paper they will be employed to define the change of variables which allows to treat the linearized problem associated to (1.13). We shall not develop a complete calculus for pseudodifferential operators of infinite order here since for our purposes we can limit ourselves to considering some particular examples of such operators, namely defined by a symbol of the form $e^{\Lambda(x,\xi)}$ for some $\Lambda \in S^{1/\kappa}_\mu(\mathbb{R}^{2n}; A)$, where $\kappa > 1$ and Λ is real-valued. It is easy to verify that $e^{\pm\Lambda}$ satisfies an estimate of the form

$$|\partial^\alpha_\xi \partial^\beta_x e^{\pm\Lambda(x,\xi)}| \leq A_1^{|\alpha+\beta|} \langle\xi\rangle^{-|\alpha|} (\alpha!\beta!)^\mu e^{2\rho_0 \langle\xi\rangle^{\frac{1}{\kappa}}} \tag{2.7}$$

for some positive constant A_1 independent of α, β, where

$$\rho_0 := \sup_{(\alpha,\beta)\in\mathbb{N}_0^{2n}} \sup_{(x,\xi)\in\mathbb{R}^{2n}} A^{-|\alpha+\beta|}(\alpha!\beta!)^{-\mu}\langle\xi\rangle^{-1/\kappa+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta\Lambda(x,\xi)|,$$

see [24, Lemma 6.2]. The estimate (2.7) guarantees that the related pseudodifferential operator

$$e^{\pm\Lambda}(x,D)u(x) = \int_{\mathbb{R}^n} e^{i\xi x\pm\Lambda(x,\xi)}\hat{u}(\xi)\, đ\xi$$

is well defined and continuous as an operator from $\gamma_0^\theta(\mathbb{R}^n)$ to $\gamma^\theta(\mathbb{R}^n)$ for every $\theta \in (\mu,\kappa)$. We shall also consider the so-called reverse operator of $e^{\pm\Lambda}(x,D)$, denoted by ${}^R\{e^{\pm\Lambda}(x,D)\}$. This operator, introduced in [25, Proposition 2.13] as the transposed of $e^{\pm\Lambda}(x,-D)$, is defined as an oscillatory integral by

$${}^R\{e^{\pm\Lambda}(x,D)\}u(x) = Os - \iint e^{i\xi(x-y)\pm\Lambda(y,\xi)}u(y)\, dyđ\xi.$$

The following continuity result holds for the operators $e^\Lambda(x,D)$ and ${}^R\{e^\Lambda(x,D)\}$.

Proposition 2.8. *Let $\Lambda \in \tilde{S}_\mu^{1/\kappa}(\mathbb{R}^{2n};A)$ for some $A>0$ and $\kappa,\mu\in\mathbb{R}$ such that $1<\mu<\kappa$ and let $\rho,m\in\mathbb{R}$ and $\theta\in(\mu,\kappa)$. Then the operators $e^\Lambda(x,D)$ and ${}^R\{e^\Lambda(x,D)\}$ map continuously $H^m_{\rho;\theta}(\mathbb{R}^n)$ into $H^m_{\rho-\delta;\theta}(\mathbb{R}^n)$ for every $\delta>0$.*

Proof. We observe that $e^\Lambda(x,D) = a(x,D)e^{\delta\langle D\rangle^{\frac{1}{\theta}}}$ for every $\delta>0$, where $a(x,\xi) = e^{\Lambda(x,\xi)-\delta\langle\xi\rangle^{\frac{1}{\theta}}}$. Since $\mu<\theta<\kappa$ we easily obtain $a\in\tilde{\Gamma}^0_\theta(\mathbb{R}^{2n})$. So we obtain from Proposition 2.7 that $e^\Lambda : H^m_{\rho;\theta}(\mathbb{R}^n)\to H^m_{\rho-\delta;\theta}(\mathbb{R}^n)$ continuously for every $m,\rho\in\mathbb{R}$. The continuity of ${}^R\{e^\Lambda(x,D)\}$ follows by similar arguments. □

In the next result we shall need to work with the weight function $\langle\xi\rangle_h = (h^2+|\xi|^2)^{1/2}$ where $h\geq 1$. We point out that we can replace $\langle\xi\rangle$ by $\langle\xi\rangle_h$ in all the previous definitions and statements, and this replacement does not change the dependence of the constants, that is, all the previous constants are independent of h. Moreover, we also need the following stronger hypothesis on $\Lambda(x,\xi)$:

$$|\partial_\xi^\alpha\partial_x^\beta\Lambda(x,\xi)| \leq C_\Lambda^{|\alpha+\beta|+1}\alpha!^\mu\beta!^\mu\langle\xi\rangle_h^{-|\alpha|}, \tag{2.8}$$

whenever $|\beta|\geq 1$. This means that $\partial_\xi^\alpha\partial_x^\beta\Lambda$ behaves like a symbol of order 0 if $\beta\neq 0$. We will show in the next Section that this condition will be fulfilled by the symbol $\tilde{\Lambda}$ appearing in the change of variable.

Theorem 2.9. *Let p be a symbol in $\Gamma^m_\theta(\mathbb{R}^{2n})$ and let Λ satisfy, for some $C_\Lambda>0$ and $\mu<\theta<\kappa$:*

$$|\partial_\xi^\alpha\Lambda(x,\xi)| \leq C_\Lambda^{|\alpha|+1}\alpha!^\mu\langle\xi\rangle_h^{\frac{1}{\kappa}-|\alpha|} \tag{2.9}$$

and (2.8) for $\beta\neq 0$. Then there exists $h_0 = h_0(C_\Lambda)\geq 1$ such that if $h\geq h_0$, then

$$e^\Lambda(x,D)p(x,D)^R\{e^{-\Lambda}(x,D)\} = p(x,D) + op\left(\sum_{1\leq|\alpha+\beta|<N}\frac{1}{\alpha!\beta!}\partial_\xi^\alpha\{\partial_\xi^\beta e^{\Lambda(x,\xi)}D_x^\beta p(x,\xi)D_x^\alpha e^{-\Lambda(x,\xi)}\}\right) + r_N(x,D) + r_\infty(x,D),$$

where r_N and r_∞ satisfy the following conditions: there exists $c' = c'(\Lambda)>0$ and for every $A>0$ there exists $C_A>0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta r_N(x,\xi)| \leq C_A A^{|\alpha+\beta|+2N}\alpha!^\theta\beta!^\theta N!^{2\theta-1}\langle\xi\rangle_h^{m-(1-\frac{1}{\kappa})N-|\alpha|}, \tag{2.10}$$

$$|\partial_\xi^\alpha \partial_x^\beta r_\infty(x,\xi)| \leq C_A A^{|\alpha+\beta|+2N} \alpha!^\theta \beta!^\theta N!^{2\theta-1} e^{-c'\langle\xi\rangle_h^{\frac{1}{\theta}}}. \tag{2.11}$$

Remark 2.10. Notice that choosing N sufficiently large depending on κ, we can consider r_N as a symbol of order 0. Concerning the remainder term r_∞, it is easy to verify that the corresponding operator possesses regularizing properties in Gevrey classes, namely it maps $(G_0^\theta)'(\mathbb{R}^n)$ into $G^\theta(\mathbb{R}^n)$. However, to prove our results it will be sufficient to regard also r_∞ as a symbol of $\tilde{\Gamma}_\theta^0(\mathbb{R}^{2n})$. In conclusion, in the computations of Section 4, choosing N large enough, we shall always consider the remainder term $r_N + r_\infty$ as a symbol of $\tilde{\Gamma}_\theta^0(\mathbb{R}^{2n})$ and apply to it Proposition 2.12 below.

Remark 2.11. The proof of Theorem 2.9 follows by applying readily in the projective Gevrey setting the same argument used in the proofs of Theorems 6.9 and 6.10 of [24] and Theorem 2 of [2] in the classical Gevrey framework. For this reason, we omit it for the sake of brevity. We just stress the fact that dealing now with projective Gevrey regular symbols p, it is possible to conclude that the remainders r_N and r_∞ also satisfy this type of estimates, cf. (2.10), (2.11).

Now we consider the conjugation with an operator of the form $e^{\Lambda_{\rho',k}}(t,D)$ where $\Lambda_{\rho',k}(t,\xi) = \rho'\langle\xi\rangle^{1/\theta} + k(T-t)\langle\xi\rangle^{2(1-\sigma)}$ for some $\rho' \in (0,\rho)$ and $k > 0$ (where $\rho > 0$ is the same index appearing in the statement of Theorem 3.1). The next result can be proved following the same argument as in the proof of [6, Proposition 3.1]. Compared to the latter result, in the present case, the conjugation can be performed for every $\rho' > 0$ since the symbol of the operator satisfies *projective* Gevrey estimates. Namely, we have the following result.

Proposition 2.12. *Let $p \in \tilde{\Gamma}_\theta^m(\mathbb{R}^2)$. Then we can write*

$$e^{\Lambda_{\rho',k}}(t,D) \circ p(x,D) \circ e^{-\Lambda_{\rho',k}}(t,D) = op\left(\sum_{\alpha<N} \frac{1}{\alpha!}\partial_\xi^\alpha e^{\Lambda_{\rho',k}(t,\xi)} D_x^\alpha p(x,\xi) e^{-\Lambda_{\rho',k}(t,\xi)}\right) + r_N(t,x,D),$$

where r_N satisfies the following condition: for every $A > 0$ there exists $C_{k,\rho',A,N} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta r_N(t,x,\xi)| \leq |p|_A C_{k,\rho',A,N} A^{\alpha+\beta} \langle\xi\rangle^{m-N(1-\frac{1}{\theta})}.$$

3. The linearized problem

Fixed $u \in \Omega \subset X_T := C^1([0,T]; H_\theta^\infty(\mathbb{R}))$, where Ω denotes a bounded set, we now consider the linear Cauchy problem

$$\begin{cases} P_u(D)v(t,x) := P(t,x,u(t,x),D_t,D_x)v(t,x) = f(t,x),\ (t,x) \in [0,T] \times \mathbb{R}, \\ v(0,x) = g(x), \end{cases} \tag{3.1}$$

in the unknown v. In this section we shall prove the following result.

Theorem 3.1. *Under the assumptions of Theorem 1.2, given $m \in \mathbb{R}$, $\rho > 0$, $\theta \in \left[\theta_0, \frac{1}{2(1-\sigma)}\right)$, $u \in \Omega \subset C^1([0,T]; H_\theta^\infty(\mathbb{R}))$, $f \in C([0,T]; H_{\rho;\theta}^m(\mathbb{R}))$ and $g \in H_{\rho;\theta}^m(\mathbb{R})$, there exists a unique solution $v \in C^1([0,T]; H_{\rho-\delta;\theta}^m(\mathbb{R}))$ for every $\delta \in (0,\rho)$ of the Cauchy problem (3.1) and the following energy estimate is satisfied:*

$$\|v(t,\cdot)\|_{H_{\rho-\delta;\theta}^m}^2 \leq C_{\Omega,\rho,T}\left(\|g\|_{H_{\rho;\theta}^m}^2 + \int_0^t \|f(\tau,\cdot)\|_{H_{\rho;\theta}^m}^2\, d\tau\right) \quad \forall t \in [0,T], \tag{3.2}$$

for some positive constant $C_{\Omega,\rho,T}$. *Moreover, if* $f \in C([0,T], H^\infty_\theta(\mathbb{R}))$, $g \in H^\infty_\theta(\mathbb{R})$, *then* v *belongs to* $C^1([0,T]; H^\infty_\theta(\mathbb{R}))$.

In order to prove the theorem above we shall follow the same method used to prove the well-posedness of the Cauchy problem for linear 3-evolution equations in $\mathcal{H}^\infty_\theta(\mathbb{R})$ in [2]. This method is based on making a suitable change of variable in order to transform the Cauchy problem (3.1) for the operator $P_u(D)$ into an equivalent Cauchy problem which turns out to be well-posed in Sobolev spaces. The transformation we have in mind will be the composition of two transformations both defined by invertible pseudodifferential operators of infinite order. Namely it will be of the form

$$Q_{\tilde{\Lambda},k,\rho'}(t,x,D) = e^{\Lambda_{\rho',k}}(t,D) \circ e^{\tilde{\Lambda}}(x,D), \tag{3.3}$$

where $\tilde{\Lambda} = \lambda_2 + \lambda_1 \in S^{2(1-\sigma)}_\mu(\mathbb{R}^2)$ for some $\mu > 1$, and $\Lambda_{\rho',k}(t,\xi) = \rho'\langle\xi\rangle_h^{\frac{1}{\theta}} + k(T-t)\langle\xi\rangle_h^{2(1-\sigma)}$ for some $\rho' \in (0,\rho), k > 0$ and $h >> 1$ to be chosen later on. Then, by the inverse transformation, we recover the solution $v = Q_{\tilde{\Lambda},k,\rho'}(t,x,D)^{-1}w$ of (3.1), where w stands for the solution of the auxiliary problem. The mapping properties of the transformations $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)$ and $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)^{-1}$ will determine the space where the Cauchy problem (3.1) is well-posed. The role of each part of the transformation $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)$ will be, broadly speaking, the following:

- in the transformation $e^{\tilde{\Lambda}}(x,D)$ the functions λ_1 and λ_2 will play two different roles: namely λ_2 will not change $a_3D^3_x$, but it will change the operator $a_2D^2_x$ into the sum of a positive operator plus a remainder of order 1 which satisfies the same assumptions as a_1D_x, plus an error of order $2(1-\sigma)$ whereas λ_1 will not change the terms of order 2 and 3, but it will turn the terms of order 1 into the sum of a positive operator, plus a remainder of order zero, plus an error of order at least $2(1-\sigma)$;
- the transformation with $e^{k(T-t)\langle D\rangle_h^{2(1-\sigma)}}$ will not change the terms of order 1, 2 and 3, but it will correct the error of order $2(1-\sigma)$, changing it into the sum of a positive operator plus a remainder of order zero;
- finally, the transformations with $e^{\rho'\langle D\rangle_h^{\frac{1}{\theta}}}$ simply serves to change the setting of the Cauchy problem from Gevrey-Sobolev spaces to standard Sobolev spaces: since $2(1-\sigma) < 1/\theta$ the leading part of $Q_{\tilde{\Lambda},k,\rho'}(t,x,\xi)$ is $e^{\rho'\langle\xi\rangle_h^{\frac{1}{\theta}}}$, then the inverse of $Q_{\tilde{\Lambda},k,\rho'}(t,x,D)$ possesses regularizing properties with respect to the spaces $H^m_{\rho;\theta}$, because $\rho' > 0$.

Working step by step, in the next subsection we define the symbol $\tilde{\Lambda}$ and briefly state its main features, then in Subsection 3.3 we perform the conjugation $Q_{\tilde{\Lambda},k,\rho'}(iP_u)Q^{-1}_{\tilde{\Lambda},k,\rho'}$, and finally in Subsection 3.4 we prove Theorem 3.1.

3.1. Change of variables

For $M_2, M_1 > 0$ and $h \geq 1$ a large parameter, we define

$$\lambda_2(x,\xi) = M_2 w\left(\frac{\xi}{h}\right) \int_0^x \langle y\rangle^{-\sigma}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right) dy, \quad (x,\xi) \in \mathbb{R}^2, \tag{3.4}$$

$$\lambda_1(x,\xi) = M_1 w\left(\frac{\xi}{h}\right) \langle\xi\rangle_h^{-1} \int_0^x \langle y\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right) dy, \quad (x,\xi) \in \mathbb{R}^2, \tag{3.5}$$

where

$$w(\xi) = \begin{cases} 0, & |\xi| \le 1, \\ -\operatorname{sgn} a_3, & |\xi| \ge 2, \end{cases} \qquad \psi(y) = \begin{cases} 1, & |y| \le \frac{1}{2}, \\ 0, & |y| \ge 1, \end{cases}$$

$|\partial_\xi^\alpha w(\xi)| \le C_w^{\alpha+1}\alpha!^\mu$, $|\partial_y^\beta \psi(y)| \le C_\psi^{\beta+1}\beta!^\mu$, with $\mu > 1$. The functions λ_1 and λ_2 have been introduced in [2]. They satisfy peculiar estimates where the powers of the weight functions $\langle\xi\rangle_h$ and $\langle x\rangle$ can be adjusted as needed thanks to the special structure of $\operatorname{supp}\psi$ and $\operatorname{supp}\psi'$. These estimates are contained in the following two lemmas which have been proved in [2].

Lemma 3.2. *Let $\lambda_2(x,\xi)$ as in (3.4). Then the following estimates hold:*

(i) $|\partial_\xi^\alpha \lambda_2(x,\xi)| \le M_2 C_{\lambda_2}^{\alpha+1}\alpha!^\mu \langle\xi\rangle_h^{-\alpha}\min\{\langle\xi\rangle_h^{2(1-\sigma)}, \langle x\rangle^{1-\sigma}\}$, *for* $\alpha \ge 0$;
(ii) $|\partial_\xi^\alpha \partial_x^\beta \lambda_2(x,\xi)| \le M_2 C_{\lambda_2}^{\alpha+\beta+1}\alpha!^\mu\beta!^\mu \langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\sigma-\beta+1}$, *for* $\alpha \ge 0, \beta \ge 1$,

where C_{λ_2} is a constant depending only on C_w, C_ψ and σ.

Lemma 3.3. *Let $\lambda_1(x,\xi)$ as in (3.5). Then*

(i) $|\partial_\xi^\alpha \lambda_1(x,\xi)| \le M_1 C_{\lambda_1}^{\alpha+1}\alpha!^\mu \langle\xi\rangle_h^{-\alpha}\min\{\langle\xi\rangle_h^{1-\sigma}, \langle\xi\rangle_h^{-1}\langle x\rangle^{1-\frac{\sigma}{2}}, \langle x\rangle^{\frac{1}{2}-\frac{\sigma}{2}}\}$, *for* $\alpha \ge 0$;
(ii) $|\partial_\xi^\alpha \partial_x^\beta \lambda_1(x,\xi)| \le M_1 C_{\lambda_1}^{\alpha+\beta+1}\alpha!^\mu\beta!^\mu \langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\frac{\sigma}{2}-\beta+1}\min\{\langle\xi\rangle_h^{-1}, \langle x\rangle^{-\frac{\sigma}{2}}\}$, *for* $\alpha \ge 0, \beta \ge 1$,

where C_{λ_1} is a constant depending only on C_w, C_ψ and σ.

Remark 3.4. Lemmas 3.2 and 3.3 imply $\lambda_2, \lambda_1 \in \mathbf{SG}_\mu^{0,1-\sigma}(\mathbb{R}^2)$. Moreover, we also have that $\lambda_1 \in S_\mu^{1-\sigma}(\mathbb{R}^2)$ and $\lambda_2 \in S_\mu^{2(1-\sigma)}(\mathbb{R}^2)$.

The following result proves the invertibility of the transformation $e^{\tilde\Lambda}(x,D)$ and expresses the inverse in terms of a composition of ${}^R\{e^{-\tilde\Lambda}(x,D)\}$ with a Neumann series, see [1, Lemma 4] for the proof. In the statement we shall denote by $\Sigma_\kappa(\mathbb{R}^2)$ the space of all symbols $\tau(x,\xi)$ satisfying for every $A > 0, c > 0$ an estimate of the form

$$|\partial_\xi^\alpha \partial_x^\beta \tau(x,\xi)| \le C_A A^{\alpha+\beta}(\alpha!\beta!)^\kappa e^{-c(\langle x\rangle^{1/k} + \langle\xi\rangle_h^{1/k})},$$

cf. [31].

Lemma 3.5. *Let $\mu > 1$. For $h \ge 1$ large enough, the operator $e^{\tilde\Lambda}(x,D)$ is invertible and its inverse is given by*

$$\{e^{\tilde\Lambda}(x,D)\}^{-1} = {}^R\{e^{-\tilde\Lambda}(x,D)\} \circ \sum_{j\ge 0}(-r(x,D))^j,$$

for some $r = \tilde r + \bar r$, where $\tilde r \in \boldsymbol{SG}_\mu^{-1,-\sigma}(\mathbb{R}^2)$, $\bar r \in \Sigma_\kappa(\mathbb{R}^2)$ for every $\kappa > 2\mu - 1$ and

$$\tilde r - \sum_{1\le\gamma\le N} \frac{1}{\gamma!}\partial_\xi^\gamma(e^{\tilde\Lambda}D_x^\gamma e^{-\tilde\Lambda}) \in \boldsymbol{SG}_\mu^{-1-N,-\sigma-\sigma N}(\mathbb{R}^2), \quad \forall N \ge 1.$$

Moreover, $\sum(-r(x,D))^j$ has symbol in $\boldsymbol{SG}_\mu^{0,0}(\mathbb{R}^2) + \Sigma_\kappa(\mathbb{R}^2)$ for every $\kappa > 2\mu - 1$. Finally, we have

$$\{e^{\tilde\Lambda}(x,D)\}^{-1} = {}^R\{e^{-\tilde\Lambda}(x,D)\} \circ op(1 - i\partial_\xi\partial_x\tilde\Lambda - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde\Lambda - [\partial_x\tilde\Lambda]^2) - [\partial_\xi\partial_x\tilde\Lambda]^2 + q_{-3}), \tag{3.6}$$

where $q_{-3} \in \boldsymbol{SG}_{\mu}^{-3,-3\sigma}(\mathbb{R}^2) + \Sigma_\kappa(\mathbb{R}^2)$.

Remark 3.6. Since we can choose $\mu > 1$ arbitrarily close to 1, we may assume $2\mu - 1 < \theta$. Therefore we can take $\kappa < \theta$ in the above lemma.

3.2. Estimates for the linearized coefficients

Before starting to prove Theorem 3.1, we need to state which type of estimates the coefficients of the linearized problem (3.1) satisfy under the assumptions of Theorem 1.2.
Since $\Omega \subset X_T$ is bounded, we have that for any $k \in \mathbb{N}$ there exists $B_k > 0$ such that

$$\sup_{w\in\Omega} \|w\|_{H^0_{k;\theta}} \leq B_k.$$

On the other hand we can write

$$D_x^\alpha u(t,x) = \int e^{i\xi x} e^{-\rho\langle\xi\rangle^{\frac{1}{\theta}}} \xi^\alpha e^{\rho\langle\xi\rangle^{\frac{1}{\theta}}} \widehat{u}(t,\xi) đ\xi.$$

Since $u(t) \in H^\infty_\theta$, then for any $\rho > 0$ Hölder inequality gives

$$\begin{aligned} |D_x^\alpha u(t,x)|^2 &\leq \int e^{-2\rho\langle\xi\rangle^{\frac{1}{\theta}}} \xi^{2\alpha} d\xi \, \|u(t)\|^2_{H^0_{\rho;\theta}} \\ &\leq \left(\frac{2\theta}{\rho}\right)^{2\theta\alpha} \alpha!^{2\theta} \|e^{-\frac{\rho}{2}\langle\cdot\rangle}\|^2_{L^2} \|u(t)\|^2_{H^0_{\rho;\theta}}. \end{aligned}$$

The above estimate implies that for any $A > 0$ there is a positive constant $C_{\Omega,A}$ such that

$$|D_x^\alpha u(t,x)| \leq C_{\Omega,A} A^\alpha \alpha!^\theta, \quad t \in [0,T], x \in \mathbb{R}, \alpha \in \mathbb{N}_0, \tag{3.7}$$

for every $u \in \Omega$. In particular, we conclude that the values of $w = u(t,x)$ lie in a fixed compact set $K_\Omega = K \subset \mathbb{C}(\approx \mathbb{R}^2)$ for every $u \in \Omega$. We shall fix this compact from now on. Using (3.7) and the fact that $a_j \in C([0,T], \Gamma^{\theta_0, \frac{j\sigma}{2}}(\mathbb{R}\times\mathbb{C})), j = 0,1,2$, in what follows we shall estimate the x-derivatives of the maps $x \mapsto a_j(t,x,u(t,x))$. For this we need the Faà di Bruno formula in several variables: let $g = (g_1,\dots,g_p) : \mathbb{R}^n \to \mathbb{R}^p$, $f : \mathbb{R}^p \to \mathbb{R}$ and $\beta \in \mathbb{N}_0^n$, then

$$D^\beta(f\circ g)(x) = \sum_{*} \frac{\beta!}{k_1!\dots k_\ell!} \{D^{k_1+\dots+k_\ell} f\}(g(x)) \prod_{j=1}^{\ell}\prod_{i=1}^{p} \left[\frac{D^{\delta_j} g_i(x)}{\delta_j!}\right]^{k_{ji}} \tag{3.8}$$

where the notation $\sum_*$ means that the sum is taken over all $\ell \in \mathbb{N}$, all sets $\{\delta_1,\dots,\delta_\ell\}$ of ℓ distinct elements of $\mathbb{N}_0^n - \{0\}$ and all $(k_1,\dots,k_\ell) \in (\mathbb{N}_0^p - \{0\})^\ell$ such that $\sum_{j=1} |k_j|\delta_j = \beta$. We also report two useful inequalities:

$$|k_1+\dots+k_\ell|!|\delta_1|!^{|k_1|}\dots|\delta_\ell|!^{|k_\ell|} \leq |\beta|!$$

and

$$\sum \frac{\beta!}{k_1!\dots k_\ell!} \lambda^{|k_1+\dots+k_\ell|} \leq C_\lambda^{|\beta|+1}, \quad \forall\, \lambda > 0,$$

where $\beta, \ell, (\delta_1, \dots, \delta_\ell), (k_1, \dots, k_\ell)$ are as in formula (3.8). For details on Faà di Bruno formula we address the reader to Proposition 4.3, Corollary 4.5 and Lemma 4.8 of [9].

Let now $\beta \in \mathbb{N}_0$, then

$$D_x^\beta(a_j(t,x,u(t,x))) = \sum_* \frac{\beta!}{k_1! \dots k_\ell!} \{D_{(x,w)}^{k_1+\dots+k_\ell} a_j\}(t,x,u(t,x)) \prod_{j=1}^{\ell} \prod_{i=1}^{3} \left[\frac{D_x^{\delta_j} g_i(t,x)}{\delta_j!} \right]^{k_{ji}},$$

where $g_1(t,x) = x$, $g_2(t,x) = Re\, u(t,x)$ and $g_3(t,x) = Im\, u(t,x)$. Applying (3.7) and the assumptions on the a_j, we get for every $A, B > 0$:

$$\begin{aligned} |D_x^\beta(a_j(t,x,u(t,x)))| &\le \sum_* \frac{\beta!}{k_1! \dots k_\ell!} C_{K,A} A^{|k_1+\dots+k_\ell|} |k_1 + \dots + k_\ell|!^{\theta_0} \langle x \rangle^{-\sigma j/2} \\ &\times \prod_{j=1}^{\ell} \prod_{i=1}^{3} \left[C_{\Omega,B} B^{\delta_j} \delta_j!^{\theta-1} \right]^{k_{ji}} \\ &\le C_{K,A} B^\beta \langle x \rangle^{-\sigma j/2} \beta! \\ &\times \sum_* \frac{|k_1 + \dots + k_\ell|!}{k_1! \dots k_\ell!} (C_{\Omega,B} A)^{|k_1+\dots+k_\ell|} \underbrace{|k_1 + \dots + k_\ell|!^{\theta-1} \prod_{j=1}^{\ell} \delta_j!^{(\theta-1)|k_j|}}_{\le \beta!^{\theta-1}} \\ &\le C_{K,A} B^\beta \langle x \rangle^{-\sigma j/2} \beta!^\theta \sum_* \frac{|k_1 + \dots + k_\ell|!}{k_1! \dots k_\ell!} (C_{\Omega,B} A)^{|k_1+\dots+k_\ell|}. \end{aligned}$$

Taking $A = C_{\Omega,B}^{-1}$ it follows

$$|D_x^\beta(a_j(t,x,u(t,x)))| \le C_1 C_{K,\Omega,B} \{C_1 B\}^\beta \beta!^\theta \langle x \rangle^{-\sigma j/2}$$

for some constant $C_1 > 0$ independent of A and B and $C_{K,\Omega,B} > 0$ which in fact depends only on B and Ω. Rescaling the constant $C_1 B$ we obtain the following result.

Lemma 3.7. *Under the assumptions of Theorem 1.2, let $\Omega \subset X_T$ be a bounded subset. Then for every $B > 0$ there exists a constant $C_{\Omega,B} > 0$ such that*

$$|D_x^\beta(a_j(t,x,u(t,x)))| \le C_{\Omega,B} B^\beta \beta!^\theta \langle x \rangle^{-\sigma j/2}, \quad t \in [0,T], x \in \mathbb{R}, \beta \in \mathbb{N}_0, \tag{3.9}$$

for every $u \in \Omega$.

3.3. The conjugation procedure

In the present subsection we perform, step by step, the conjugations needed to obtain the operator $Q_{\tilde{\Lambda},k,\rho'}(iP_u)Q_{\tilde{\Lambda},k,\rho'}^{-1}$.

3.3.1. Conjugation with $e^{\tilde{\Lambda}}$

Now we perform the conjugation of iP_u by the operator $e^{\tilde{\Lambda}}(x,D)$, with $\tilde{\Lambda}(x,\xi) = \lambda_2(x,\xi) + \lambda_1(x,\xi)$. In the next computation, by abuse, we shall denote by $a_3(t,D)$ and $a_j(t,x,u,D)$ for $j = 1,2$, the operators $a_3(t)D_x^3$ and $a_j(t,x,u)D_x^j, j = 1,2$, respectively, and by a_1, a_2, a_3 their symbols, sometimes omitting the dependence on the variables t, x, u and ξ.

- Conjugation of $ia_3(t,D)$: Since a_3 does not depend on x, Theorem 2.9 simplifies into (omitting (t,x,D) in the notation)

$$e^{\tilde{\Lambda}}(x,D)\circ ia_3(t,D)\circ^R\{e^{-\tilde{\Lambda}}(x,D)\} = ia_3(t,D) + \mathrm{op}\left(\partial_\xi\{ia_3D_x(-\tilde{\Lambda})\} + \frac{1}{2}\partial_\xi^2\{ia_3[D_x^2(-\tilde{\Lambda}) + (D_x\tilde{\Lambda})^2]\} + q_3 + r_\infty\right).$$

Since x-derivatives kill the ξ-growth given by the integrals of $\tilde{\Lambda}$, we can conclude that q_3 has order zero. Composing with the Neumman series we get

$$\begin{aligned}
e^{\tilde{\Lambda}}(x,D)ia_3(t,D)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \mathrm{op}\left(ia_3 - \partial_\xi(a_3\partial_x\tilde{\Lambda}) + \frac{i}{2}\partial_\xi^2[a_3(\partial_x^2\tilde{\Lambda} - (\partial_x\tilde{\Lambda})^2)] + q_3 + r_\infty\right)\\
&\quad\circ\mathrm{op}\left(1 - i\partial_\xi\partial_x\tilde{\Lambda} - \frac{1}{2}\partial_\xi^2(\partial_x^2\tilde{\Lambda} - [\partial_x\tilde{\Lambda}]^2) - [\partial_\xi\partial_x\tilde{\Lambda}]^2 + q_{-3}\right)\\
&= ia_3(t,D) + \mathrm{op}\left(-\partial_\xi(a_3\partial_x\tilde{\Lambda}) + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\tilde{\Lambda} - \{\partial_x\tilde{\Lambda}\}^2)\} + a_3\partial_\xi\partial_x\tilde{\Lambda} - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde{\Lambda}\right)\\
&\quad + \mathrm{op}\left(i\partial_\xi(a_3\partial_x\tilde{\Lambda})\partial_\xi\partial_x\tilde{\Lambda} - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde{\Lambda} + [\partial_x\tilde{\Lambda}]^2) + 2[\partial_\xi\partial_x\tilde{\Lambda}]^2\} + \tilde{r}_0\right)\\
&= ia_3(t,D) + \mathrm{op}\left(-\partial_\xi a_3\partial_x\tilde{\Lambda} + \frac{i}{2}\partial_\xi^2\{a_3[\partial_x^2\tilde{\Lambda} - (\partial_x\tilde{\Lambda})^2]\} - i\partial_\xi a_3\partial_\xi\partial_x^2\tilde{\Lambda}\right)\\
&\quad + \mathrm{op}\left(i\partial_\xi(a_3\partial_x\tilde{\Lambda})\partial_\xi\partial_x\tilde{\Lambda} - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\tilde{\Lambda} + [\partial_x\tilde{\Lambda}]^2) + 2(\partial_\xi\partial_x\tilde{\Lambda})^2\} + \tilde{r}_0\right),
\end{aligned}$$

where $\tilde{r}_0 \in C([0,T];\tilde{\Gamma}_\theta^0(\mathbb{R}^2))$. From now on we are going to denote by $\tilde{r}_0$ all remainders of class $C([0,T];\tilde{\Gamma}_\theta^0(\mathbb{R}^2))$ satisfying uniform estimates with respect to $u \in \Omega$. Writing $\tilde{\Lambda} = \lambda_2 + \lambda_1$ and noticing that $D_x\lambda_1$ has order -1 we get

$$\begin{aligned}
e^{\tilde{\Lambda}}(x,D)ia_3(t,D)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= ia_3(t,D)\\
&\quad + \mathrm{op}\left(-\partial_\xi a_3\partial_x\lambda_2 - \partial_\xi a_3\partial_x\lambda_1 + \frac{i}{2}\partial_\xi^2\{a_3(\partial_x^2\lambda_2 - \{\partial_x\lambda_2\}^2)\} - i\partial_\xi a_3\partial_\xi\partial_x^2\lambda_2\right)\\
&\quad + \mathrm{op}\left(i\partial_\xi(a_3\partial_x\lambda_2)\partial_\xi\partial_x\lambda_2 - \frac{i}{2}a_3\{\partial_\xi^2(\partial_x^2\lambda_2 + [\partial_x\lambda_2]^2) + 2[\partial_\xi\partial_x\lambda_2]^2\} + \tilde{r}_0\right).
\end{aligned}$$

For simplicity we write in short

$$\begin{aligned}
d_1(t,x,\xi) &= \frac{1}{2}\partial_\xi^2\{a_3(\partial_x^2\lambda_2 - \{\partial_x\lambda_2\}^2)\} - \partial_\xi a_3\partial_\xi\partial_x^2\lambda_2\\
&\quad + \partial_\xi(a_3\partial_x\lambda_2)\partial_\xi\partial_x\lambda_2 - \frac{1}{2}a_3\{\partial_\xi^2(\partial_x^2\lambda_2 + [\partial_x\lambda_2]^2) + 2[\partial_\xi\partial_x\lambda_2]^2\}.
\end{aligned}$$

Hence

$$e^{\tilde{\Lambda}}(x,D)ia_3(t,D)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} = ia_3(t,D) + \mathrm{op}\left(-\partial_\xi a_3\partial_x\lambda_2 - \partial_\xi a_3\partial_x\lambda_1 + id_1 + \tilde{r}_0\right).$$

Notice that d_1 is a real valued symbol of order 1 which does not depend on λ_1. Namely, we have the following estimates: for every $A > 0$ there exists $C_{\lambda_2,A} > 0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta d_1(t,x,\xi)| \le C_{\lambda_2,A}A^{\alpha+\beta}\alpha!^\theta\beta!^\theta\langle\xi\rangle_h^{1-\alpha}\langle x\rangle^{-\sigma}. \tag{3.10}$$

- Conjugation of $ia_2(t,x,u,D)$: for $N \in \mathbb{N}$ such that $2 - N(2\sigma - 1) \leq 0$, Theorem 2.9 and (3.9) give

$$e^{\tilde{\Lambda}}(x,D) \circ ia_2(t,x,u,D) \circ {}^R\{e^{-\tilde{\Lambda}}(x,D)\} = ia_2(t,x,u,D) + \mathrm{op}\underbrace{\left(\sum_{1 \leq \alpha+\beta < N} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \{\partial_\xi^\beta e^{\tilde{\Lambda}} D_x^\beta ia_2 D_x^\alpha e^{-\tilde{\Lambda}}\} \right)}_{=:(ia_2)_N} + \tilde{r}_0.$$

Composing with the Neumann series and using the fact that $\partial_x \lambda_1$ has order -1 we get

$$\begin{aligned} e^{\tilde{\Lambda}}(x,D) \circ ia_2(t,x,u,D) \circ \{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \mathrm{op}(ia_2 + (ia_2)_N + \tilde{r}_0 + \tilde{r}) \circ \mathrm{op}(1 - i\partial_\xi \partial_x \lambda_2 + q_{-2}) \\ &= ia_2(t,x,u,D) + \mathrm{op}((ia_2)_N + a_2 \circ \partial_\xi \partial_x \lambda_2 - i(ia_2)_N \circ \partial_\xi \partial_x \lambda_2 + \tilde{r}_0) \\ &= ia_2(t,x,u,D) + \mathrm{op}(\underbrace{(ia_2)_N - i(ia_2)_N \partial_\xi \partial_x \lambda_2}_{=:(ia_2)_{\tilde{\Lambda}}} + a_2 \partial_\xi \partial_x \lambda_2 + \tilde{r}_0). \end{aligned}$$

Moreover, in view of (3.9), we have the following estimates: for every $A > 0$ there exists $C_{\tilde{\Lambda},\Omega,A} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta (ia_2)_{\tilde{\Lambda}}(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,A} A^{\alpha+\beta} \alpha!^\theta \beta!^\theta \langle \xi \rangle_h^{2-(2\sigma-1)-\alpha} \langle x \rangle^{-\sigma}, \tag{3.11}$$

for every $u \in \Omega$.

- Conjugation of $ia_1(t,x,u,D)$: working as in the previous conjugation, we get

$$\begin{aligned} e^{\tilde{\Lambda}}(x,D) \circ (ia_1)(t,x,u,D) \circ \{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \mathrm{op}(ia_1 + (ia_1)_{\tilde{\Lambda}} + r_1) \circ \sum_{j \geq 0} (-r(x,D))^j \\ &= ia_1(t,x,u,D) + \mathrm{op}((ia_1)_{\tilde{\Lambda}} + \tilde{r}_0), \end{aligned}$$

where we have the following estimates: for every $A > 0$ there exists $C_{\tilde{\Lambda},\Omega,A} > 0$ such that for every $u \in \Omega$:

$$|\partial_\xi^\alpha \partial_x^\beta (ia_1)_{\tilde{\Lambda}}(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,A} A^{\alpha+\beta} \alpha!^\theta \beta!^\theta \langle \xi \rangle_h^{2(1-\sigma)-\alpha} \langle x \rangle^{-\sigma/2}. \tag{3.12}$$

- Conjugation of $ia_0(t,x,u)$: $e^{\tilde{\Lambda}}(x,D) \circ (ia_0)(t,x,u) \circ \{e^{\tilde{\Lambda}}(x,D)\}^{-1} = \tilde{r}_0$.

Gathering all the previous computations we get (omitting (t,x,u,D) in the notation)

$$\begin{aligned} e^{\tilde{\Lambda}}(x,D)(iP_u)\{e^{\tilde{\Lambda}}(x,D)\}^{-1} &= \partial_t + ia_3(t,D) + \mathrm{op}\left(-\partial_\xi a_3 \partial_x \lambda_2 - \partial_\xi a_3 \partial_x \lambda_1 + id_1\right) \\ &+ ia_2(t,x,u,D) + \mathrm{op}((ia_2)_{\tilde{\Lambda}} + a_2 \partial_\xi \partial_x \lambda_2) + ia_1(t,x,u,D) + \mathrm{op}((ia_1)_{\tilde{\Lambda}} + \tilde{r}_0), \end{aligned}$$

where $d_1, (ia_2)_{\tilde{\Lambda}}$ and $(ia_1)_{\tilde{\Lambda}}$ satisfy the estimates (3.10), (3.11), (3.12) for every $u \in \Omega$.

3.3.2. Conjugation by $e^{\Lambda_{\rho',k}}(t,D)$, with $\Lambda_{\rho',k}(t,\xi) = \rho' \langle \xi \rangle_h^{\frac{1}{\theta}} + k(T-t)\langle \xi \rangle_h^{2(1-\sigma)}$

- Conjugation of ∂_t: $e^{\Lambda_{\rho',k}}(t,D) \circ \partial_t \circ e^{-\Lambda_{\rho',k}}(t,D) = \partial_t + k\langle D \rangle_h^{2(1-\sigma)}$.
- Conjugation of $ia_3(t,D)$: since a_3 does not depend on x, we simply have

$$e^{\Lambda_{\rho',k}}(t,D) \circ ia_3(t,D) \circ e^{-\Lambda_{\rho',k}}(t,D) = ia_3(t,D).$$

- Conjugation of $\operatorname{op}\{ia_2 - \partial_\xi a_3 \partial_x \lambda_2\}$:

$$
\begin{aligned}
e^{\Lambda_{\rho',k}}(t,D) \circ \operatorname{op}(ia_2 - \partial_\xi a_3 \partial_x \lambda_2) \circ e^{-\Lambda_{\rho',k}}(t,D) &= ia_2(t,x,u,D) \\
&\quad - \operatorname{op}(\partial_\xi a_3 \partial_x \lambda_2) + (b_{2,\rho',k} + \tilde{r}_0)(t,x,u,D)
\end{aligned}
$$

where $b_{2,\rho',k}$ satisfies: for any $A > 0$ there exists $C_{\lambda_2,\Omega,\rho',k,A} > 0$ such that (for every $u \in \Omega$)

$$
|\partial_\xi^\alpha \partial_x^\beta b_{2,\rho',k}(t,x,u,\xi)| \leq C_{\lambda_2,\Omega,\rho',k,A} A^{\alpha+\beta} (\alpha!\beta!)^\theta \langle \xi \rangle_h^{2-(1-\frac{1}{\theta})-\alpha} \langle x \rangle^{-\sigma}. \tag{3.13}
$$

- Conjugation of $(ia_2)_{\tilde{\Lambda}}(t,x,u,D)$:

$$
e^{\Lambda_{\rho',k}}(t,D) \circ (ia_2)_{\tilde{\Lambda}}(t,x,u,D) \circ e^{-\Lambda_{\rho',k}}(t,D) = \{(ia_2)_{\rho',k,\tilde{\Lambda}} + \tilde{r}_0\}(t,x,u,D),
$$

where $(ia_2)_{\rho',k,\tilde{\Lambda}}$ satisfies: for any $A > 0$ there exists $C_{\tilde{\Lambda},\Omega,\rho',k,A} > 0$ such that (for every $u \in \Omega$)

$$
|\partial_\xi^\alpha \partial_x^\beta (ia_2)_{\rho',k,\tilde{\Lambda}}(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,\rho',k,A} A^{\alpha+\beta} (\alpha!\beta!)^\theta \langle \xi \rangle_h^{2-(2\sigma-1)-\alpha} \langle x \rangle^{-\sigma}. \tag{3.14}
$$

- Conjugation of $\operatorname{op}\{ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2\}$: we have

$$
\begin{aligned}
&e^{\Lambda_{\rho',k}}(t,D) \circ \operatorname{op}(ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2) \circ e^{-\Lambda_{\rho',k}}(t,D) \\
&\qquad = \operatorname{op}(ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2 + b_{1,\rho',k} + \tilde{r}_0),
\end{aligned}
$$

where $b_{1,\rho',k}$ satisfies: for any $A > 0$ there exists $C_{\tilde{\Lambda},\Omega,\rho',k,A} > 0$ such that (for every $u \in \Omega$)

$$
|\partial_\xi^\alpha \partial_x^\beta b_{1,\rho',k}(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,\rho',k,A} A^{\alpha+\beta} (\alpha!\beta!)^\theta \langle \xi \rangle_h^{1-(1-\frac{1}{\theta})-\alpha} \langle x \rangle^{-\sigma/2}. \tag{3.15}
$$

- Conjugation of $(ia_1)_{\tilde{\Lambda}}(t,x,u,D)$:

$$
e^{\Lambda_{\rho',k}}(t,D) \circ (ia_1)_{\tilde{\Lambda}}(t,x,u,D) \circ e^{-\Lambda_{\rho',k}}(t,D) = \{(ia_1)_{\rho',k,\tilde{\Lambda}} + \tilde{r}_0\}(t,x,u,D),
$$

where $(ia_1)_{\rho',k,\tilde{\Lambda}}$ satisfies: for any $A > 0$ there exists $C_{\tilde{\Lambda},\Omega,\rho',k,A} > 0$ such that (for every $u \in \Omega$)

$$
|\partial_\xi^\alpha \partial_x^\beta (ia_1)_{\rho',k,\tilde{\Lambda}}(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,\rho',k,A} A^{\alpha+\beta} (\alpha!\beta!)^\theta \langle \xi \rangle_h^{2(1-\sigma)-\alpha} \langle x \rangle^{-\sigma/2}. \tag{3.16}
$$

Finally, gathering all the previous computations we obtain the following expression for the conjugated operator (provided that the parameter h is sufficiently large)

$$
\begin{aligned}
Q_{\tilde{\Lambda},k,\rho'}(iP_u)Q^{-1}_{\tilde{\Lambda},k,\rho'} &= \partial_t + k\langle D \rangle_h^{2(1-\sigma)} + ia_3(t,D) \\
&\quad + \operatorname{op}(ia_2 - \partial_\xi a_3 \partial_x \lambda_2 + b_{2,\rho',k} + (ia_2)_{\rho',k,\tilde{\Lambda}}) \\
&\quad + \operatorname{op}(ia_1 - \partial_\xi a_3 \partial_x \lambda_1 + id_1 + a_2 \partial_\xi \partial_x \lambda_2 + b_{1,\rho',k} + (ia_1)_{\rho',k,\tilde{\Lambda}}) \\
&\quad + \tilde{r}_0(t,x,u,D),
\end{aligned} \tag{3.17}
$$

where $b_{2,\rho',k}$ satisfies (3.13), $(ia_2)_{\rho',k,\tilde{\Lambda}}$ satisfies (3.14), $b_{1,\rho',k}$ satisfies (3.15), $(ia_1)_{\rho',k,\tilde{\Lambda}}$ satisfies (3.16), $\tilde{r}_0$ is a projective symbol of order zero satisfying uniform estimates with respect to $u \in \Omega$.

3.4. Proof of Theorem 3.1

This Subsection is devoted to the proof of Theorem 3.1. First of all we need some estimates from below for the terms appearing in the operator (3.17) in order to apply to these terms Fefferman-Phong and sharp Gårding inequalities. Let us start with the terms $\partial_\xi a_3(t,\xi)\partial_x\lambda_j(x,\xi) = 3a_3(t)\xi^2\partial_x\lambda_j(x,\xi)$, $j=1,2$.
For $|\xi|>2h$, by (3.4) and (3.5) we have

$$\begin{aligned}-\partial_\xi a_3\partial_x\lambda_2(x,\xi) &= 3M_2|a_3(t)|\xi^2\langle x\rangle^{-\sigma}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\\ &= 3M_2|a_3(t)|\xi^2\langle x\rangle^{-\sigma} - 3M_2|a_3(t)|\xi^2\langle x\rangle^{-\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right],\\ -\partial_\xi a_3\partial_x\lambda_1(x,\xi) &= 3M_1|a_3(t)|\xi^2\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\\ &= 3M_1|a_3(t)|\xi^2\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}} - 3M_1|a_3(t)|\xi^2\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right].\end{aligned}$$

Since $\langle x\rangle \geq \frac{1}{2}\langle\xi\rangle_h^2$ on the support of $(1-\psi)(\langle x\rangle\langle\xi\rangle_h^{-2})$, we have

$$-3M_2|a_3(t)|\xi^2\langle x\rangle^{-\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right] \geq -2^\sigma 3C'M_2\langle\xi\rangle_h^{2(1-\sigma)},$$

and

$$-3M_1|a_3(t)|\xi^2\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right] \geq -2^{\frac{\sigma}{2}}3C'M_1\langle\xi\rangle_h^{1-\sigma},$$

where $C' = \sup_{t\in[0,T]}|a_3(t)|$. In this way we may write ($|\xi|>2h$)

$$Q_{\tilde\Lambda,k,\rho'}\circ(iP_u)\circ Q^{-1}_{\tilde\Lambda,k,\rho'} = \partial_t + ia_3(t)D_x^3 + \tilde a_2(t,x,u,D) + \tilde a_1(t,x,u,D) + \tilde a_{2(1-\sigma)}(t,x,D) + r_0(t,x,u,D),$$

where r_0 is an operator of order 0 and

$$\begin{aligned}Re\,\tilde a_2 &= -Im\,a_2 + 3M_2|a_3(t)|\xi^2\langle x\rangle^{-\sigma} + Re\,b_{2,\rho',k} + Re\,(ia_2)_{\rho',k,\tilde\Lambda},\\ Im\,\tilde a_2 &= Re\,a_2 + Im\,b_{2,\rho',k} + Im\,(ia_2)_{\rho',k,\tilde\Lambda},\\ Re\,\tilde a_1 &= -Im\,a_1 + 3|a_3(t)|\xi^2M_1\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}} + Re\,a_2\partial_\xi\partial_x\lambda_2 + Re\,b_{1,\rho',k} + Re\,(ia_1)_{\rho',k,\tilde\Lambda},\\ \tilde a_{2(1-\sigma)} &= k\langle\xi\rangle_h^{2(1-\sigma)} - 3|a_3(t)|\xi^2M_2\langle x\rangle^{-\sigma}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right] - 3|a_3(t)|\xi^2M_1\langle\xi\rangle_h^{-1}\langle x\rangle^{-\frac{\sigma}{2}}\left[1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\right].\end{aligned}$$

Now we decompose $iIm\,\tilde a_2$ into its Hermitian and anti-Hermitian part:

$$iIm\,\tilde a_2 = \frac{iIm\,\tilde a_2 + (iIm\,\tilde a_2)^*}{2} + \frac{iIm\,\tilde a_2 - (iIm\,\tilde a_2)^*}{2} = H_{Im\,\tilde a_2} + A_{Im\,\tilde a_2};$$

we have that $2Re\,\langle A_{Im\,\tilde a_2}u,u\rangle = 0$, while $H_{Im\,\tilde a_2}$ has symbol

$$\sum_{\alpha\geq1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha Im\,\tilde a_2 = \underbrace{\sum_{\alpha\geq1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha Re\,a_2}_{=:c(t,x,u,\xi)} + \underbrace{\sum_{\alpha\geq1}\frac{i}{2\alpha!}\partial_\xi^\alpha D_x^\alpha\{Im\,b_{2,\rho',k} + Im\,(ia_2)_{\rho',k,\tilde\Lambda}\}}_{=:e(t,x,u,\xi)}.$$

The hypothesis on a_2 implies

$$|\partial_\xi^\alpha \partial_x^\beta c(t,x,u,\xi)| \leq C_{\Omega,A} A^{\alpha+\beta}(\alpha!\beta!)^\theta \langle\xi\rangle_h^{1-\alpha}\langle x\rangle^{-\sigma},$$

whereas from (3.13), (3.14) and using the fact that $2(1-\sigma) \leq \frac{1}{\theta}$ we obtain

$$|\partial_\xi^\alpha \partial_x^\beta e(t,x,u,\xi)| \leq C_{\tilde{\Lambda},\Omega,\rho',k,A} A^{\alpha+\beta}(\alpha!\beta!)^\theta \langle\xi\rangle_h^{\frac{1}{\theta}}\langle x\rangle^{-\sigma}.$$

We are ready to get the desired estimates from below. Using the above decomposition we get

$$\begin{aligned} e^\Lambda \circ (iP_u) \circ \{e^\Lambda\}^{-1} = \partial_t + ia_3(t)D_x^3 + Re\, \tilde{a}_2(t,x,u,D) + A_{Im\, \tilde{a}_2}(t,x,u,D) \\ + (\tilde{a}_1 + c + e)(t,x,u,D) + \tilde{a}_{2(1-\sigma)}(t,x,D) + r_0(t,x,u,D). \end{aligned}$$

Note that $\langle\xi\rangle_h^2 \leq 2\xi^2$ provided that $|\xi| > 2h$. In the next we shall fix $A = 1$ in the estimates and we shall omit the dependence on A in the constants. Estimating the terms of order 2 we get

$$Re\, \tilde{a}_2 \geq \left(M_2 \frac{3C_{a_3}}{2} - C_\Omega - C_{\lambda_2,\Omega,\rho',k} h^{-(1-\frac{1}{\theta})} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(2\sigma-1)} \right) \langle\xi\rangle_h^2 \langle x\rangle^{-\sigma},$$

where C_{a_3} is the constant appearing in the statement of Theorem 1.2. For the terms of order 1 we obtain

$$Re\,(\tilde{a}_1 + c + e) \geq \left(M_1 \frac{3C_{a_3}}{2} - C_\Omega - C_{\Omega,\lambda_2} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(1-\frac{1}{\theta})} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(2\sigma-1)} \right) \langle\xi\rangle_h \langle x\rangle^{-\frac{\sigma}{2}}.$$

Finally, for the terms of order $\leq 2(1-\sigma)$ we have

$$\begin{aligned} Re\, \tilde{a}_{2(1-\sigma)} &\geq k\langle\xi\rangle_h^{2(1-\sigma)} - 2^\sigma 3C_{a_3} M_2 \langle\xi\rangle_h^{2(1-\sigma)} - 2^{\frac{\sigma}{2}} 3C_{a_3} M_1 \langle\xi\rangle_h^{1-\sigma} \\ &\geq \left(k - 2^\sigma 3C_{a_3} M_2 - 2^{\frac{\sigma}{2}} 3C_{a_3} M_1 h^{-(1-\sigma)} \right) \langle\xi\rangle_h^{2(1-\sigma)}. \end{aligned} \tag{3.18}$$

From the previous lower bound estimates we obtain the following proposition.

Proposition 3.8. *There exist constants $M_2, M_1, k > 0$ and $h_0 = h_0(k, M_2, M_1, T, \Omega, \rho') > 0$ such that for every $h \geq h_0$ the Cauchy problem associated to the conjugated operator (3.17) is well-posed in $H^m(\mathbb{R})$ for every $u \in \Omega$. More precisely, for any Cauchy data $\tilde{f} \in C([0,T]; H^m(\mathbb{R}))$ and $\tilde{g} \in H^m(\mathbb{R})$, there exists a unique solution $w \in C([0,T]; H^m(\mathbb{R})) \cap C^1([0,T]; H^{m-3}(\mathbb{R}))$ such that the following energy estimate holds: there exists a constant $C_{\Omega,\rho',T} > 0$ depending on Ω, $\rho' > 0$ and $T > 0$ such that*

$$\|w(t)\|_{H^m}^2 \leq C_{\Omega,\rho',T} \left(\|\tilde{g}\|_{H^m}^2 + \int_0^t \|\tilde{f}(\tau)\|_{H^m}^2 d\tau \right), \quad t \in [0,T].$$

Proof. First we take $M_2 > 0$ large in order to get

$$M_2 \frac{3C_{a_3}}{2} - C_\Omega > 0, \tag{3.19}$$

then we set $M_1 = M_1(M_2) > 0$ in such a way that

$$M_1 \frac{3C_{a_3}}{2} - C_\Omega - C_{\Omega,\lambda_2} > 0. \tag{3.20}$$

Thereafter we choose $k = k(M_2) > 0$ such that

$$k - 2^{\sigma} 3C_{a_3} M_2 > 0. \tag{3.21}$$

Making the parameter h_0 large enough, we obtain

$$M_2 \frac{3C_{a_3}}{2} - C_\Omega - C_{\lambda_2,\Omega,\rho',k} h^{-(1-\frac{1}{\theta})} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(2\sigma-1)} \geq 0,$$
$$M_1 \frac{3C_{a_3}}{2} - C_\Omega - C_{\Omega,\lambda_2} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(1-\frac{1}{\theta})} - C_{\tilde{\Lambda},\Omega,\rho',k} h^{-(2\sigma-1)} \geq 0,$$
$$k - 2^{\sigma} 3C_{a_3} M_2 - 2^{\frac{\sigma}{2}} 3C_{a_3} M_1 h^{-(1-\sigma)} \geq 0.$$

With these choices $Re\, \tilde{a}_2(t, x, u, \xi), Re\, (\tilde{a}_1 + c + e)(t, x, u, \xi), Re\, \tilde{a}_{2(1-\sigma)}(t, x, \xi)$ are non-negative for large $|\xi|$. Applying the Fefferman-Phong inequality, cf. [16], to $Re\, \tilde{a}_2$ we have

$$Re\langle Re\, \tilde{a}_2(t, x, u, D)w, w\rangle_{L^2} \geq -C\|w\|^2_{L^2}, \quad w \in \mathscr{S}(\mathbb{R}).$$

By the sharp Gårding inequality, cf. [28, Theorem 4.4], we also obtain that

$$Re\langle(\tilde{a}_1 + c + e)(t, x, u, D)w, w\rangle_{L^2} \geq -C\|w\|^2_{L^2}, \quad w \in \mathscr{S}(\mathbb{R})$$

and

$$Re\langle \tilde{a}_2(1-\sigma)(t, x, D)w, w\rangle_{L^2} \geq -C\|w\|^2_{L^2}, \quad w \in \mathscr{S}(\mathbb{R}).$$

The constant $C > 0$ that we just wrote in the above inner product estimates depends on a finite number of seminorms of the symbols, in this way we have that C depends on $\Omega, \rho', T, \tilde{\Lambda}$ and k. As a consequence we get the energy estimate

$$\frac{d}{dt}\|w(t)\|^2_{L^2} \leq C_{\Omega,\rho',T}(\|w(t)\|^2_{L^2} + \|(iP)_\Lambda w(t)\|^2_{L^2}),$$

which gives the well-posedness in $H^m(\mathbb{R})$. □

Remark 3.9. We underline that the assumption $|a_3(t)| \geq C_{a_3} > 0, \forall t \in [0, T]$ is crucial in the choice of M_2, M_1. If a_3 may vanish for some $t \in [0, T]$, then some Levi type conditions are needed on a_2, a_1 to let the choice of M_2, M_1 work, see [4].

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Given $m \in \mathbb{R}$ and $\theta > 1$, take $f \in C([0, T], H^m_{\rho;\theta}(\mathbb{R}))$ and $g \in H^m_{\rho;\theta}(\mathbb{R})$ for some $\rho > 0$. Let $M_2, M_1, k, h_0 > 0$ so that Proposition 3.8 holds. Since $\tilde{\Lambda}$ and $k(T-t)\langle\cdot\rangle_h^{2(1-\sigma)}$ have order $2(1-\sigma) < \frac{1}{\theta}$, we have by Proposition 2.8 that

$$f_{\tilde{\Lambda},k,\rho'} := Q_{\tilde{\Lambda},k,\rho'}(t, x, D)f \in C([0, T]; H^m(\mathbb{R}))$$
$$g_{\tilde{\Lambda},k,\rho'} := Q_{\tilde{\Lambda},k,\rho'}(0, x, D)g \in H^m(\mathbb{R}),$$

provided that $\rho' < \rho$. Proposition 3.8 ensures that the Cauchy problem associated with the operator in (3.17), call it $P_{\tilde{\Lambda},k,\rho',u}$, is well posed in Sobolev spaces $H^m(\mathbb{R})$. Hence, there exists a unique $w \in C([0, T]; H^m(\mathbb{R}))$ satisfying

$$\begin{cases} P_{\tilde{\Lambda},k,\rho',u} w(t,x) = f_{\tilde{\Lambda},k,\rho'}(t,x), \\ w(0,x) = g_{\tilde{\Lambda},k,\rho'}(x), \end{cases}$$

and

$$\|w(t)\|^2_{H^m} \leq C_{\Omega,\rho',T} \left(\|g_{\tilde{\Lambda},k,\rho'}\|^2_{H^m} + \int_0^t \|f_{\tilde{\Lambda},k,\rho'}(\tau)\|^2_{H^m} d\tau \right), \quad t \in [0,T]. \tag{3.22}$$

Setting $v = \{Q_{\tilde{\Lambda},k,\rho'}(t,x,D)\}^{-1} w$ we obtain a solution for the original problem (3.1). Let us now study which space the solution v belongs to. We have

$$\begin{aligned} v(t,x) &= \{Q_{\tilde{\Lambda},k,\rho'}(t,x,D)\}^{-1} w(t,x) \\ &= {}^R\{e^{-\tilde{\Lambda}}\}(x,D) \sum_j (-r(x,D))^j e^{-k(T-t)\langle D\rangle_h^{2(1-\sigma)}} e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} w(t,x), \quad w \in H^m(\mathbb{R}). \end{aligned}$$

Since $e^{-\rho'\langle D\rangle_h^{\frac{1}{\theta}}} w =: v_1 \in H^m_{\rho';\theta}(\mathbb{R})$, we get

$$v(t,x) = {}^R\{e^{-\tilde{\Lambda}}(x,D)\} \sum_j (-r(x,D))^j e^{-k(T-t)\langle D\rangle_h^{2(1-\sigma)}} v_1, \quad v_1 \in H^m_{\rho';\theta}(\mathbb{R}),$$

but $e^{-k(T-t)\langle D\rangle_h^{2(1-\sigma)}} v_1 = \underbrace{e^{-k(T-t)\langle D\rangle_h^{2(1-\sigma)}} e^{-\delta_1\langle D\rangle_h^{\frac{1}{\theta}}}}_{\text{order zero}} e^{\delta_1\langle D\rangle_h^{\frac{1}{\theta}}} v_1 =: v_2 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R})$, for every $\delta_1 > 0$, so

$$v(t,x) = {}^R\{e^{-\tilde{\Lambda}}(x,D)\} \underbrace{\sum_j (-r(x,D))^j}_{\text{order zero}} v_2 = {}^R\{e^{-\tilde{\Lambda}}(x,D)\} v_3, \quad v_3 \in H^m_{\rho'-\delta_1;\theta}(\mathbb{R}).$$

By Proposition 2.8, ${}^R\{e^{-\tilde{\Lambda}}(x,D)\}$ maps $H^m_{\rho;\theta}$ into $H^m_{\rho-\delta_2;\theta}$, for every $\delta_2 > 0$, hence we finally obtain ($\delta = \delta_1 + \delta_2$) that $v(t,\cdot) \in H^m_{\rho'-\delta;\theta}(\mathbb{R})$ for all $\delta > 0, t \in [0,T]$. We remark that the solution exhibits an arbitrarily small loss δ in the coefficient of the exponential weight: the solution is then less regular than the Cauchy data. Moreover, denoting $\rho'' = \rho' - \delta$, from (3.22) we obtain that v satisfies the following energy estimate

$$\begin{aligned} \|v(t)\|^2_{H^m_{\rho'';\theta}} &= \|\{e^{\Lambda}(t,\cdot,D)\}^{-1} w(t)\|^2_{H^m_{\rho'';\theta}} \leq C_{\rho',T} \|w(t)\|^2_{H^m} \\ &\leq C_{\rho,T} C_{\Omega,\rho',T} \left(\|g_{\tilde{\Lambda},k,\rho'}\|^2_{H^m} + \int_0^t \|f_{\tilde{\Lambda},k,\rho'}(\tau)\|^2_{H^m} d\tau \right) \\ &\leq C_{\Omega,\rho',T} \left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau)\|^2_{H^m_{\rho;\theta}} d\tau \right), \quad t \in [0,T]. \end{aligned}$$

Finally, let us notice that if the data are valued in $H^m_{\rho;\theta}(\mathbb{R})$ for every $\rho > 0$, then the solution belongs to $H^m_{\rho'';\theta}(\mathbb{R})$ for every $\rho'' \in (0,\rho)$, that is $v \in C([0,T]; H^\infty_\theta(\mathbb{R}))$. □

The argument of the proof of Theorem 3.1, suitably simplified, provides a well-posedness result in projective Gevrey-Sobolev spaces also for linear 3-evolution equations, that is when the coefficients of the operator do not depend on u. Since also this result is new in the literature we state it here below as a separate result.

Corollary 3.10. *Let P be a linear differential operator of the form (1.7) and assume that $a_3 \in C([0,T];\mathbb{R})$ is such that $|a_3(t)| \geq C_{a_3} > 0$ for all $t \in [0,T]$ and for some constant C_{a_3}. Let moreover $\sigma \in \left(\frac{1}{2},1\right)$ and $\theta_0 < \frac{1}{2(1-\sigma)}$ such that for $j = 0,1,2$ the coefficients a_j satisfy the following assumptions: for every $A > 0$ there exists $C_A > 0$ such that*

$$|\partial_x^\beta a_j(t,x)| \leq C_A A^\beta \beta!^{\theta_0} \langle x \rangle^{-\frac{j\sigma}{2}},$$

for every $x \in \mathbb{R}, t \in [0,T]$ and $\beta \in \mathbb{N}_0$. Then for every $m \in \mathbb{R}$, $\rho > 0$, $\theta \in \left[\theta_0, \frac{1}{2(1-\sigma)}\right)$ and $f \in C([0,T];H^m_{\rho;\theta}(\mathbb{R}))$, $g \in H^m_{\rho;\theta}(\mathbb{R})$, there exists a unique solution $v \in C^1([0,T];H^m_{\rho-\delta;\theta}(\mathbb{R}))$ for every $\delta \in (0,\rho)$ of the Cauchy problem (1.6) and the following energy estimate is satisfied:

$$\|v(t,\cdot)\|^2_{H^m_{\rho-\delta;\theta}} \leq C_{\rho,T}\left(\|g\|^2_{H^m_{\rho;\theta}} + \int_0^t \|f(\tau,\cdot)\|^2_{H^m_{\rho;\theta}}\,d\tau\right) \qquad \forall t \in [0,T], \tag{3.23}$$

for some positive constant $C_{\rho,T}$. Moreover, if $f \in C([0,T],H^\infty_\theta(\mathbb{R}))$ and $g \in H^\infty_\theta(\mathbb{R})$, then v belongs to $C^1([0,T];H^\infty_\theta(\mathbb{R}))$.

4. The quasilinear problem

In this section we consider the quasilinear Cauchy problem (1.13) and prove Theorem 1.2. First of all, by Theorem 2.3, it is easy to verify that the space

$$X_T := C^1([0,T];H^\infty_\theta(\mathbb{R}))$$

is a tame Fréchet space endowed with the family of seminorms

$$\|u\|_k = \sup_{t\in[0,T]} \left\{|u(t,\cdot)|_k + |D_t u(t,\cdot)|_k\right\}, \qquad k \in \mathbb{N}_0,$$

for every $\theta > 1$. Let us consider, for every $u \in X_T$, the map

$$\begin{aligned} J(u) := \; & u(t,x) - g(x) + i\int_0^t a_3(s)D_x^3 u(s,x)ds + i\int_0^t a_2(s,x,u(s,x))D_x^2 u(s,x)ds \\ & + i\int_0^t a_1(s,x,u(s,x))D_x u(s,x)ds + i\int_0^t a_0(s,x,u(s,x))u(s,x)ds \\ & - i\int_0^t f(s,x)ds. \end{aligned} \tag{4.1}$$

Remark 4.1. By Lemma 3.7, we have $a_j(t,x,u(t,x))\xi^j \in C([0,T];\Gamma^j_\theta(\mathbb{R}^2))$, then from Proposition 2.7 we conclude that $a_j(s,x,u(s,x))D_x^j u(s,x) \in C([0,T];H^\infty_\theta(\mathbb{R}))$. This implies that the map J maps X_T into itself.

As anticipated in the introduction we shall prove the existence of a unique solution $u \in C^1([0,T^*];H^\infty_\theta(\mathbb{R}))$ for some $T^* \in (0,T]$ of the Cauchy problem (1.13) by showing the existence of a unique solution $u \in C^1([0,T^*];H^\infty_\theta(\mathbb{R}))$ of the integral equation

$$J(u) \equiv 0 \text{ in } [0, T^*] \times \mathbb{R}. \tag{4.2}$$

This will be achieved using Theorem 2.5. It is not difficult to prove that J is tame together with all its derivatives. To apply the Nash-Moser Theorem we only need to prove that the equation $DJ(u)v = h$ has a unique solution $v := S(u, h) \in X_T$ for all $u, h \in X_T$ and that the map

$$S: \ X_T \times X_T \to X_T : (u, h) \to v = S(u, h) \tag{4.3}$$

is smooth tame, where $DJ(u)v$ stands for the derivative of J at u in the direction v.

Remark 4.2. We claim that $\lim_{\varepsilon\to 0} a_j(s, x, u(s, x) - \varepsilon v(s, x)) = a_j(s, x, u(s, x))$, where the limit is taken with respect to the topology of $C([0, T]; H^\infty_\theta(\mathbb{R}))$. Indeed, first we write

$$\begin{aligned} a_j(s, x, u + \varepsilon v) - a_j(s, x, u) &= \int_0^\varepsilon \frac{d}{dr}\{a_j(s, x, u + rv)\}dr \\ &= v \int_0^\varepsilon \partial_w a_j(s, x, u + rv)dr \\ &= v \cdot \sigma_\varepsilon(s, x). \end{aligned}$$

Observe that $\sigma_\varepsilon \in C([0, T]; \tilde{\Gamma}^0_\theta(\mathbb{R}^2))$. Therefore, by Proposition 2.7, we get $\sigma_\varepsilon v \in C([0, T]; H^\infty_\theta(\mathbb{R}))$. Moreover, since the norms $|\sigma(s)|_A$ are bounded by a constant of the form $\varepsilon C_{\Omega,A}$, Ω being a bounded neighborhood of u, we are able to conclude

$$|\sigma_\varepsilon(s, x)v|_k \to 0, \quad \text{as } \varepsilon \to 0,$$

for every $k \in \mathbb{N}_0$, which finalizes the proof of our claim. In the same manner one gets

$$\lim_{\varepsilon\to 0} \frac{a_j(s, x, u(s, x) + \varepsilon v(s, x)) - a_j(s, x, u(s, x))}{\varepsilon} = \partial_w a_j(s, x, u(s, x))v(s, x)$$

in $C([0, T], H^\infty_\theta(\mathbb{R}))$. We shall use extensively these two types of limits in the sequel.

Proposition 4.3. *For every $u, h \in X_T$, there exists a unique $v \in X_T$ solution of the equation $DJ(u)v = h$, and the function v satisfies for every $k \in \mathbb{N}$ the following estimate:*

$$|v(t, \cdot)|^2_k \le C_{\Omega,k,T} \left(|h(0)|^2_{k+1} + \int_0^t |D_t h(\tau, \cdot)|^2_{k+1}\, d\tau \right), \quad \forall t \in [0, T], \tag{4.4}$$

where Ω stands for some bounded open neighborhood of u.

Proof. By the definition (4.1) of the map J, let us compute the derivative of J, for $u, v \in X_T$:

$$\begin{aligned} DJ(u)v = &\lim_{\varepsilon\to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\ = &\lim_{\varepsilon\to 0} \Big\{ v + i \int_0^t a_3(s) D^3_x v(s)ds + i \sum_{j=0}^{2} \int_0^t \frac{a_j(s, x, u + \varepsilon v) - a_j(s, x, u)}{\varepsilon} D^j_x u(s)ds \end{aligned}$$

$$
\begin{aligned}
&+i\sum_{j=0}^{2}\int_0^t a_j(s,x,u+\varepsilon v)D_x^j v(s)ds\Big\} \\
&= v + i\int_0^t a_3(s)D_x^3 v(s)ds + i\sum_{j=0}^{2}\int_0^t (\partial_w a_j)(s,x,u)v(s)D_x^j u(s)ds \\
&+i\sum_{j=0}^{2}\int_0^t a_j(s,x,u)D_x^j v(s)ds \\
&= v + i\int_0^t a_3(s)D_x^3 v(s)ds + i\int_0^t a_2(s,x,u)D_x^2 v(s)ds + i\int_0^t a_1(s,x,u)D_x v(s)ds \\
&+i\int_0^t \underbrace{\Big(a_0(s,x,u) + \sum_{j=0}^{2}(\partial_w a_j)(s,x,u)D_x^j u\Big)}_{:=\ \tilde a_0(s,x,u)} v(s)ds =: J_{0,u,0}(v)
\end{aligned}
$$

where, given $u, g, f \in X_T$, the map $J_{g,u,f}: X_T \to X_T$ is defined by

$$
\begin{aligned}
J_{g,u,f}v :=\ & v(t,x) - g(x) + i\int_0^t a_3(s)D_x^3 v(s,x)ds + i\int_0^t a_2(s,x,u(s,x))D_x^2 v(s,x)ds \\
&+i\int_0^t a_1(s,x,u(s,x))D_x v(s,x)ds + i\int_0^t \tilde a_0(s,x,u(s,x))v(s,x)ds - i\int_0^t f(s,x)ds.
\end{aligned}
$$

Of course, v solves $J_{g,u,f}(v) \equiv 0$ if and only if it solves the linearized Cauchy problem

$$
\begin{cases}
\tilde P_u(D)v(t,x) = f(t,x) \\
v(0,x) = g(x),
\end{cases}
$$

where $\tilde P_u(D)$ is obtained from $P_u(D)$ substituting a_0 with $\tilde a_0$.
Writing

$$
J_{0,u,0}(v) - h = J_{0,u,0}(v) - h_0 - i\int_0^t D_t h(s,x)ds = J_{h_0,u,D_t h}(v)
$$

with $h_0 := h(0,x)$, we see that v is a solution of $DJ(u)v = h$ if and only if it is a solution of $J_{h_0,u,D_t h}(v) = 0$, or equivalently of the linearized Cauchy problem

$$
\begin{cases}
\tilde P_u(D)v(t,x) = D_t h(t,x) \\
v(0,x) = h_0(x).
\end{cases}
\tag{4.5}
$$

Summing up, the solutions to $DJ(u)v = h$ in X_T coincide with the solutions to (4.5).
The Cauchy problem (4.5) fulfills the assumptions of Theorem 3.1, indeed, on one hand the operators $P_u(D)$ and $\tilde P_u(D)$ have the same coefficients but for the terms of order 0 for which no decay assumptions are required; on the other hand, clearly $D_t h \in C([0,T]; H_\theta^\infty(\mathbb{R}))$ and $h_0 \in H_\theta^\infty(\mathbb{R})$. We obtain by Theorem 3.1

a unique solution $v \in C([0,T]; H^\infty_\theta(\mathbb{R}))$ of (4.5) which satisfies an energy estimate of the form (3.2) for every $\rho, \delta > 0$ with $0 < \delta < \rho$. Taking $\rho = k+1$ and $\delta = 1$ in (3.2), $k \in \mathbb{N}$, we obtain (4.4). □

Lemma 4.4. *The map S defined by (4.3) is smooth tame.*

Proof. We have to prove that S and its derivatives $D^m S$ are tame maps for any positive integer m. Let us first prove that S is tame. First of all, notice that if we take u in a bounded set $\Omega \subset X_T$, from (4.4) we get

$$\sup_{t\in[0,T]} |v(t,\cdot)|_k \le C_{\Omega,k,T} \|h\|_{k+1} \tag{4.6}$$

for every $k \in \mathbb{N}$ and for some $C_{\Omega,k,T} > 0$. Moreover, from the equation it follows that

$$\begin{aligned}|D_t v(t,\cdot)|_k &= \left| -a_3(t) D_x^3 v(t,\cdot) - \sum_{j=1}^{2} a_j(t,\cdot,u) D_x^j v(t,\cdot) + \tilde{a}_0(t,\cdot,u) v(t,\cdot) + D_t h(t,\cdot) \right|_k \\ &\le C(|v(t,\cdot)|_{k+1} + \|h\|_k)\end{aligned}$$

for some $C > 0$ depending on the set Ω and on the coefficients. Hence

$$\|S(u,h)\|_k = \sup_{t\in[0,T]} (|v(t,\cdot)|_k + |D_t v(t,\cdot)|_k) \le C_{\Omega,k,T} \|h\|_{k+1} \le C_{\Omega,k,T} \|(u,h)\|_{k+1} \tag{4.7}$$

for some (possibly larger) constant $C_{\Omega,k,T} > 0$, and so S is tame.
Let us now consider the first derivative of S, defined for $(u,h), (u_1,h_1) \in X_T \times X_T$ as

$$DS(u,h)(u_1,h_1) = \lim_{\varepsilon\to 0} \frac{S(u+\varepsilon u_1, h+\varepsilon h_1) - S(u,h)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{v_\varepsilon - v}{\varepsilon} = \lim_{\varepsilon\to 0} w_\varepsilon,$$

where $w_\varepsilon := \varepsilon^{-1}(v_\varepsilon - v)$ and $v_\varepsilon = S(u+\varepsilon u_1, h+\varepsilon h_1)$ is the solution of the Cauchy problem

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1}(D) v = D_t(h+\varepsilon h_1) \\ v(0,x) = h(0,x) + \varepsilon h_1(0,x). \end{cases} \tag{4.8}$$

Since v_ϵ, v solve the Cauchy problems (4.5) and (4.8) respectively, it is easy to check that the function w_ε satisfies

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1} w_\varepsilon = f_\varepsilon \\ w_\varepsilon(0,x) = h_1(0,x) \end{cases} \tag{4.9}$$

with (omitting (t,x) in the notation for brevity's sake)

$$f_\varepsilon := D_t h_1 - \frac{a_2(u+\varepsilon u_1) - a_2(u)}{\varepsilon} D_x^2 v - \frac{a_1(u+\varepsilon u_1) - a_1(u)}{\varepsilon} D_x v - \frac{\tilde{a}_0(u+\varepsilon u_1) - \tilde{a}_0(u)}{\varepsilon} v.$$

If we prove that the sequence $\{w_\varepsilon\}_\varepsilon$ is a Cauchy sequence in X_T, then we obtain that w_ε converges to some w in X_T; this function w, which is on one hand the first derivative of S, is on the other hand the solution to

$$\begin{cases} \tilde{P}_u(D) w = f_1 \\ w(0,x) = h_1(0,x) \end{cases}$$

with

$$f_1 := \lim_{\varepsilon\to 0} f_\varepsilon = D_t h_1 - \partial_w a_2(u) u_1 D_x^2 v - \partial_w a_1(u) u_1 D_x v - \partial_w \tilde{a}_0(u) u_1 v,$$

so, taking u in a bounded set Ω, by Theorem 3.1 it satisfies the energy estimate

$$|w(t,\cdot)|_k^2 \leq C_{\Omega,k,T}\Big(|h_1(0,\cdot)|_{k+1}^2 + \int_0^t |f_1(\tau,\cdot)|_{k+1}^2 d\tau\Big).$$

Now if we take u_1 in a bounded set Ω_1, by (4.6) we get

$$\begin{aligned} |w(t,\cdot)|_k &\leq C_{\Omega,k,T}(|h_1(0,\cdot)|_{k+1} + \sup_{t\in[0,T]} |f_1(t,\cdot)|_{k+1}) \\ &\leq C_{\Omega,\Omega_1,k,T} \sup_{t\in[0,T]} (|h_1(t,\cdot)|_{k+1} + |D_t h_1(t,\cdot)|_{k+1} + |v(t,\cdot)|_{k+2}) \\ &\leq C_{\Omega,\Omega_1,k,T}\,(\|h_1\|_{k+1} + \|h\|_{k+3}) \end{aligned}$$

for some positive constant $C_{\Omega,\Omega_1,k,T}$ depending on Ω, Ω_1, k, T and on the coefficients. Also

$$D_t w = -a_3(t) D_x^3 w - a_2(t,x,u) D_x^2 w - a_1(t,x,u) D_x w - \tilde{a}_0(t,x,u) w + f_1$$

satisfies a similar estimate, so the first derivative DS (coinciding with w) is tame.
Thus, we only need to prove that $\{w_\varepsilon\}_{\varepsilon\in[0,1]}$ is a Cauchy sequence in X_T to conclude that DS is a tame map.
To this aim, arguing as before, let us consider w_{ε_1} and w_{ε_2} solutions of the Cauchy problems

$$\tilde{P}_{u+\varepsilon_i u_1}(D) w_{\varepsilon_i} = f_{\varepsilon_i}, \qquad w_{\varepsilon_i}(0,x) = h_1(0,x), \quad i = 1,2;$$

then $w_{\varepsilon_1} - w_{\varepsilon_2}$ solves

$$\begin{cases} \tilde{P}_{u+\varepsilon_1 u_1}(D)(w_{\varepsilon_1} - w_{\varepsilon_2}) = f_{\varepsilon_1} - f_{\varepsilon_2} + f_{\epsilon_1,\epsilon_2} \\ (w_{\varepsilon_1} - w_{\varepsilon_2})(0,x) = 0 \end{cases}$$

with (omitting (t,x) in the notation)

$$\begin{aligned} f_{\epsilon_1,\epsilon_2} :&= \big(a_2(u+\varepsilon_2 u_1) - a_2(u+\varepsilon_1 u_1)\big) D_x^2 w_{\varepsilon_2} \\ &+ \big(a_1(u+\varepsilon_2 u_1) - a_1(u+\varepsilon_1 u_1)\big) D_x w_{\varepsilon_2} + \big(\tilde{a}_0(u+\varepsilon_2 u_1) - \tilde{a}_0(u+\varepsilon_1 u_1)\big) w_{\varepsilon_2} \end{aligned}$$

and the energy estimate (3.2) gives

$$|(w_{\varepsilon_1} - w_{\varepsilon_2})(t,\cdot)|_k \leq C_{\Omega,\Omega_1,k,T} \sup_{t\in[0,T]} \Big(|f_{\varepsilon_1}(t,\cdot) - f_{\varepsilon_2}(t,\cdot)|_{k+1} + |f_{\epsilon_1,\epsilon_2}(t,\cdot)|_{k+1}\Big). \tag{4.10}$$

By Lagrange theorem, there exist $\tilde{u}_j, j = 0,1,2$, between $u + \varepsilon_1 u_1$ and $u + \varepsilon_2 u_1$ such that, for all $t \in [0,T]$,

$$\begin{aligned} |f_{\epsilon_1,\epsilon_2}(t,\cdot)|_{k+1} &\leq |\varepsilon_1 - \varepsilon_2| \sup_{t\in[0,T]} \big(|\partial_w a_2(t,\cdot,\tilde{u}_2) u_1(t,\cdot) D_x^2 w_{\varepsilon_2}(t,\cdot)|_{k+1} \\ &\quad + |\partial_w a_1(t,\cdot,\tilde{u}_1) u_1(t,\cdot) D_x w_{\varepsilon_2}(t,\cdot)|_{k+1} + |\partial_w a_0(t,\cdot,\tilde{u}_0) u_1(t,\cdot) w_{\varepsilon_2}(t,\cdot)|_{k+1}\big) \\ &\leq |\varepsilon_1 - \varepsilon_2| C_{\Omega,\Omega_1,k,T} |w_{\varepsilon_2}|_{k+2} \end{aligned}$$

with $C_{\Omega,\Omega_1,k,T}$ independent of $\varepsilon_1, \varepsilon_2 \in [0,1]$, where we used the algebra property of H^∞_θ spaces: namely, we know that $H^m_{\rho,\theta}(\mathbb{R})$ is an algebra if $m > 1/2$, see for instance [12]. Hence, for every $f, g \in H^\infty_\theta$ we have $fg \in H^\infty_\theta$ and, taking $m > \frac{n}{2}$ we may write

$$\|fg\|_{H^0_{\rho;\theta}} \le \|fg\|_{H^m_{\rho,\theta}} \le C\|f\|_{H^m_{\rho;\theta}}\|g\|_{H^m_{\rho;\theta}} \le C_2\|f\|_{H^0_{\rho+\varepsilon;\theta}}\|g\|_{H^0_{\rho+\varepsilon;\theta}}.$$

From the energy inequality for the linearized problem we see that $|w_\varepsilon|_k$ is bounded with respect to $\varepsilon \in [0,1]$ for every $k \in \mathbb{N}_0$. Hence $f_{\epsilon_1,\epsilon_2} \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$ in the $H^\infty_\theta(\mathbb{R})$ topology. In the same manner one gets $f_{\varepsilon_1} - f_{\varepsilon_2} \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$.
This gives that $\{w_\varepsilon\}_\varepsilon$ is a Cauchy sequence in X_T and therefore we can conclude that DS is a tame map. To conclude the proof it is sufficient to repeat the previous computations in an inductive procedure similar to the one in the proof of [3, Theorem 1.3, Step 4]. □

We are now ready for the final step of this paper, that is the proof of Theorem 1.2.

Proof of Theorem 1.2. As described at the beginning of this section the existence of a unique local solution $u \in C^1([0,T^*]; H^\infty_\theta(\mathbb{R}))$ of the Cauchy problem (1.13) is equivalent to the existence of a unique solution $u \in C^1([0,T^*]; H^\infty_\theta(\mathbb{R}))$ of the equation

$$\begin{aligned} u(t,x) = g(x) &- i\int_0^t a_3(s)D_x^3u(s,x)ds - i\int_0^t a_2(s,x,u(s,x))D_x^2u(s,x)ds \\ &- i\int_0^t a_1(s,x,u(s,x))D_xu(s,x)ds - i\int_0^t a_0(s,x,u(s,x))u(s,x)ds \\ &+ i\int_0^t f(s,x)ds. \end{aligned} \tag{4.11}$$

Equation (4.11) provides the first order Taylor expansion of u:

$$\begin{aligned} u(t,x) &= g(x) - it\left(a_3(0)D_x^3g(x) + a_2(0,x,g(x))D_x^2g(x)\right. \\ &\quad + \left. a_1(0,x,g(x))D_xg(x) + a_0(0,x,g(x))g(x) - f(0,x)\right) + o(t) \\ &=: w(t,x) + o(t), \qquad \text{as } t \to 0. \end{aligned} \tag{4.12}$$

If t is sufficiently small, the function $w \in X_T$ is in a neighborhood of the solution u we are looking for. The idea of the proof is then the following: we first approximate $Jw \in X_T$ by a function ϕ_ε such that $\phi_\varepsilon(t) \equiv 0$ for $0 \le t \le T_\varepsilon \le T$; then, we apply Theorem 2.5, in particular the fact that J is a bijection of a neighborhood U of w onto a neighborhood V of Jw. More precisely, we show that $\phi_\varepsilon \in V$, and then by the local invertibility of J there will be $u \in U$ such that $Ju = \phi_\varepsilon \equiv 0$ in $[0, T_\varepsilon]$ and hence u is the local in time solution of the Cauchy problem (1.13). Let us construct ϕ_ε: given $\rho \in C^\infty(\mathbb{R})$ with $0 \le \rho \le 1$ and

$$\rho(s) = \begin{cases} 0, & s \le 1 \\ 1, & s \ge 2, \end{cases}$$

we define

$$\phi_\varepsilon(t,x) := \int_0^t \rho\left(\frac{s}{\varepsilon}\right)(\partial_t Jw)(s,x)ds. \tag{4.13}$$

We immediately see that $\phi_\varepsilon \equiv 0$ for $0 \le t \le \varepsilon$. We are going to prove that, for every fixed neighborhood V of Jw in $X_T = C^1([0,T]; H^\infty(\mathbb{R}))$, we have $\phi_\varepsilon \in V$ if ε is sufficiently small. To this aim let us notice that

$$Jw(t) - \phi_\varepsilon(t) = \int_0^t (\partial_t Jw)(s)ds + \underbrace{Jw(0)}_{=0} - \int_0^t \rho\left(\frac{s}{\varepsilon}\right)(\partial_t Jw)(s)ds = \int_0^t \left[1 - \rho\left(\frac{s}{\varepsilon}\right)\right](\partial_t Jw)(s)ds.$$

Hence

$$\|Jw - \phi_\varepsilon\|_k \le \int_0^{2\varepsilon} |(\partial_t Jw)(s)|_k \, ds + \sup_{t\in[0,2\varepsilon]} |(\partial_t Jw)(t)|_k. \tag{4.14}$$

Now we compute explicitly $\partial_t(Jw(t,x))$ and estimate its k-seminorms for small values of t. From (4.1) we get

$$\partial_t(Jw(t,x)) = \partial_t w + ia_3(t)D_x^3 w + \sum_{j=0}^{2} ia_j(t,x,w)D_x^j w - if(t,x).$$

Using the definition of w in (4.12) we get

$$\begin{aligned}
\partial_t(Jw(t,x)) = &-ia_3(0)D_x^3 g - \sum_{j=0}^{2} ia_j(0,x,g)D_x^j g + if(0,x) \\
&+ia_3(t)D_x^3 g + ta_3(t)D_x^3\Big(a_3(0)D_x^3 g + \sum_{j=0}^{2} a_j(0,x,g)D_x^j g - f(0,x)\Big) \\
&+ia_2(t,x,w)D_x^2 g + ta_2(t,x,w)D_x^2\Big(a_3(0)D_x^3 g + \sum_{j=0}^{2} a_j(0,x,g)D_x^j g - f(0,x)\Big) \\
&+ia_1(t,x,w)D_x g + ta_1(t,x,w)D_x\Big(a_3(0)D_x^3 g + \sum_{j=0}^{2} a_j(0,x,g)D_x^j g - f(0,x)\Big) \\
&+ia_0(t,x,w)g + ta_0(t,x,w)\Big(a_3(0)D_x^3 g + \sum_{j=0}^{2} a_j(0,x,g)D_x^j g - f(0,x)\Big) \\
&-if(t,x) \\
= \quad & i[a_3(t) - a_3(0)]D_x^3 g + i\sum_{j=0}^{2}\big[a_j(t,x,w) - a_j(0,x,g)\big]D_x^j g \\
&+a_3(t)tD_x^3\left[a_3(0)D_x^3 g + \sum_{j=0}^{2} a_j(0,x,g)D_x^j g - f(0,x)\right] \\
&+\sum_{j=0}^{2} a_j(t,x,w)tD_x^j\left[a_3(0)D_x^3 g + \sum_{s=0}^{2} a_s(0,x,g)D_x^s g - f(0,x)\right] \\
&+i\big(f(0,x) - f(t,x)\big).
\end{aligned}$$

Now observe that for every $k \in \mathbb{N}_0$ we have:

- for every $\varepsilon_3 > 0$ there exists $\delta_3 > 0$ depending on g, a_3 and k such that $|(a_3(t) - a_3(0))D_x^3 g|_k \leq \varepsilon_3$ for every $t \in [0, \delta_3]$, since a_3 is continuous;
- for every $\varepsilon_2 > 0$ there exists $\delta_2 > 0$ depending on g, a_0, a_1, a_2 and k such that

$$\sum_{j=0}^{2} |\,(a_j(t,x,w) - a_j(0,x,g))\, D_x^j g|_k \leq \varepsilon_2$$

for every $t \in [0, \delta_2]$, since $(t,x) \mapsto a_j(t,x,w(t,x))$ belongs to $C([0,T];\gamma^\theta(\mathbb{R}))$ and henceforth $(t,x) \mapsto a_j(t,x,w(t,x))\xi^j$ belongs to $C([0,T];\Gamma^j_\theta(\mathbb{R}^2))$;
- $t \sup\limits_{t\in[0,T]} |a_3(t)| \cdot |a_3(0)D_x^6 g + \sum\limits_{j=0}^{2} D_x^3(a_j(0,x,g)D_x^j g) - D_x^3 f(0,x)|_k \leq C(a_3, a_2, a_1, a_0, g, f, k)t$;
- $t \sum\limits_{j=0}^{2} \left| a_j(t,x,w) D_x^j \left(a_3(0)D_x^3 g + \sum\limits_{s=0}^{2} a_s(0,x,g) D_x^s g - f(0,x) \right) \right|_k \leq C(a_3, a_2, a_1, a_0, g, f, k)t$;
- since $f \in C([0,T]; H^\infty_\theta(\mathbb{R}))$, then for every $\epsilon_1 > 0$ there exist $\delta_1 > 0$ depending on f and k such that $|f(0,x) - f(t,x)|_k \leq \varepsilon_1$ for every $t \in [0, \delta_1]$.

In this way we are able to conclude that for every $\tilde{\varepsilon} > 0$ there exists $\tilde{\delta}_k$ depending on $\tilde{\varepsilon}, k, f, g$ and on the coefficients such that

$$|(\partial_t Jw)(t)|_k \leq \tilde{\varepsilon}, \quad \forall\, t \in [0, \tilde{\delta}_k].$$

If $2\varepsilon < \tilde{\delta}_k$, from (4.14) we obtain

$$\|Jw - \phi_\varepsilon\|_k \leq 2\tilde{\varepsilon}.$$

Now let V be an open neighborhood of Jw. Recalling that the topology of X_T is defined by the metric

$$d(u,v) = \sum_{k\geq 0} \frac{1}{2^{k+1}} \frac{\|u-v\|_k}{1 + \|u-v\|_k}, \quad \forall u, v \in X_T,$$

we see that there exists $r > 0$ such that the ball $\{d(u, Jw) < r : u \in X_T\} \subset V$. Take $K \in \mathbb{N}_0$ such that $\sum_{k>K} 2^{-(k+1)} \leq \frac{r}{2}$ and choose $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon} < \frac{r}{4K}$. Then, if $2\varepsilon < \tilde{\delta} = \min\limits_{0\leq k\leq K} \tilde{\delta}_k$ we infer that

$$d(Jw, \phi_\varepsilon) \leq \sum_{k\leq K} \|Jw - \phi_\varepsilon\|_k + \sum_{k>K} \frac{1}{2^{k+1}} \leq \frac{r}{2} + \frac{r}{2} = r.$$

So, if $\varepsilon > 0$ is sufficiently small then $\phi_\varepsilon \in V$.
Now suppose in addition that $V = J(U)$ where U is an open neighborhood of w and that $J : U \to V$ is bijective. Then there exists $u \in U$ such that $Ju = \phi_\varepsilon$. In particular, this proves that $u \in C^1([0,\varepsilon]; H^\infty_\theta(\mathbb{R}))$ is a local solution of the Cauchy problem (1.13). Uniqueness follows by standard arguments. Indeed, if u, v are two solutions of the Cauchy problem (1.13), then $w := u - v$ solves the linear Cauchy problem

$$\tilde{\tilde{P}} w = 0, \qquad w(0,x) = 0,$$

for an operator $\tilde{\tilde{P}}$ which is exactly as $P_u(D)$ except for the term a_0 which is substituted by another term satisfying the same assumptions. From the uniqueness of the solution to the linearized Cauchy problem we get $w = 0$, that is $u = v$.
To prove that $T^*(g, f)$ is a lower semi-continuous function of the initial data g, f we follow the ideas

presented in [13]. Let us assume $\tilde{u}$ to be the solution of (1.13) with initial data $\tilde{g}$, $\tilde{f}$ and life span $\tilde{T} \leq T$. Then we consider the map

$$Q : C^1([0, \tilde{T}]; H^\infty_\theta(\mathbb{R})) \to H^\infty_\theta(\mathbb{R}) \times C([0, \tilde{T}]; H^\infty_\theta(\mathbb{R}))$$

defined by $Q(u) = (u(0), P_u u)$ for all $u \in C^1([0, \tilde{T}]; H^\infty_\theta(\mathbb{R}))$.
The derivative of Q is $DQ(u)v = (v(0), \tilde{P}_u v)$ where $\tilde{P}_u$ is the operator

$$\tilde{P}_u = D_t + D_x^3 + a_2(t, x, u)D_x^2 + a_1(t, x, u)D_x + \tilde{a}_0(t, x, u),$$

$$\tilde{a}_0(t, x, u) = a_0(t, x, u) + \sum_{j=0}^{2} (\partial_w a_j)(t, x, u)D_x^j u.$$

So, the equation $DQ(u)v = (h, w)$ is equivalent to the following linear Cauchy problem

$$\begin{cases} \tilde{P}_u v = w \\ v(0) = h. \end{cases}, \qquad t \in [0, \tilde{T}], x \in \mathbb{R}.$$

Hence, similarly to what we did for the map J defined by (4.1), we can show that Q is locally invertible. Therefore we obtain open neighborhoods $U_{\tilde{g}}$ of $\tilde{g}$, $U_{\tilde{f}}$ of $\tilde{f}$ and $U_{\tilde{u}}$ of $\tilde{u}$ such that for all $(g, f) \in U_{\tilde{g}} \times U_{\tilde{f}}$ there exists a unique $u \in U_{\tilde{u}}$ satisfying $Q(u) = (g, f)$, that is $(u(0), P_u u) = (g, f)$. This means that u solves the Cauchy problem for every $(t, x) \in [0, \tilde{T}] \times \mathbb{R}$. In particular, the life span $T^*(g, f)$ is not smaller than $\tilde{T}$ provided that (g, f) is close enough to $(\tilde{g}, \tilde{f})$, so $T^*(g, f)$ is a lower semi-continuous function in the $H^\infty_\theta(\mathbb{R}) \times C([0, \tilde{T}]; H^\infty_\theta(\mathbb{R}))$ topology. □

Remark 4.5. We stress the fact that the life span T^* may be small. In fact, we need $T^* = \varepsilon$ small enough in order to conclude that the function ϕ_ϵ (defined by (4.13)) belongs to some suitable open neighborhood of Jw (w given by (4.12)) provided by the Nash-Moser theorem.

Remark 4.6. In Theorem 1.2 we assume that the coefficient of the third order term is independent of x. Indeed, if a_3 depends on x, even allowing its derivatives with respect to x to decay like $\langle x \rangle^{-m}$ for $m >> 0$, we obtain

$$e^{\rho' \langle D \rangle^{1/\theta}} (ia_3(t, x)D_x^3) e^{-\rho' \langle D \rangle^{1/\theta}} = ia_3(t, x)D_x^3 + \text{op}\left(\rho' \partial_\xi \langle \xi \rangle^{\frac{1}{\theta}} \cdot \partial_x a_3(t, x)\xi^3\right) + \text{l.o.t}$$

with $\rho' \partial_\xi \langle \xi \rangle^{\frac{1}{\theta}} \cdot \partial_x a_3(t, x)\xi^3 \sim \langle \xi \rangle^{2+\frac{1}{\theta}} \langle x \rangle^{-m}$. This term has order $2 + \frac{1}{\theta} > 2$ and cannot be controlled by other lower order terms whose order does not exceed 2.

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