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A remark on boundedness of manifolds embedded with small codimension

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Abstract For embedded projective manifolds of small codimension we give an explicit bound for their degree, depending on the Castelnuovo–Mumford regularity of their structure sheaf. As an application, we obtain bounds for the degree of such manifolds whose structure sheaf is arithmetically Cohen–Macaulay (in a weak sense) and whose canonical map is not birational.

Keywords small codimension, boundedness, Castelnuovo–Mumford regularity

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1 Introduction

Embedded projective manifolds $X \subset \mathbb{P}^N$ of dimension n and codimension c are called *of small codimension* if $c \leqslant n$. It is well known that they have many special properties, as shown e. g. by the famous Barth–Larsen Theorems [1]. In their seminal paper [4], Ellingsrud and Peskine proved that surfaces not of general type in $\mathbb{P}^4$ have bounded degree. The same statement was later proved for threefolds in $\mathbb{P}^5$, [2]. In [9] Schneider showed that a similar result holds when $n \geqslant c+2$, using the Fulton–Lazarsfeld Theorem on the positivity properties of Chern classes of ample vector bundles [5]. Thus, for codimension two, the boundedness problem for manifolds not of general type was solved completely. Schneider's arguments apply for the case of subcanonical manifolds, a condition which is implied by the Barth–Larsen Theorem when $n \geqslant c+2$. In [6] the authors refined Schneider's approach and obtained an explicit numerical bound for the degree of manifolds of small codimension in some special cases, e. g. when their anticanonical bundle is pseudoeffective.

This short note may be seen as an appendix to [6]. Its aim is to relate boundedness for manifolds of small codimension with the Castelnuovo–Mumford regularity of their structure sheaf. Precisely, we prove the following:

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Theorem. Let $X \subset \mathbb{P}^N$ be smooth, connected, of dimension n and of small codimension. If $\mathcal{O}_X$ is m-regular, then the degree of X satisfies

$$\deg(X) < \max[(2n+1)^n, (2m+n)^n].$$

As a corollary, we see that manifolds of small codimension whose structure sheaf is arithmetically Cohen–Macaulay (in a weak sense) and such that their canonical map is not birational have degree bounded by $(3n+2)^n$.

We hope that our remark may be useful in proving results in the cases where very little is known, namely when $c \geqslant 3$ and either $n = c$ or $n = c+1$.

We work over $\mathbb{C}$ and use standard notation and terminology. For instance, K_X denotes the canonical class and H a hyperplane section of $X \subset \mathbb{P}^N$.

2 Castelnuovo–Mumford regularity and boundedness

Let $X \subseteq \mathbb{P}^N$ be a connected manifold of dimension n, codimension c and degree $\deg(X)$. A coherent sheaf $\mathcal{F}$ on X is called *m-regular* if $H^i(X, \mathcal{F}(m-i)) = 0$, for all $i > 0$. This condition, called Castelnuovo–Mumford regularity, was introduced by Mumford [8] for his proof of the existence of the Hilbert Scheme. See [7] for a discussion of subsequent developments and appropriate references. One of Mumford's results says that if $\mathcal{F}$ is m-regular, then it is also m'-regular for any $m' > m$ and $\mathcal{F}(m)$ is globally generated.

We say that $X \subset \mathbb{P}^N$ as above is *m-regular* if its ideal sheaf is so, as a sheaf on $\mathbb{P}^N$.

If the Hilbert polynomial P_X is given, there are explicit bounds for the regularity of X in terms of the coefficients of P_X. This was the original reason for which Mumford introduced this concept. Conversely, if the regularity of X is fixed, say equal to m, then the degree of X is bounded by $\deg(X) \leqslant m^c$. This follows easily from the fact that $\mathcal{I}_X(m)$ is globally generated, so that X may be scheme-theoretically defined by equations of degree at most m.

So, bounding the regularity of X or its degree are equivalent tasks. The regularity of $\mathcal{O}_X$ is a weaker condition than the regularity of X, see Remark 1. Therefore, in general, bounding the regularity of $\mathcal{O}_X$ is not enough to ensure boundedness of $\deg(X)$, see Remark 2.

Remark 1 (i) If $\mathcal{O}_X$ is m-regular, then $m \geqslant 0$. Indeed, by Serre duality

$$0 = H^n(X, \mathcal{O}_X(m-n)) \cong H^0(X, \mathcal{O}_X(K_X + (n-m)H)).$$

By the Kodaira Vanishing Theorem and an easy induction on n, this gives $n - m \leqslant n$.

(ii) X is $(m+1)$-regular if and only if $\mathcal{O}_X$ is m-regular and the restriction map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$ is onto. This follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_X(m-i) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(m-i) \longrightarrow \mathcal{O}_X(m-i) \longrightarrow 0.$$

(iii) If $mH - K_X$ is ample, then $\mathcal{O}_X$ is $(m+n)$-regular. This follows from the Kodaira Vanishing Theorem.

3 The case of small codimension

Let us recall that $X \subset \mathbb{P}^N$ is a connected manifold of dimension n, codimension c and degree $\deg(X)$. We show that if X has small codimension the condition that $\mathcal{O}_X$ is of bounded regularity is enough to ensure that the degree of X is bounded.

Remark 2 When $c \geqslant n+1$, one can not hope to bound the degree of X if the regularity of $\mathcal{O}_X$ is bounded. Indeed, let $X(d) \subset \mathbb{P}^{2n+1}$ be the isomorphic projection of the Veronese variety $v_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$, for any $d \geqslant 2$. Clearly, $\mathcal{O}_{X(d)}$ is $(n+1)$-regular, but the degree of $X(d)$ is d^n.

The proof of the theorem stated in the introduction relies on Proposition 1 from [6], which we recall for reader's convenience.

Lemma 1 ([6], Prop. 1) *Let $X \subset \mathbb{P}^N$ be as above, with $c \leqslant n$. Assume that there exists a non-negative constant a such that $K_X \cdot H^{n-1} \leqslant a \cdot \deg(X)$. Then $\deg(X) < (a+2n+1)^n$.*

Corollary 1 *Let $X \subset \mathbb{P}^N$ be as above, with $c \leqslant n$. Assume that there exists a non-negative constant m such that $mH - K_X$ is pseudoeffective. Then we have $\deg(X) < (m+2n+1)^n$.*

The examples in Remark 2 show that the condition $c \leqslant n$ in the corollary is essential (for any non-negative m).

Define the following sequence of manifolds, starting with $X_0 = X \subset \mathbb{P}^N$: for $i \geqslant 0$, let $X_{i+1} =: X_i \cap H_i$, where H_i is a general hyperplane in $\mathbb{P}^{N-i}$.

Lemma 2 *Assume that $H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}(b)) = 0$ for some integer b. Then $K_X \cdot H^{n-1} \leqslant (2b-n+1) \cdot \deg(X)$.*

Proof If we let g be the genus of the smooth curve X_{n-1}, by adjunction formula we get $K_X \cdot H^{n-1} = 2(g-1) - (n-1)\deg(X)$ and the Riemann–Roch Theorem yields $b \cdot \deg(X) + 1 - g \geqslant 0$. The result follows. □

Lemma 3 *If $\mathcal{O}_X$ is m-regular, then $\mathcal{O}_{X_{n-1}}$ is also m-regular.*

Proof It follows by induction on j, using the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_j}(m-i-1) \longrightarrow \mathcal{O}_{X_j}(m-i) \longrightarrow \mathcal{O}_{X_{j+1}}(m-i) \longrightarrow 0 \,. \qquad \square$$

Proof of the Theorem. By Lemma 3, $\mathcal{O}_{X_{n-1}}$ is m-regular. Therefore

$$H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}(m-1)) = 0.$$

From Lemma 2, we get $K_X \cdot H^{n-1} \leqslant (2m-n-1) \cdot \deg(X)$. Finally, by Lemma 1, we obtain $\deg(X) < (2n+1)^n$ if $2m-n-1 \leqslant 0$ and $\deg(X) < (2m+n)^n$ otherwise. □

We say that $\mathcal{O}_X$ is *weakly arithmetically Cohen–Macaulay* (waCM) if

$$H^i(X, \mathcal{O}_X(j)) = 0 \quad \text{for } 0 < i < n \text{ and } 2 \leqslant j \leqslant n.$$

Corollary 2 *Assume that $X \subset \mathbb{P}^N$ is as above, with $c \leqslant n$. Moreover, assume that $\mathcal{O}_X$ is waCM and the canonical map of X is not birational. Then $\deg(X) < (3n+2)^n$.*

Proof We show that $\mathcal{O}_X$ is $(n+1)$-regular, so the Theorem applies.

We have $H^i(X, \mathcal{O}_X(n+1-i)) = 0$ for $0 < i < n$. For $i = n$, $H^n(X, \mathcal{O}_X(1)) \cong H^0(X, \mathcal{O}_X(K_X - H))$ by Serre duality. The last group is zero since the canonical map of X is not birational. □

Remark 3 For manifolds of small codimension, the same argument shows the following statement:

Suppose that there is a constant m_0 (which depends only on n) such that $H^i(X, \mathcal{O}_X(m)) = 0$ for $0 < i < n$ and every $m \geqslant m_0$. Then one of the following holds:

(i) the canonical map of X is birational; or
(ii) the degree of X is bounded.

For a proof it is enough to remark that we may assume $m_0 \geqslant 2$ and therefore $\mathcal{O}_X$ is $(m_0 + n - 1)$-regular.

For surfaces in $\mathbb{P}^4$ this becomes:

If $H^1(X, \mathcal{O}_X(m)) = 0$ for some $m \geqslant 2$, then either the canonical map of X is birational, or $\deg(X) < 4(m+2)^2$. In fact, it was proved in [3] that in this case there is a constant M such that for $\deg(X) > M$, the canonical map of X is birational.

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