Contents

1. Introduction

The starting point of this work was the notion of a torsor which comes from principle bundles in classical geometry. A torsor is a principal homogeneous space over a group, i.e. a G-set X where G is a group acting freely and transitively over X. An idea due to Baer which goes back to the 1920's allows to reformulate the definition of a torsor without specifying the group G: a torsor is a set, sometimes called herd, X together with a structure $X \times X \times X \to X$ satisfying some parallelogram relations (see [Ba, p. 202] or [Pr, p. 170]).
A noncommutative analogue is the notion of a Hopf-Galois object as introduced by Kreimer and Takeuchi [KT]. Let H be a Hopf algebra, flat over the base ring k, a (right) H-Galois object A is a (right) H-comodule algebra such that the Galois map $\beta : A \otimes A \to A \otimes H$ given by $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is bijective (where $\delta : A \to A \otimes H : x \mapsto x_{(0)} \otimes x_{(1)}$ is the H-comodule structure of A) and $A^{co(H)} = \{x \in A \mid \delta(x) = x \otimes 1_H\} = k$.
A similar concept for the noncommutative case was introduced by Grunspan in [G] as the notion of a quantum torsor. Together with the definition, Grunspan gives the proof that every quantum torsor gives rise to two Hopf algebras over which it is a bi-Galois extension of the base field. Conversely, Schauenburg in [Sch4] proves that every Hopf-Galois extension of the base field is a quantum torsor in the sense of Grunspan.
The axioms defining a quantum torsor were simplified allowing to prove anyway a correspondence between faithfully flat torsors and Hopf-(bi)Galois object (see [Sch1]). Moreover, Schauenburg in [Sch4] could prove that the two Hopf algebras coming from a torsor are Morita-Takeuchi equivalent, i.e. their categories of comodules are equivalent. Another equivalence between module categories has been studied in [BMV] and it is related with Morita contexts defined in the pure categorical setting. This gave the hint to investigate a special class of herds at this level of generality.

The simplified version of the torsor axioms admits a generalization to arbitrary Galois extensions (not only of the base ring or field) and gave rise to different results which we try to summarize here. Hopf Galois extensions of an arbitrary algebra B by introducing the notion of a B-torsor in [Sch1], Galois extensions by bialgebroids by means of A-B-torsors in [Ho, Chapter 5] and [BB], Galois extensions by corings using the notion of a pretorsor in [BB] and Galois comodules of corings arising from entwining structures using the notion of a bimodule herd in [BV]. Generalizing the notion of pretorsor given in [BB], pretorsors over two adjunctions are introduced in [BM, Section 4]. Such categorical setting is the one we choose for this work trying to develop the notion of pretorsor and herd at this pure general level.

The first aim of this thesis is to give a unified and self-contained treatment of a number of known results related to the theory of herds. This gives us the technical tools to deal with the second aim of our work which is to obtain some new results about herds and coherds in the pure categorical setting. A herd at this level is a pretorsor with respect to a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$. This is given by two monads $\mathbb{A}$ and $\mathbb{B}$ over two different categories, two bimodule functors P and Q with respect to the monads and functorial morphisms σ^A and σ^B satisfying linearity and compatibility conditions. In particular we refer to the study of the special class of tame herds, which yields a correspondence with Galois functors (generalizing the historical results we started with). Moreover, under the regularity assumption on the formal dual structure related to a herd, one can construct a coherd. Conversely, beginning with a coherd over a regular formal codual structure, a herd can be obtained. By applying twice this process, one can start with a formal dual structure and a herd, construct a formal codual structure and a coherd and then compute also a new formal dual structure. Under the extra assumption that the starting formal dual structure is also a Morita context, the final formal dual structure computed from a tame herd comes out to be closely related to the starting one. We consider a few cases in which the monads are in fact isomorphic. As an application, we develop some examples. The first is given by a right Galois comodule from which we derive the herd. Then we simplify the setting and we study the Schauenburg case of A/k a faithfully flat Hopf-Galois extension with respect to H. In this example we can compute the comonads associated to the herd and the coherd. The last example is a non trivial example of a coherd. It allows to compute the two monads associated and the equivalence between the module categories with respect to these monads. Finally we investigate the bicategory of balanced bimodule functors which are one of the most useful tools in this work and is inspired by the balanced bimodules in the classical sense.

We developed the portions of the theory of herds, resp. coherds, we found more suitable for our purposes.

In the first part we collect some well-known results including proofs. It is about equalizers and coequalizers, contractible equalizers and coequalizers and notation for adjunctions.

Then we concentrate, in the second section, the needed materials for the sequel about monads. Similarly we do for comonads. At this point we also include the Beck's Tripleability Theorem and the generalized version which introduce the Eilenberg-Moore comparison functor and the categories of modules and comodules.

We reserve a short section to the notion of distributive laws and above all the correspondence between distributive laws and liftings of monads and comonads.

The next section introduces the notion of pretorsor and herd bringing all the details and the results needed to prove the equivalence between herds and Galois functors in the tame case. The same has been done for the dual case of copretorsors and coherds.

Later on a section dedicated to a new fundamental functor built from a herd and a coherd respectively and then the theorem relating the starting formal dual structure and the one obtained after applying the two processes from a herd and from a coherd.

One section is dedicated to the equivalence between the module categories obtained from a copretorsor and to the equivalence between the comodule categories obtained from a pretorsor.

The following section is a collection of the examples we provide in this work, about herds and coherds first and then applications of Beck's Theorem and of its generalization. In particular, the example of a coherd was produced during some useful discussions with T. Brzeziński. In the subsection dedicated to Galois comodules we need some material related to Gabriel-Popescu theorem which is contained in the appendix.

The last part is the first outcome of a joint work with J. Gomez-Torrecillas. It is devoted to the introduction of the bicategory of balanced bimodule functors $BIM(\mathsf{C})$. First we fix some notation and terminology about 2-categories and bicategories. Then we define the bicategory of balanced bimodule functors and finally we study how it can be related to entwined modules and comodules.

2. Preliminaries

2.1. **Some results on equalizers and coequalizers.** In the following, most of the computations are justified. We denote by the name of a functorial morphism, its naturality property.

Definitions 2.1. Let $\alpha : B \to C$ be a functorial morphism. We say that α is

- a *functorial monomorphism*, or simply a monomorphism, if for every $\beta, \gamma : A \to B$ such that $\alpha \circ \beta = \alpha \circ \gamma$ we have $\beta = \gamma$.
- a *functorial regular monomorphism,* or simply a regular monomorphism, if α is the equalizer of two functorial morphisms.
- a *functorial epimorphism*, or simply an epimorphism, if for every $\beta, \gamma : C \to D$ such that $\beta \circ \alpha = \gamma \circ \alpha$ we have $\beta = \gamma$.
- a *regular epimorphism*, or simply a regular epimorphism, if α is the coequalizer of two functorial morphisms.

Definition 2.2. A parallel pair $\alpha, \beta : F \to F'$ is said to be *reflexive* if the two arrows have a common right inverse $\delta : F' \to F$.

Definition 2.3. A *reflexive equalizer* is an equalizer of a reflexive parallel pair.

Definition 2.4. A *reflexive coequalizer* is a coequalizer of a reflexive parallel pair.

Lemma 2.5. *Let F, G, H be functors and let $f : F \to G$, $g : G \to H$ and $h : F \to H$ be functorial morphisms such that $h = g \circ f$. Assume that f is a functorial isomorphism. Then h is a regular epimorphism if and only if g is a regular epimorphism.*

Proof. First, let us assume that g is a regular epimorphism, i.e.
$(H, g) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$. Then we have

$$h \circ f^{-1} \circ \alpha = g \circ f \circ f^{-1} \circ \alpha = g \circ f \circ f^{-1} \circ \beta = h \circ f^{-1} \circ \beta.$$

Now, let $\chi : F \to X$ be a functorial morphism such that $\chi \circ f^{-1} \circ \alpha = \chi \circ f^{-1} \circ \beta$. By the universal property of the coequalizer $(H, g) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$, there exists a unique functorial morphism $\overline{\chi} : H \to X$ such that $\overline{\chi} \circ g = \chi$. Then, by composing to the right with f we get

$$\overline{\chi} \circ h = \overline{\chi} \circ g \circ f = \chi \circ f^{-1} \circ f = \chi.$$

Moreover, let χ' be another functorial morphism such that $\chi' \circ h = \chi$. Since we also have $\overline{\chi} \circ h = \chi$ we have

$$\chi' \circ g \circ f = \chi' \circ h = \chi = \overline{\chi} \circ h = \overline{\chi} \circ g \circ f$$

and since $g \circ f$ is an epimorphism, we deduce that $\chi' = \overline{\chi}$ so that

$$(H, h) = \mathrm{Coequ}_{\mathrm{Fun}} \left(f^{-1} \circ \alpha, f^{-1} \circ \beta \right).$$

Conversely, let us assume that h is a regular epimorphism, i.e. $(H, h) = \mathrm{Coequ}_{\mathrm{Fun}} (\alpha, b)$. Then we have

$$g \circ f \circ a = h \circ a = h \circ b = g \circ f \circ b.$$

Now, let $\xi : G \to X$ be a functorial morphism such that $\xi \circ f \circ a = \xi \circ f \circ b$. By the universal property of $(H, h) = \mathrm{Coequ}_{\mathrm{Fun}} (\alpha, b)$, there exists a unique functorial morphism $\overline{\xi} : H \to X$ such that $\overline{\xi} \circ h = \xi \circ f$, i.e. $\overline{\xi} \circ g \circ f = \xi \circ f$ and since f is an isomorphism we deduce that $\overline{\xi} \circ g = \xi$. Let us assume that there exists another functorial morphism $\xi' : H \to X$ such that $\xi' \circ g = \xi$. Since we also have $\overline{\xi} \circ g = \xi$ we get that

$$\xi' \circ h = \xi' \circ g \circ f = \xi \circ f = \overline{\xi} \circ g \circ f = \overline{\xi} \circ h$$

and since h is an epimorphism, we deduce that $\xi' = \overline{\xi}$. Therefore, $(H, g) = \mathrm{Coequ}_{\mathrm{Fun}} (f \circ a, f \circ b)$. $\square$

LEMMA 2.6. *Let F, G, H be functors and let $f : F \to G$, $g : G \to H$ and $h : F \to H$ be functorial morphisms such that $h = g \circ f$. Assume that g is a functorial isomorphism. Then h is a regular epimorphism if and only if f is a regular epimorphism.*

Proof. Assume first that f is a regular epimorphism, i.e. $(G, f) = \mathrm{Coequ}_{\mathrm{Fun}} (\alpha, \beta)$. Then we have

$$h \circ \alpha = g \circ f \circ \alpha = g \circ f \circ \beta = h \circ \beta.$$

Let $\xi : F \to X$ be a functorial morphism such that $\xi \circ \alpha = \xi \circ \beta$. By the universal property of the coequalizer $(G, f) = \mathrm{Coequ}_{\mathrm{Fun}} (\alpha, \beta)$, there exists a unique functorial morphism $\overline{\xi}$ such that $\overline{\xi} \circ f = \xi$. Then we have

$$\overline{\xi} \circ g^{-1} \circ h = \overline{\xi} \circ g^{-1} \circ g \circ f = \overline{\xi} \circ f = \xi$$

so that ξ factorizes through h via $\overline{\xi} \circ g^{-1}$. Moreover, if there exists another functorial morphism $\xi' : F \to X$ such that $\xi' \circ h = \xi$, since we also have $\xi = \overline{\xi} \circ g^{-1} \circ h$ we have

$$\xi' \circ g \circ f = \xi' \circ h = \xi = \overline{\xi} \circ g^{-1} \circ h = \overline{\xi} \circ g^{-1} \circ g \circ f = \overline{\xi} \circ f$$

and since f is epi we get

$$\xi' \circ g = \overline{\xi}$$

from which we deduce that

$$\xi' = \overline{\xi} \circ g^{-1}.$$

Therefore we obtained

$$(H, h) = \mathrm{Coequ}_{\mathrm{Fun}} (\alpha, \beta).$$

Conversely, let now assume that h is a regular epimorphism, i.e. $(H, h) = \mathrm{Coequ}_{\mathrm{Fun}} (a, b)$. Then we have

$$g \circ f \circ a = h \circ a = h \circ b = g \circ f \circ b$$

and since g is mono we get that

$$f \circ a = f \circ b.$$

Let now $\chi : F \to X$ be a functorial morphism such that $\chi \circ a = \chi \circ b$. by the universal property of the coequalizer $(H, h) = \mathrm{Coequ}_{\mathrm{Fun}}(a, b)$ there exists a unique functorial morphism $\overline{\chi} : H \to X$ such that $\overline{\chi} \circ h = \chi$ and hence $\overline{\chi} \circ g \circ f = \chi$ so that χ factorizes through f via $\overline{\chi} \circ g$. Moreover, let $\chi' : F \to X$ be another functorial morphism such that $\chi' \circ f = \chi$. Since we also have $\overline{\chi} \circ h = \chi$ we have

$$\chi' \circ g^{-1} \circ h = \chi' \circ g^{-1} \circ g \circ f = \chi' \circ f = \chi = \overline{\chi} \circ h$$

and since h is epi we get that $\chi' \circ g^{-1} = \overline{\chi}$ from which we deduce that

$$\chi' = \overline{\chi} \circ g.$$

Therefore $(G, f) = \mathrm{Coequ}_{\mathrm{Fun}}(a, b)$. □

LEMMA 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let $F, F' : \mathcal{A} \to \mathcal{B}$ be functors and $\alpha, \beta : F \to F'$ be functorial morphisms. If, for every $X \in \mathcal{A}$, there exists* $\mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$, *then there exists the coequalizer* $(C, c) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$ *in the category of functors. Moreover, for any object X in $\mathcal{A}$, we have* $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$.

Proof. Define a functor $C : \mathcal{A} \to \mathcal{B}$ with object map $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$ for every $X \in \mathcal{A}$. For a morphism $f : X \to X'$ in $\mathcal{A}$, naturality of α and β implies that

$$(F'f) \circ (\alpha X) = (\alpha X') \circ (Ff) \text{ and } (F'f) \circ (\beta X) = (\beta X') \circ (Ff)$$

and hence

$$\begin{aligned}(cX') \circ (F'f) \circ (\alpha X) &= (cX') \circ (\alpha X') \circ (Ff) \overset{\text{ccoequ}}{=} (cX') \circ (\beta X') \circ (Ff) \\ &= (cX') \circ (F'f) \circ (\beta X)\end{aligned}$$

i.e. $(cX') \circ (F'f)$ coequalizes the parallel morphisms βX and αX. In light of this fact, by the universal property of the coequalizer (CX, cX), $Cf : CX \to CX'$ is defined as the unique morphism in $\mathcal{B}$ such that $(Cf) \circ (cX) = (cX') \circ (F'f)$. By construction, c is a functorial morphism $F' \to C$ such that $c \circ \alpha = c \circ \beta$. It remains to prove universality of c. Let $H : \mathcal{A} \to \mathcal{B}$ be a functor and let $\chi : F' \to H$ be a functorial morphism such that $\chi \circ \alpha = \chi \circ \beta$. Then, for any object X in $\mathcal{A}$, $(\chi X) \circ (\alpha X) = (\chi X) \circ (\beta X)$. Since $\subset$ $(CX, cX) = \mathrm{Coequ}_{\mathcal{B}}(\alpha X, \beta X)$, there is a unique morphism $\xi X : CX \to HX$ such that $(\xi X) \circ (cX) = \chi X$. The proof is completed by proving naturality of ξX in X. Take a morphism $f : X \to X'$ in $\mathcal{A}$. Since c and χ functorial morphisms,

$$\begin{aligned}&(Hf) \circ (\xi X) \circ (cX) = (Hf) \circ (\chi X) \overset{\chi}{=} (\chi X') \circ (F'f) \\ &= (\xi X') \circ (cX') \circ (F'f) = (\xi X') \circ (Cf) \circ (cX).\end{aligned}$$

Since cX is a epimorphism, we get that ξ is a functorial morphism. □

LEMMA 2.8 ([BM, Lemma 2.1]). *Let $\mathcal{C}$ and $\mathcal{K}$ be categories, let G,$G' : \mathcal{C} \to \mathcal{K}$ be functors and γ, $\theta : G \to G'$ be functorial morphisms. If, for every $X \in \mathcal{C}$, there exists* $\mathrm{Equ}_{\mathcal{K}}(\gamma X, \theta X)$, *then there exists the equalizer* $(E, i) = \mathrm{Equ}_{\mathrm{Fun}}(\gamma, \theta)$ *in the category of functors. Moreover, for any object X in $\mathcal{C}$,* $(EX, iX) = \mathrm{Equ}_{\mathcal{K}}(\gamma X, \theta X)$.

LEMMA 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let F,$F' : \mathcal{A} \to \mathcal{B}$ be functors, and let $\alpha, \beta : F \to F'$ be functorial morphisms. Assume that, for every $X \in \mathcal{A}$, $\mathcal{B}$ has coequalizers of αX and βX and let* $(Q, q) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$. *Under these assumptions, for any functor $P : \mathcal{D} \to \mathcal{A}$,* $\mathrm{Coequ}_{\mathrm{Fun}}(\alpha P, \beta P) = (QP, qP)$.

Proof. Clearly $(qP)\circ(\alpha P) = (qP)\circ(\beta P)$. Let $y : F'P \to Y$ be a functorial morphism such that $y \circ (\alpha P) = y \circ (\beta P)$. Then,

$$(yD) \circ (\alpha PD) = (yD) \circ (\beta PD)$$

and hence, since by Lemma 2.7 $(QPD, qPD) = \mathrm{Coequ}_{\mathcal{B}}(\alpha PD, \beta PD)$, there exists a unique $d_D : QPD \to YD$ such that

$$d_D \circ (qPD) = yD.$$

Let us prove that the assignment $D \mapsto d_D$ yields a functorial morphism $d : QP \to Y$. Let $h : D \to D'$ be a morphism in $\mathcal{D}$. We compute

$$\begin{aligned} (Yh) \circ d_D \circ (qPD) &= (Yh) \circ (yD) \overset{y}{=} (yD') \circ (F'Ph) \\ &= d_{D'} \circ (qPD') \circ (F'Ph) \overset{q}{=} d_{D'} \circ (QPh) \circ (qPD). \end{aligned}$$

Since qPD is an epimorphism, we conclude. $\square$

Lemma 2.10 ([BM, Lemma 2.2]). *Let $G, G' : \mathcal{C} \to \mathcal{K}$ be functors, and let $\gamma, \theta : G \to G'$ be functorial morphisms. Assume that every pair of parallel morphisms in $\mathcal{K}$ has an equalizer and let $(E, i) = \mathrm{Equ}_{\mathrm{Fun}}(\gamma, \theta)$. Under these assumptions, for any functor $P : \mathcal{D} \to \mathcal{C}$, $\mathrm{Equ}_{\mathrm{Fun}}(\gamma P, \theta P) = (EP, iP)$.*

Lemma 2.11. *Consider the following serially commutative diagram in an arbitrary category $\mathcal{K}$*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "n"'] \arrow[d, shift left, "m"] & B \arrow[r, "i"] \arrow[d, shift right, "n'"'] \arrow[d, shift left, "m'"] & C \arrow[d, shift right, "n''"'] \arrow[d, shift left, "m''"] \\
A' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, "e"'] & B' \arrow[r, "i'"] \arrow[d, "e'"'] & C' \arrow[d, "e''"'] \\
A'' \arrow[r, shift left, "f''"] \arrow[r, shift right, "g''"'] & B'' \arrow[r, "i''"] & C''
\end{tikzcd}$$

Assume that all columns are coequalizers and also the first and second rows are coequalizers. Then also the third row is a coequalizer.

Proof. In order to see that the third row is a fork, note that, by commutativity of the diagram and fork property of the second row,

$$\begin{aligned} i'' \circ f'' \circ e &= i'' \circ e' \circ f' = e'' \circ i' \circ f' = e'' \circ i' \circ g' = i'' \circ e' \circ g' \\ &= i'' \circ g'' \circ e. \end{aligned}$$

Since e is an epimorphism, this proves that the third row is a fork that is $i'' \circ f'' = i'' \circ g''$.

To conclude we want to prove the universality of i''. To do so, let us take any morphism $x : B'' \to X$ such that $x \circ f'' = x \circ g''$. Then we want to prove that there exist a unique functorial morphism $z : C'' \to X$ such that $z \circ i'' = x$.

We observe

$$x \circ e' \circ f' = x \circ f'' \circ e = x \circ g'' \circ e = x \circ e' \circ g'.$$

so we get that

$$x \circ e' \circ f' = x \circ e' \circ g'.$$

Since the second row is a coequalizer by assumption, there is a unique morphism $y : C' \to X$ such that

$$y \circ i' = x \circ e'. \tag{1}$$

We calculate

$$\begin{aligned} y \circ n'' \circ i &= y \circ i' \circ n' \overset{(1)}{=} x \circ e' \circ n' = x \circ e' \circ m' \overset{(1)}{=} y \circ i' \circ m' \\ &= y \circ m'' \circ i \end{aligned}$$

and since i is an epimorphism we get that

$$y \circ n'' = y \circ m''.$$

Since the third column is a coequalizer, there exists a unique morphism $z : C'' \to X$ such that

$$z \circ e'' = y.$$

Then

$$z \circ i'' \circ e' = z \circ e'' \circ i' = y \circ i' = x \circ e'$$

so we get that $z \circ i'' = x$. Since $e'' \circ i' = i'' \circ e'$ and e'', e', i' are epimorphism, we deduce that i'' is epimorphism and hence z is unique with respect to $z \circ i'' = x$. $\square$

Corollary 2.12. *Let F, $F' : \mathcal{A} \to \mathcal{B}$ be functors and α, $\beta : F \to F'$ be functorial morphisms. Assume that, for every $X \in \mathcal{A}$, $\mathcal{B}$ has coequalizers of αX and βX hence there exists $(Q, q) = \mathrm{Coequ}_{\mathrm{Fun}}(\alpha, \beta)$, cf. Lemma 2.7. Assume that $(P, p) = \mathrm{Coequ}_{\mathcal{A}}(f, g)$ of morphisms $f, g : X \to Y$ in $\mathcal{A}$ and that both F and F' preserve $\mathrm{Coequ}_{\mathcal{A}}(f, g)$. Then also Q preserves $\mathrm{Coequ}_{\mathcal{A}}(f, g)$.*

Proof. The following diagram (in $\mathcal{B}$) is serially commutative by naturality

$$\begin{tikzcd}
FX \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] \arrow[d, shift right, "\alpha X"'] \arrow[d, shift left, "\beta X"] & FY \arrow[r, "Fp"] \arrow[d, shift right, "\alpha Y"'] \arrow[d, shift left, "\beta Y"] & FP \arrow[d, shift right, "\alpha P"'] \arrow[d, shift left, "\beta P"] \\
F'X \arrow[r, shift left, "F'f"] \arrow[r, shift right, "F'g"'] \arrow[d, "qX"'] & F'Y \arrow[r, "F'p"] \arrow[d, "qY"'] & F'P \arrow[d, "qP"'] \\
QX \arrow[r, shift left, "Qf"] \arrow[r, shift right, "Qg"'] & QY \arrow[r, "Qp"] & QP
\end{tikzcd}$$

The columns are coequalizers by Lemma 2.7. The first and second rows are coequalizers by the assumption that F and F' preserve coequalizers. Thus the third row is a coequalizer by Lemma 2.11. $\square$

Lemma 2.13 ([BM, Lemma 2.5]). *Consider the following serially commutative diagram in an arbitrary category $\mathcal{K}$*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "e"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "e'"'] & C \arrow[d, "e''"'] \\ A' \arrow[r, "i'"] \arrow[d, shift right, "n"'] \arrow[d, shift left, "m"] & B' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, shift right, "n'"'] \arrow[d, shift left, "m'"] & C' \arrow[d, shift right, "n''"'] \arrow[d, shift left, "m''"] \\ A'' \arrow[r, "i''"] & B'' \arrow[r, shift left, "f''"] \arrow[r, shift right, "g''"'] & C'' \end{tikzcd}$$

Assume that all columns are equalizers and also the second and third rows are equalizers. Then also the first row is an equalizer.

Proof. Dual to Lemma 2.11. $\square$

Corollary 2.14. *Let G,$G' : \mathcal{C} \to \mathcal{K}$ be functors and γ, $\theta : G \to G'$ be functorial morphisms. Assume that, for every $X \in \mathcal{C}$, $\mathcal{K}$ has equalizers of γX and θX hence there exists $(E, e) = \mathrm{Equ}_{\mathrm{Fun}}(\gamma, \theta)$, cf. Lemma 2.8. Assume that $(I, i) = \mathrm{Equ}_{\mathcal{C}}(f, g)$ of morphisms $f, g : X \to Y$ in $\mathcal{C}$ and that both G and G' preserve* $\mathrm{Equ}_{\mathcal{C}}(f, g)$. *Then also E preserves* $\mathrm{Equ}_{\mathcal{C}}(f, g)$.

Proof. Dual to Corollary 2.12. $\square$

Lemma 2.15. *Let $Z, Z', W, W' : \mathcal{A} \to \mathcal{B}$ be functors, let $a, b : Z \to W$ and $a', b' : Z' \to W'$ be functorial morphisms, let $\varphi : Z \to Z'$ and $\psi : W \to W'$ be functorial isomorphisms such that*

$$\psi \circ a = a' \circ \varphi \qquad \textit{and} \qquad \psi \circ b = b' \circ \varphi.$$

Assume that there exist $(E, i) = \mathrm{Equ}_{\mathrm{Fun}}(a, b)$ and $(E', i') = \mathrm{Equ}_{\mathrm{Fun}}(a', b')$. Then φ induces an isomorphism $\widehat{\varphi} : E \to E'$ such that $\varphi \circ i = i' \circ \widehat{\varphi}$.

$$\begin{tikzcd} E \arrow[r, dotted, "\hat{\varphi}"] \arrow[d, "i"'] & E' \arrow[d, "i'"] \\ Z \arrow[r, "\varphi"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & Z' \arrow[d, shift right, "a'"'] \arrow[d, shift left, "b'"] \\ W \arrow[r, "\psi"] & W' \end{tikzcd}$$

Proof. Let us define $\widehat{\varphi}$. Let us compute

$$a' \circ \varphi \circ i = \psi \circ a \circ i \overset{\mathrm{def}i}{=} \psi \circ b \circ i = b' \circ \varphi \circ i$$

and since $(E', i') = \mathrm{Equ}_{\mathrm{Fun}}(a', b')$ there exists a unique functorial morphism $\widehat{\varphi} : E \to E'$ such that

$$i' \circ \widehat{\varphi} = \varphi \circ i.$$

Note that $\widehat{\varphi}$ is mono since so are i and i' and φ is an isomorphism. Consider $\varphi^{-1} : Z' \to Z$ and $\psi^{-1} : W' \to W$. Then we have

$$a \circ \varphi^{-1} = \psi^{-1} \circ a' \qquad \text{and} \qquad b \circ \varphi^{-1} = \psi^{-1} \circ b'.$$

Let us compute

$$a \circ \varphi^{-1} \circ i' = \psi^{-1} \circ a' \circ i' \overset{\text{def } i'}{=} \psi^{-1} \circ b' \circ i' = b \circ \varphi^{-1} \circ i'$$

and since $(E, i) = \mathrm{Equ}_{\mathrm{Fun}}(a, b)$ there exists a unique functorial morphism $\widehat{\varphi}' : E' \to E$ such that

$$i \circ \widehat{\varphi}' = \varphi^{-1} \circ i'.$$

Then we have

$$i \circ \widehat{\varphi}' \circ \widehat{\varphi} = \varphi^{-1} \circ i' \circ \widehat{\varphi} = \varphi^{-1} \circ \varphi \circ i = i$$

and since i is a monomorphism we deduce that

$$\widehat{\varphi}' \circ \widehat{\varphi} = \mathrm{Id}_E.$$

Similarly

$$i' \circ \widehat{\varphi} \circ \widehat{\varphi}' = \varphi \circ i \circ \widehat{\varphi}' = \varphi \circ \varphi^{-1} \circ i' = i'$$

and since i' is a monomorphism we obtain that

$$\widehat{\varphi} \circ \widehat{\varphi}' = \mathrm{Id}_{E'}.$$

□

LEMMA 2.16. *Let $K : \mathcal{B} \to \mathcal{A}$ be a full and faithful functor and let $f, g : X \to Y$ be morphisms in $\mathcal{B}$. If $(KE, Ke) = \mathrm{Equ}_{\mathcal{A}}(Kf, Kg)$ then $(E, e) = \mathrm{Equ}_{\mathcal{B}}(f, g)$.*

Proof. Since K is faithful, from $(Kf) \circ (Ke) = (Kg) \circ (Ke)$ we get that $f \circ e = g \circ e$. Let $h : Z \to X$ be a morphism in $\mathcal{B}$ such that $f \circ h = g \circ h$. Then in $\mathcal{A}$ we get $(Kf) \circ (Kh) = (Kg) \circ (Kh)$ and hence there exists a unique morphism $\xi : KZ \to KE$ such that $(Ke) \circ \xi = (Kh)$. Since $\xi \in \mathrm{Hom}_{\mathcal{A}}(KZ, KE)$ and K is full, there exists a morphism $\zeta \in \mathrm{Hom}_{\mathcal{B}}(Z, E)$ such that $\xi = K\zeta$. Since K is faithful, from $(Ke) \circ (K\zeta) = Kh$ we get $e \circ \zeta = h$. From the uniqueness of ξ, the one of ζ easily follows. □

LEMMA 2.17. *Let $\alpha, \gamma : F \to G$ be functorial morphisms where $F, G : \mathcal{A} \to \mathcal{B}$ are functors. Assume that, for every $X \in \mathcal{A}$ there exists $\mathrm{Equ}_{\mathcal{B}}(\alpha X, \gamma X)$. Let $(E, i) = \mathrm{Equ}_{\mathrm{Fun}}(\alpha, \gamma)$, where $i : E \to F$. Then, for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ we have that*

$$\left(\mathrm{Hom}_{\mathcal{B}}(Y, EX), \mathrm{Hom}_{\mathcal{B}}(Y, iX)\right) = \mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X), \mathrm{Hom}_{\mathcal{B}}(Y, \gamma X)\right)$$

which means that

$$\left(\mathrm{Hom}_{\mathcal{B}}(-, E), \mathrm{Hom}_{\mathcal{B}}(-, i)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{B}}(-, \alpha), \mathrm{Hom}_{\mathcal{B}}(-, \gamma)\right)$$

where

$$\mathrm{Hom}_{\mathcal{B}}(-, E) \text{ and } \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{B}}(-, \alpha), \mathrm{Hom}_{\mathcal{B}}(-, \gamma)\right) : \mathcal{B}^{op} \times \mathcal{A} \to \mathrm{Sets}.$$

Proof. We have that

$$\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X) \circ \mathrm{Hom}_{\mathcal{B}}(Y, iX) = \mathrm{Hom}_{\mathcal{B}}(Y, (\alpha X) \circ (iX))$$
$$= \mathrm{Hom}_{\mathcal{B}}(Y, (\gamma X) \circ (iX)) = \mathrm{Hom}_{\mathcal{B}}(Y, \gamma X) \circ \mathrm{Hom}_{\mathcal{B}}(Y, iX)$$

i.e. $\mathrm{Hom}_{\mathcal{B}}(Y, iX)$ equalizes $\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X)$ and $\mathrm{Hom}_{\mathcal{B}}(Y, \gamma X)$, for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Let now $\zeta : Z \to \mathrm{Hom}_{\mathcal{B}}(Y, FX)$ be a map such that $\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X) \circ \zeta = \mathrm{Hom}_{\mathcal{B}}(Y, \gamma X) \circ \zeta$. Then, for every $X \in \mathcal{A}$, $Y \in \mathcal{B}$ and for every $z \in Z$ we have

$$\begin{aligned}(\alpha X) \circ \zeta(z) &= \mathrm{Hom}_{\mathcal{B}}(Y, \alpha X)(\zeta(z)) = \mathrm{Hom}_{\mathcal{B}}(Y, \gamma X) \circ (\zeta(z)) \\ &= (\gamma X) \circ \zeta(z).\end{aligned}$$

Then, for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ there exists a unique morphism $\theta_z : Y \to EX$ in $\mathcal{B}$ such that

$$(iX) \circ \theta_z = \zeta(z)$$

i.e.

$$\mathrm{Hom}_{\mathcal{B}}(Y, iX)(\theta_z) = \zeta(z).$$

The assignment $z \mapsto \theta_z$ defines a map $\theta : Z \to \mathrm{Hom}_{\mathcal{B}}(Y, EX)$ such that $\mathrm{Hom}_{\mathcal{B}}(Y, iX) \circ \theta = \zeta$. □

2.2. **Contractible Equalizers and Coequalizers.**

Definition 2.18. Let $\mathcal{C}$ be a category. A *contractible (*or *split) equalizer* is a eightuple $(Z, X, Y, d, d_0, d_1, s, t)$ where

$$\begin{tikzcd} Z \arrow[r, shift left, "d"] & X \arrow[l, shift left, "s"] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "t"'] \end{tikzcd}$$

such that

$$\begin{aligned} t \circ d_0 &= \mathrm{Id}_X \\ s \circ d &= \mathrm{Id}_Z \\ t \circ d_1 &= d \circ s \\ d_0 \circ d &= d_1 \circ d. \end{aligned}$$

Proposition 2.19. *Let $\mathcal{C}$ be a category and let $(Z, X, Y, d, d_0, d_1, s, t)$ be a contractible equalizer. Then $(Z, d) = \mathrm{Equ}_{\mathcal{C}}(d_0, d_1)$.*

Proof. Let $\xi : L \to X$ be such that

$$d_0 \circ \xi = d_1 \circ \xi$$

then

$$\xi = \mathrm{Id}_X \circ \xi = t \circ d_0 \circ \xi = t \circ d_1 \circ \xi = d \circ (s \circ \xi).$$

Let

$$\xi' = s \circ \xi : L \to Z$$

so that

$$\xi = d \circ \xi'.$$

Let now $\xi'' : L \to Z$ be such that $d \circ \xi'' = \xi$. Then

$$\xi'' = \mathrm{Id}_Z \circ \xi'' = s \circ d \circ \xi'' = s \circ \xi = \xi'.$$

□

Proposition 2.20. *Let $\mathcal{C}$ be a category, let $(Z, X, Y, d, d_0, d_1, s, t)$ be a contractible equalizer and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then*

$$\begin{tikzcd} FZ \arrow[r, shift left, "Fd"] & FX \arrow[l, shift left, "Fs"] \arrow[r, shift left=2, "Fd_0"] \arrow[r, shift right=2, "Fd_1"'] & FY \arrow[l, "Ft"'] \end{tikzcd}$$

is a contractible equalizer in $\mathcal{D}$.

Proof. Since functors preserve composition, the statement is proved. □

Proposition 2.21. *Assume that*

$$\begin{tikzcd} Z \arrow[r, "d"] & X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y \end{tikzcd}$$

is an equalizer and there exists $t : Y \to X$ such that

$$\begin{aligned} t \circ d_0 &= \mathrm{Id}_X \\ d_1 \circ t \circ d_1 &= d_0 \circ t \circ d_1 \end{aligned}$$

Then there exists $s : X \to Z$ such that $(Z, X, Y, d, d_0, d_1, s, t)$ is a contractible equalizer.

Proof. Since $d_1 \circ t \circ d_1 = d_0 \circ t \circ d_1$ and $(Z, d) = \mathrm{Equ}\,(d_0, d_1)$, there exists $s : X \to Z$ such that

$$t \circ d_1 = d \circ s.$$

Let us compute

$$d \circ s \circ d = t \circ d_1 \circ d = t \circ d_0 \circ d = d$$

and since d is mono we get

$$s \circ d = \mathrm{Id}_Z.$$

□

Definition 2.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. An *F-contractible equalizer pair* is a parallel pair

$$\begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ such that there exists a contractible equalizer

$$\begin{tikzcd} D \arrow[r, shift left, "d"] & FX \arrow[l, shift left, "s"] \arrow[r, shift left=2, "Fd_0"] \arrow[r, shift right=2, "Fd_1"'] & FY \arrow[l, "t"'] \end{tikzcd}$$

in $\mathcal{D}$.

All the previous results can be considered in the opposite category so that they give the dual notion, namely contractible coequalizers.

Definition 2.23. Let $\mathcal{C}$ be a category. A *contractible coequalizer* is a eightuple $(C, X, Y, c, d_0, d_1, u, v)$ where

$$\begin{tikzcd} X \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "v"'] \arrow[r, shift left, "c"] & C \arrow[l, shift left, "u"] \end{tikzcd}$$

such that

$$\begin{aligned} d_0 \circ v &= \mathrm{Id}_Y \\ d_1 \circ v &= u \circ c \\ c \circ u &= \mathrm{Id}_C \\ c \circ d_0 &= c \circ d_1. \end{aligned}$$

PROPOSITION 2.24 ([BW, Proposition 2 (a)]). *Let $\mathcal{C}$ be a category and let $(C, X, Y, c, d_0, d_1, u, v)$ be a contractible coequalizer. Then $(C, c) = \mathrm{Coequ}_{\mathcal{C}}(d_0, d_1)$.*

Proof. Dual to Proposition 2.24. □

DEFINITION 2.25. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. An *F-contractible coequalizer pair* is a parallel pair

$$\begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ such that there exists a contractible coequalizer

$$\begin{tikzcd} FX \arrow[r, shift left=2, "Fd_0"] \arrow[r, shift right=2, "Fd_1"'] & FY \arrow[l, "v"'] \arrow[r, shift left, "c"] & C \arrow[l, shift left, "u"] \end{tikzcd}$$

in $\mathcal{D}$.

2.3. **Adjunction.**

2.26. Let $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ be functors. Recall that L is called a *left adjoint* of R, or R is called a *right adjoint* of L if there exists functorial morphisms

$$\eta : \mathrm{Id}_{\mathcal{B}} \to RL \qquad \text{and} \qquad \epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$$

such that

$$(\epsilon L) \circ (L\eta) = L \text{ and } (R\epsilon) \circ (\eta R) = R.$$

In this case we also say that (L, R) is an *adjunction* and η is called the *unit* of the adjunction while ϵ is called the *counit* of the adjunction. Let

$$a_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(LY, X) \to \mathrm{Hom}_{\mathcal{B}}(Y, RX)$$

be the isomorphism of the adjunction (L, R). Then, for every $\xi \in \mathrm{Hom}_{\mathcal{A}}(LY, X)$ and for every $\zeta \in \mathrm{Hom}_{\mathcal{B}}(Y, RX)$ we also have

$$a_{X,Y}(\xi) = (R\xi) \circ (\eta Y) \qquad \text{and} \qquad a_{X,Y}^{-1}(\zeta) = (\epsilon X) \circ (L\zeta).$$

Moreover, for every $X \in \mathcal{A}, Y \in \mathcal{B}$, unit and counit of the adjunction are given by

$$\eta Y = a_{LY,Y}(\mathrm{Id}_{LY}) \qquad \text{and} \qquad \epsilon X = a_{X,RX}^{-1}(\mathrm{Id}_{RX}).$$

2.27. Let (L, R) be an adjunction. Then L preserves colimits and thus coequalizers and R preserves limits and thus equalizers. We also say that L is right exact and that R is left exact.

LEMMA 2.28. *Let (L, R) be an adjunction with unit η and counit ϵ, where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. For every $Y' \in \mathcal{B}$ the following conditions are equivalent:*

(1) $\mathcal{L}_{-,Y'} = a_{LY',-}^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(-, \eta Y')$ *is a functorial isomorphism*

(2) $\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')$ *is a functorial isomorphism*
(3) $\eta Y'$ *is an isomorphism (η is a functorial isomorphism).*

Proof. Since (L,R) is an adjunction, $a_{X,Y}:\mathrm{Hom}_{\mathcal{A}}(LY,X)\to\mathrm{Hom}_{\mathcal{B}}(Y,RX)$ is an isomorphism for every $X\in\mathcal{A}$ and for every $Y\in\mathcal{B}$, so that (1) is equivalent to (2).
(3) $\Rightarrow$ (2) Let $\eta^{-1}Y'$ be the two-sided inverse of $\eta Y'$. Then $\mathrm{Hom}_{\mathcal{B}}(-,\eta^{-1}Y')$ is the inverse of the functor $\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')$. In fact, let $f\in\mathrm{Hom}_{\mathcal{B}}(Y,Y')$ and compute

$$\begin{aligned}\left[\mathrm{Hom}_{\mathcal{B}}\left(-,\eta^{-1}Y'\right)\circ\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')\right](f)&=\mathrm{Hom}_{\mathcal{B}}\left(-,\eta^{-1}Y'\right)(\eta Y'\circ f)\\&=\left(\eta^{-1}Y'\right)\circ(\eta Y')\circ f=f\end{aligned}$$

and

$$\begin{aligned}\left[\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')\circ\mathrm{Hom}_{\mathcal{B}}\left(-,\eta^{-1}Y'\right)\right](f)&=\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')\left(\left(\eta^{-1}Y'\right)\circ f\right)\\&=(\eta Y')\circ\left(\eta^{-1}Y'\right)\circ f=f.\end{aligned}$$

Thus $\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')$ is a functorial isomorphism.
(2) $\Rightarrow$ (3) Since $\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')$ is a functorial isomorphism, in particular $\mathrm{Hom}_{\mathcal{B}}(RLY',\eta Y'):\mathrm{Hom}_{\mathcal{B}}(RLY',Y')\to\mathrm{Hom}_{\mathcal{B}}(RLY',RLY')$ is an isomorphism. Thus, there exists $f\in\mathrm{Hom}_{\mathcal{B}}(RLY',Y')$ such that $(\eta Y')\circ f=\mathrm{Id}_{RLY'}$ which implies that $\eta Y'$ is an epimorphism. Moreover we also have $\mathrm{Hom}_{\mathcal{B}}(Y',\eta Y')(\mathrm{Id}_{Y'})=\eta Y'=(\eta Y')\circ f\circ(\eta Y')=\mathrm{Hom}_{\mathcal{B}}(Y',\eta Y')(f\circ(\eta Y'))$. Since $\mathrm{Hom}_{\mathcal{B}}(-,\eta Y')$ is a functorial isomorphism, also $\mathrm{Hom}_{\mathcal{B}}(Y',\eta Y')$ is an isomorphism. Thus we deduce that $\mathrm{Id}_{Y'}=f\circ(\eta Y')$ which implies that $\eta Y'$ is also a monomorphism and moreover $\eta Y'$ has a two-sided inverse $f:RLY'\to Y'$. □

REMARK 2.29. Note that, for every $f\in\mathrm{Hom}_{\mathcal{B}}(Y,Y')$ we have

$$\begin{aligned}\mathcal{L}_{Y,Y'}(f)&=\left[a^{-1}_{LY',Y}\circ\mathrm{Hom}_{\mathcal{B}}(Y,\eta Y')\right](f)=a^{-1}_{LY',Y}(\eta Y'\circ f)\\&=(\epsilon LY')\circ(L\eta Y')\circ(Lf)\overset{(L,R)\mathrm{adj}}{=}Lf.\end{aligned}$$

LEMMA 2.30. *Let (L,R) be an adjunction with unit η and counit ϵ, where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. For every $X\in\mathcal{A}$ the following conditions are equivalent:*
(1) $\mathcal{R}_{X,-}=a_{-,RX}\circ\mathrm{Hom}_{\mathcal{A}}(\epsilon X,-)$ *is a functorial isomorphism*
(2) $\mathrm{Hom}_{\mathcal{A}}(\epsilon X,-)$ *is a functorial isomorphism*
(3) ϵX *is an isomorphism (ϵ is a functorial isomorphism).*

Proof. Dual to proof of Lemma 2.28. □

REMARK 2.31. Note that, for every $f\in\mathrm{Hom}_{\mathcal{A}}(X,X')$ we have

$$\begin{aligned}\mathcal{R}_{X,X'}(f)&=\left[a_{X',RX}\circ\mathrm{Hom}_{\mathcal{A}}(\epsilon X,X')\right](f)=a_{X',RX}(f\circ\epsilon X)=R(f\circ\epsilon X)\circ(\eta RX)\\&=(Rf)\circ(R\epsilon X)\circ(\eta RX)\overset{(L,R)\mathrm{adj}}{=}Rf.\end{aligned}$$

PROPOSITION 2.32. *Let (L,R) be an adjunction with unit η and counit ϵ, where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Then R is full and faithful if and only if ϵ is a functorial isomorphism.*

Proof. To be full and faithful for R means that the map

$$\phi:\mathrm{Hom}_{\mathcal{A}}(X,X')\longrightarrow\mathrm{Hom}_{\mathcal{B}}(RX,RX')$$

$$f \mapsto Rf$$

is bijective for every $X, X' \in \mathcal{A}$. Since this $\phi(f) = R(f) = \mathcal{R}_{X,X'}(f)$, ϕ is an isomorphism if and only if $\mathcal{R}_{X,X'}$ is an isomorphism for every $X, X' \in \mathcal{A}$ and, by Lemma 2.30, if and only if ϵX is an isomorphism for every $X \in \mathcal{A}$. $\square$

LEMMA 2.33. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ such that L is an equivalence of categories. Then R is also an equivalence of categories.*

Proof. By assumption $L : \mathcal{B} \to \mathcal{A}$ is an equivalence of categories with $R' : \mathcal{A} \to \mathcal{B}$. Then it is well-known that (L, R') is an adjunction. By the uniqueness of the adjoint, we have that $R \simeq R'$ which is an equivalence. Thus R is also an equivalence of categories. $\square$

3. Monads

DEFINITION 3.1. A *monad* on a category $\mathcal{A}$ is a triple $\mathbb{A} = (A, m_A, u_A)$, where $A : \mathcal{A} \to \mathcal{A}$ is a functor, $m_A : AA \to A$ and $u_A : \mathcal{A} \to A$ are functorial morphisms satisfying the associativity and the unitality conditions:

$$m_A \circ (m_A A) = m_A \circ (A m_A) \quad \text{and} \quad m_A \circ (A u_A) = A = m_A \circ (u_A A).$$

DEFINITION 3.2. A *morphism between two monads* $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ on a category $\mathcal{A}$ is a functorial morphism $\varphi : A \to B$ such that

$$\varphi \circ m_A = m_B \circ (\varphi\varphi) \quad \text{and} \quad \varphi \circ u_A = u_B.$$

EXAMPLE 3.3. Let $(\mathcal{A}, m_{\mathcal{A}}, u_{\mathcal{A}})$ be an R-ring where R is an algebra. Then

- $\mathcal{A}$ is an R-R-bimodule
- $m_{\mathcal{A}} : \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$ is a morphism of R-R-bimodules
- $u_{\mathcal{A}} : R \to \mathcal{A}$ is a morphism of R-R-bimodules satisfying the following

$$m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes_R \mathcal{A}) = m_{\mathcal{A}} \circ (\mathcal{A} \otimes_R m_{\mathcal{A}}), m_{\mathcal{A}} \circ (\mathcal{A} \otimes_R u_{\mathcal{A}}) = r_{\mathcal{A}} \text{ and } m_{\mathcal{A}} \circ (u_{\mathcal{A}} \otimes_R \mathcal{A}) = l_{\mathcal{A}}$$

where $r_{\mathcal{A}} : \mathcal{A} \otimes_R R \to \mathcal{A}$ and $l_{\mathcal{A}} : R \otimes_R \mathcal{A} \to \mathcal{A}$ are the right and left constraints. Let

$$\begin{aligned} A &= - \otimes_R \mathcal{A} : Mod\text{-}R \to Mod\text{-}R \\ m_A &= - \otimes_R m_{\mathcal{A}} : - \otimes_R \mathcal{A} \otimes_R \mathcal{A} \to - \otimes_R \mathcal{A} \\ u_A &= (- \otimes_R u_{\mathcal{A}}) \circ r_-^{-1} : - \to - \otimes_R R \to - \otimes_R \mathcal{A} \end{aligned}$$

We prove that $\mathbb{A} = (A, m_A, u_A)$ is a monad on the category $Mod\text{-}R$. For every $M \in Mod\text{-}R$ we compute

$$\begin{aligned} [m_A \circ (m_A A)](M) &= (M \otimes_R m_{\mathcal{A}}) \circ (M \otimes_R \mathcal{A} \otimes_R m_{\mathcal{A}}) = M \otimes_R [m_{\mathcal{A}} \circ (\mathcal{A} \otimes_R m_{\mathcal{A}})] \\ &= M \otimes_R [m_{\mathcal{A}} \circ (m_{\mathcal{A}} \otimes_R \mathcal{A})] = (M \otimes_R m_{\mathcal{A}}) \circ (M \otimes_R m_{\mathcal{A}} \otimes_R \mathcal{A}) \\ &= [m_A \circ (A m_A)](M) \end{aligned}$$

$$\begin{aligned} [m_A \circ (A u_A)](M) &= (M \otimes_R m_{\mathcal{A}}) \circ \left[(M \otimes_R u_{\mathcal{A}}) \circ r_M^{-1} \right] \otimes_R \mathcal{A} \\ &= (M \otimes_R m_{\mathcal{A}}) \circ (M \otimes_R u_{\mathcal{A}} \otimes_R \mathcal{A}) \circ \left(r_M^{-1} \otimes_R \mathcal{A} \right) \\ &= (M \otimes_R [m_{\mathcal{A}} \circ (u_{\mathcal{A}} \otimes_R \mathcal{A})]) \circ \left(r_M^{-1} \otimes_R \mathcal{A} \right) \\ &= (M \otimes_R l_{\mathcal{A}}) \circ \left(r_M^{-1} \otimes_R \mathcal{A} \right) = M \otimes_R \mathcal{A} = AM \end{aligned}$$

and

$$
\begin{aligned}
[m_A \circ (u_A A)](M) &= (M \otimes_R m_{\mathcal{A}}) \circ (M \otimes_R \mathcal{A} \otimes_R u_{\mathcal{A}}) \circ r^{-1}_{M \otimes_R \mathcal{A}} \\
&= (M \otimes_R [m_{\mathcal{A}} \circ (\mathcal{A} \otimes_R u_{\mathcal{A}})]) \circ r^{-1}_{M \otimes_R \mathcal{A}} \\
&= (M \otimes_R r_{\mathcal{A}}) \circ r^{-1}_{M \otimes_R \mathcal{A}} = M \otimes_R \mathcal{A} = AM.
\end{aligned}
$$

PROPOSITION 3.4 ([H]). *Let (L, R) be an adjunction with unit η and counit ϵ where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $\mathbb{RL} = (RL, R\epsilon L, \eta)$ is a monad on the category $\mathcal{B}$.*

Proof. We have to prove that

$(R\epsilon L) \circ (RLR\epsilon L) = (R\epsilon L) \circ (R\epsilon LRL)$ and $(R\epsilon L) \circ RL\eta = RL = (R\epsilon L) \circ (\eta RL)$.

In fact we have

$$(R\epsilon L) \circ (RLR\epsilon L) \overset{\epsilon}{=} (R\epsilon L) \circ (R\epsilon LRL)$$

and

$$(R\epsilon L) \circ RL\eta \overset{(L,R)}{=} RL \overset{(L,R)}{=} (R\epsilon L) \circ (\eta RL).$$

$\square$

DEFINITION 3.5. A *left module functor* for a monad $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ is a pair $\left(Q, {}^A\mu_Q\right)$ where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and ${}^A\mu_Q : AQ \to Q$ is a functorial morphism satisfying:

$${}^A\mu_Q \circ \left(A\,{}^A\mu_Q\right) = {}^A\mu_Q \circ (m_A Q) \quad \text{and} \quad Q = {}^A\mu_Q \circ (u_A Q).$$

DEFINITION 3.6. A *right module functor* for a monad $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ is a pair $\left(P, \mu_P^A\right)$ where $P : \mathcal{A} \to \mathcal{B}$, is a functor and $\mu_P^A : PA \to P$ is a functorial morphism such that

$$\mu_P^A \circ \left(\mu_P^A A\right) = \mu_P^A \circ (P m_A) \quad \text{and} \quad P = \mu_P^A \circ (P u_A).$$

REMARK 3.7. Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(Q, {}^A\mu_Q\right)$ be a left $\mathbb{A}$-module functor and $\left(P, \mu_P^A\right)$ be a right $\mathbb{A}$-module functor. By the unitality property of ${}^A\mu_Q$ and μ_P^A we deduce that they are both epimorphism.

DEFINITION 3.8. For two monads $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ and $\mathbb{B} = (B, m_B, u_B)$ on a category $\mathcal{B}$, a $\mathbb{A}$-$\mathbb{B}$-*bimodule functor* is a triple $\left(Q, {}^A\mu_Q, \mu_Q^B\right)$, where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and $\left(Q, {}^A\mu_Q\right)$ is a left $\mathbb{A}$-module functor, $\left(Q, \mu_Q^B\right)$ is a right $\mathbb{B}$-module functor such that in addition

$${}^A\mu_Q \circ \left(A\mu_Q^B\right) = \mu_Q^B \circ \left({}^A\mu_Q B\right).$$

DEFINITION 3.9. A *module* for a monad $\mathbb{A} = (A, m_A, u_A)$ on a category $\mathcal{A}$ is a pair $\left(X, {}^A\mu_X\right)$ where $X \in \mathcal{A}$ and ${}^A\mu_X : AX \to X$ is a morphism in $\mathcal{A}$ such that

$${}^A\mu_X \circ \left(A\,{}^A\mu_X\right) = {}^A\mu_X \circ (m_A X) \quad \text{and} \quad X = {}^A\mu_X \circ (u_A X).$$

A *morphism between two $\mathbb{A}$-modules* $\left(X, {}^A\mu_X\right)$ and $\left(X', {}^A\mu_{X'}\right)$ is a morphism $f : X \to X'$ in $\mathcal{A}$ such that

$${}^A\mu_{X'} \circ (Af) = f \circ {}^A\mu_X.$$

We will denote by ${}_{\mathbb{A}}\mathcal{A}$ the category of $\mathbb{A}$-modules and their morphisms. This is the so-called *Eilenberg-Moore category* which is sometimes also denoted by $\mathcal{A}^{\mathbb{A}}$.

Remark 3.10. Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$. From the unitality property of ${}^A\mu_X$ we deduce that ${}^A\mu_X$ is epi for every $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ and that $u_A X$ is mono for every $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$, i.e. u_A is a monomorphism.

Definition 3.11. Corresponding to a monad $\mathbb{A} = (A, m_A, u_A)$ on $\mathcal{A}$, there is an adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ where ${}_{\mathbb{A}}U$ is the forgetful functor and ${}_{\mathbb{A}}F$ is the free functor

$$\begin{array}{rccc} {}_{\mathbb{A}}U: & {}_{\mathbb{A}}\mathcal{A} & \to & \mathcal{A} \\ & \left(X, {}^A\mu_X\right) & \to & X \\ & f & \to & f \end{array} \qquad \begin{array}{rccc} {}_{\mathbb{A}}F: & \mathcal{A} & \to & {}_{\mathbb{A}}\mathcal{A} \\ & X & \to & (AX, m_A X) \\ & f & \to & Af. \end{array}$$

Note that ${}_{\mathbb{A}}U {}_{\mathbb{A}}F = A$. The unit of this adjunction is given by the unit u_A of the monad $\mathbb{A}$:

$$u_A : \mathcal{A} \to {}_{\mathbb{A}}U {}_{\mathbb{A}}F = A.$$

The counit $\lambda_A : {}_{\mathbb{A}}F {}_{\mathbb{A}}U \to {}_{\mathbb{A}}\mathcal{A}$ of this adjunction is defined by setting

$${}_{\mathbb{A}}U\left(\lambda_A\left(X, {}^A\mu_X\right)\right) = {}^A\mu_X \text{ for every } \left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}.$$

Therefore we have

$$(\lambda_{A\mathbb{A}}F) \circ ({}_{\mathbb{A}}F u_A) = {}_{\mathbb{A}}F \qquad \text{and} \qquad ({}_{\mathbb{A}}U \lambda_A) \circ (u_{A\mathbb{A}}U) = {}_{\mathbb{A}}U.$$

Proposition 3.12. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then ${}_{\mathbb{A}}U$ is a faithful functor. Moreover, given $Z, W \in {}_{\mathbb{A}}\mathcal{A}$ we have that*

$$Z = W \textit{ if and only if } {}_{\mathbb{A}}U(Z) = {}_{\mathbb{A}}U(W) \textit{ and } {}_{\mathbb{A}}U(\lambda_A Z) = {}_{\mathbb{A}}U(\lambda_A W).$$

In particular, if $F, G : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ are functors, we have

$$F = G \textit{ if and only if } {}_{\mathbb{A}}UF = {}_{\mathbb{A}}UG \textit{ and } {}_{\mathbb{A}}U(\lambda_A F) = {}_{\mathbb{A}}U(\lambda_A G)$$

Proposition 3.13. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$. Then $({}_{\mathbb{A}}U, ({}_{\mathbb{A}}U\lambda_A))$ is a left $\mathbb{A}$-module functor.*

Proof. We have to prove that

$$\begin{aligned} ({}_{\mathbb{A}}U_A\lambda) \circ (A {}_{\mathbb{A}}U_A\lambda) &= ({}_{\mathbb{A}}U_A\lambda) \circ (m_{A\mathbb{A}}U) \text{ and} \\ ({}_{\mathbb{A}}U_A\lambda) \circ (u_{A\mathbb{A}}U) &= {}_{\mathbb{A}}U. \end{aligned}$$

Let us consider $\left(X, {}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$. We have to show that

$$\left({}_{\mathbb{A}}U_A\lambda\left(X, {}^A\mu_X\right)\right) \circ \left(A {}_{\mathbb{A}}U_A\lambda\left(X, {}^A\mu_X\right)\right) = \left({}_{\mathbb{A}}U_A\lambda\left(X, {}^A\mu_X\right)\right) \circ \left(m_{A\mathbb{A}}U\left(X, {}^A\mu_X\right)\right)$$

and that

$$\left({}_{\mathbb{A}}U_A\lambda\left(X, {}^A\mu_X\right)\right) \circ \left(u_{A\mathbb{A}}U\left(X, {}^A\mu_X\right)\right) = {}_{\mathbb{A}}U\left(X, {}^A\mu_X\right)$$

i.e. that

$${}^A\mu_X \circ \left(A {}^A\mu_X\right) = {}^A\mu_X \circ (m_A X) \qquad \text{and} \qquad {}^A\mu_X \circ (u_A X) = X$$

which hold since $\left(X, {}^A\mu_X\right)$ is an $\mathbb{A}$-module. $\square$

Proposition 3.14. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(X, {}^A\mu_X\right)$ be a module for $\mathbb{A}$. Then we have*

$$\left(X, {}^A\mu_X\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_X, m_A X\right).$$

In particular if $\left(Q, {}^A\mu_Q\right)$ is a left $\mathbb{A}$-module functor, then we have

$$\left(Q, {}^A\mu_Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A^A\mu_Q, m_A Q\right).$$

Proof. Note that

$$\begin{tikzcd} AAX \arrow[r, shift left=2, "m_A X"] \arrow[r, shift right=2, "{}^A\mu_X"'] & AX \arrow[l, "u_A AX"'] \arrow[r, shift left, "{}^A\mu_X"] & X \arrow[l, shift left, "u_A X"] \end{tikzcd}$$

is a contractible coequalizer and thus, by Proposition 2.24, $\left(X, {}^A\mu_X\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_X, m_A X\right)$. Let now $\left(Q, {}^A\mu_Q\right)$ be a left $\mathbb{A}$-module functor where $Q : \mathcal{B} \to \mathcal{A}$. Then, by the foregoing, for every $Y \in \mathcal{B}$ we have that

$$\left(QY, {}^A\mu_Q Y\right) = \left(QY, {}^A\mu_{QY}\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_{QY}, m_A QY\right) = \mathrm{Coequ}_{\mathcal{A}}\left(A^A\mu_Q Y, m_A QY\right).$$

Then, by Lemma 2.7, we have that $\left(Q, {}^A\mu_Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A^A\mu_Q, m_A Q\right)$. □

Corollary 3.15. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ be the associated adjunction. Then $({}_{\mathbb{A}}U, ({}_{\mathbb{A}}U\lambda_A))$ is a left $\mathbb{A}$-module functor and*

$$({}_{\mathbb{A}}U, ({}_{\mathbb{A}}U\lambda_A)) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A_{\mathbb{A}}U\lambda_A, m_{A\mathbb{A}}U\right).$$

Proof. By Proposition 3.13 $({}_{\mathbb{A}}U, ({}_{\mathbb{A}}U\lambda_A))$ is a left $\mathbb{A}$-module functor. By Proposition 3.14 we get that $({}_{\mathbb{A}}U, ({}_{\mathbb{A}}U\lambda_A)) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A_{\mathbb{A}}U\lambda_A, m_{A\mathbb{A}}U\right)$. □

Proposition 3.16. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(P, \mu_P^A\right)$ be a right $\mathbb{A}$-module functor, then we have*

$$\left(P, \mu_P^A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu_P^A A, P m_A\right). \tag{2}$$

Proof. Note that

$$\begin{tikzcd} PAA \arrow[r, shift left=2, "P m_A"] \arrow[r, shift right=2, "\mu_P^A A"'] & PA \arrow[l, "P A u_A"'] \arrow[r, shift left, "\mu_P^A"] & P \arrow[l, shift left, "P u_A"] \end{tikzcd}$$

is a contractible coequalizer and thus, by Proposition 2.24, $\left(P, \mu_P^A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu_P^A A, P m_A\right)$. □

Lemma 3.17. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\left(Q, {}^A\mu_Q\right)$ be a left and $\left(P, \mu_P^A\right)$ be a right $\mathbb{A}$-module functors where $Q : \mathcal{Q} \to \mathcal{A}$ and $P : \mathcal{A} \to \mathcal{P}$. Let $F : \mathcal{X} \to \mathcal{Q}$ and $G : \mathcal{P} \to \mathcal{B}$ be functors. Then*

(1) *$\left(QF, {}^A\mu_Q F\right)$ is a left $\mathbb{A}$-module functor and*
(2) *$\left(GP, G\mu_P^A\right)$ is a right $\mathbb{A}$-module functor.*

Proof. From

$${}^A\mu_Q \circ \left(A^A\mu_Q\right) = {}^A\mu_Q \circ (m_A Q) \quad \text{and} \quad Q = {}^A\mu_Q \circ (u_A Q)$$

we deduce that

$${}^A\mu_Q F \circ \left(A^A\mu_Q F\right) = {}^A\mu_Q F \circ (m_A QF) \quad \text{and} \quad QF = {}^A\mu_Q F \circ (u_A QF)$$

and from

$$\mu_P^A \circ \left(\mu_P^A A\right) = \mu_P^A \circ (Pm_A) \quad \text{and} \quad P = \mu_P^A \circ (PA)$$

we deduce that

$$G\mu_P^A \circ \left(G\mu_P^A A\right) = G\mu_P^A \circ (GPm_A) \quad \text{and} \quad GP = G\mu_P^A \circ (GPA)\,.$$

$\square$

PROPOSITION 3.18. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ be the adjunction associated. Then ${}_{\mathbb{A}}U$ reflects isomorphisms.*

Proof. Let $f : \left(X, {}^{A}\mu_X\right) \to \left(Y, {}^{A}\mu_Y\right)$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}Uf$ has a two-sided inverse f^{-1} in $\mathcal{A}$. Since

$${}^{A}\mu_{X'} \circ (Af) = f \circ {}^{A}\mu_X$$

we get that

$$f^{-1} \circ {}^{A}\mu_{X'} = {}^{A}\mu_X \circ \left(Af^{-1}\right).$$

$\square$

LEMMA 3.19 ([BMV, Lemma 4.1]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$, let $\left(P, \mu_P^A\right)$ be a right $\mathbb{A}$-module functor and let $\left(Q, {}^{A}\mu_Q\right)$ be a left $\mathbb{A}$-module functor where $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$. Then any coequalizer preserved by PA is also preserved by P and any coequalizer preserved by AQ is also preserved by Q.*

Proof. Consider the following coequalizer

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "z"] & Z \end{tikzcd}$$

in the category $\mathcal{A}$ and assume that PA preserves it. By applying to it the functors PA and P we get the following diagram in $\mathcal{B}$

$$\begin{tikzcd} PAX \arrow[r, shift left, "PAf"] \arrow[r, shift right, "PAg"'] \arrow[d, shift left, "\mu_P^A X"] & PAY \arrow[r, "PAz"] \arrow[d, shift left, "\mu_P^A Y"] & PAZ \arrow[d, shift left, "\mu_P^A Z"] \\ PX \arrow[r, shift left, "Pf"] \arrow[r, shift right, "Pg"'] \arrow[u, shift left, "Pu_A X"] & PY \arrow[r, "Pz"] \arrow[u, shift left, "Pu_A Y"] & PZ. \arrow[u, shift left, "Pu_A Z"] \end{tikzcd}$$

By assumption, the first row is a coequalizer. Assume that there exists a morphism $h : PY \to H$ such that

$$h \circ (Pf) = h \circ (Pg)\,.$$

Then, by composing with $\mu_P^A X$ we get

$$h \circ (Pf) \circ \left(\mu_P^A X\right) = h \circ (Pg) \circ \left(\mu_P^A X\right)$$

and since μ_P^A is a functorial morphism we obtain

$$h \circ \left(\mu_P^A Y\right) \circ (PAf) = h \circ \left(\mu_P^A Y\right) \circ (PAg)\,.$$

Since $(PAZ, PAz) = \mathrm{Coequ}_{\mathcal{B}}(PAf, PAg)$, there exists a unique morphism $k : PAZ \to H$ such that

$$k \circ (PAz) = h \circ \left(\mu_P^A Y\right). \tag{3}$$

By composing with Pu_AY we get

$$k\circ(PAz)\circ(Pu_AY)=h\circ\left(\mu_P^AY\right)\circ(Pu_AY)$$

and thus

$$k\circ(Pu_AZ)\circ(Pz)=h.$$

Let $l:=k\circ(Pu_AZ):PZ\to H$. Then we have

$$\begin{aligned}l\circ(Pz)=k\circ(Pu_AZ)\circ(Pz)&\overset{u_A}{=}k\circ(PAz)\circ(Pu_AY)\\&\overset{(3)}{=}h\circ\left(\mu_P^AY\right)\circ(Pu_AY)=h.\end{aligned}$$

Let $l':PZ\to H$ be another morphism such that

$$l'\circ(Pz)=h.$$

Then we have

$$\begin{aligned}l\circ\left(\mu_P^AZ\right)\circ(PAz)&=l\circ(Pz)\circ\left(\mu_P^AY\right)=h\circ\left(\mu_P^AY\right)\\&=l'\circ(Pz)\circ\left(\mu_P^AY\right)=l'\circ\left(\mu_P^AZ\right)\circ(PAz).\end{aligned}$$

Since PA preserves coequalizers, we have that PAz is an epimorphism. Since μ_P^AZ is also an epimorphism, we deduce that $l=l'$. Therefore we obtain that $(PZ,Pz)=\mathrm{Coequ}_{\mathcal{B}}(Pf,Pg)$. The second statement can be proved similarly. We consider the above coequalizer

$$\begin{tikzcd}X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "z"] & Z\end{tikzcd}$$

in the category $\mathcal{B}$ and assume that AQ preserves it. By applying to it the functors AQ and Q we get the following diagram in $\mathcal{A}$

$$\begin{tikzcd}AQX \arrow[r, shift left, "AQf"] \arrow[r, shift right, "AQg"'] \arrow[d, shift left, "{}^A\mu_QX"] & AQY \arrow[r, "AQz"] \arrow[d, shift left, "{}^A\mu_QY"] & AQZ \arrow[d, shift left, "{}^A\mu_QZ"] \\ QX \arrow[r, shift left, "Qf"] \arrow[r, shift right, "Qg"'] \arrow[u, shift left, "u_AQX"] & QY \arrow[r, "Qz"] \arrow[u, shift left, "u_AQY"] & QZ \arrow[u, shift left, "u_AQZ"]\end{tikzcd}$$

By assumption, the first row is a coequalizer. Assume that there exists a morphism $h:QY\to H$ such that

$$h\circ(Qf)=h\circ(Qg).$$

Then, by composing with ${}^A\mu_QX$ we get

$$h\circ(Qf)\circ\left({}^A\mu_QX\right)=h\circ(Qg)\circ\left({}^A\mu_QX\right)$$

and since ${}^A\mu_Q$ is a functorial morphism we obtain

$$h\circ\left({}^A\mu_QY\right)\circ(AQf)=h\circ\left({}^A\mu_QY\right)\circ(AQg).$$

Since $(AQZ,AQz)=\mathrm{Coequ}_{\mathcal{B}}(AQf,AQg)$, there exists a unique morphism $k:AQZ\to H$ such that

$$k\circ(AQz)=h\circ\left({}^A\mu_QY\right). \tag{4}$$

By composing with u_AQY we get

$$k\circ(AQz)\circ(u_AQY)=h\circ\left({}^A\mu_QY\right)\circ(u_AQY)$$

and thus

$$k \circ (u_A QZ) \circ (Qz) = h.$$

Let $l := k \circ (u_A QZ) : QZ \to H$. Then we have

$$\begin{aligned} l \circ (Qz) = k \circ (u_A QZ) \circ (Qz) &\overset{u_A}{=} k \circ (AQz) \circ (u_A QY) \\ &\overset{(4)}{=} h \circ \left({}^A\mu_Q Y\right) \circ (u_A QY) = h. \end{aligned}$$

Let $l' : QZ \to H$ be another morphism such that

$$l' \circ (Qz) = h.$$

Then we have

$$\begin{aligned} l \circ \left({}^A\mu_Q Z\right) \circ (AQz) &= l \circ (Qz) \circ \left({}^A\mu_Q Y\right) = h \circ \left({}^A\mu_Q Y\right) \\ &= l' \circ (Qz) \circ \left({}^A\mu_Q Y\right) = l' \circ \left({}^A\mu_Q Z\right) \circ (AQz). \end{aligned}$$

Since AQ preserves coequalizers, we have that AQz is an epimorphism. Since ${}^A\mu_Q Z$ is also an epimorphism, we deduce that $l = l'$. Therefore we obtain that $(QZ, Qz) = \mathrm{Coequ}_{\mathcal{B}}(Qf, Qg)$. □

Lemma 3.20 ([BMV, Lemma 4.2]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $f, g : \left(X, {}^A\mu_X\right) \to \left(Y, {}^A\mu_Y\right)$ be morphisms in ${}_{\mathbb{A}}\mathcal{A}$. Assume that there exists $(C, c) = \mathrm{Coequ}_{\mathcal{A}}({}_{\mathbb{A}}Uf, {}_{\mathbb{A}}Ug)$ and assume that AA preserves coequalizers. Then there exists $(\Gamma, \gamma) = \mathrm{Coequ}_{{}_{\mathbb{A}}\mathcal{A}}(f, g)$ and ${}_{\mathbb{A}}U(\Gamma, \gamma) = (C, c)$.*

Proof. Since AA preserves coequalizers and (A, m_A) is a right $\mathbb{A}$-module functor, also A preserves coequalizers by Lemma 3.19, in particular, A preserves (C, c). Since

$$\begin{aligned} c \circ {}^A\mu_Y \circ (A_{\mathbb{A}}Uf) &\overset{f \in {}_{\mathbb{A}}\mathcal{A}}{=} c \circ ({}_{\mathbb{A}}Uf) \circ {}^A\mu_X \\ &\overset{c\,\mathrm{coequ}}{=} c \circ ({}_{\mathbb{A}}Ug) \circ {}^A\mu_X \overset{g \in {}_{\mathbb{A}}\mathcal{A}}{=} c \circ {}^A\mu_Y \circ (A_{\mathbb{A}}Ug) \end{aligned}$$

by the universal property of the coequalizer (AC, Ac) there exists a unique morphism ${}^A\mu_C : AC \to C$ such that

$$c \circ {}^A\mu_Y = {}^A\mu_C \circ (Ac).$$

Moreover, by composing with $u_A Y$ this identity we get

$$c = {}^A\mu_C \circ (Ac) \circ (u_A Y) \overset{u_A}{=} {}^A\mu_C \circ (u_A C) \circ c.$$

Since c is an epimorphism we obtain

$$C = {}^A\mu_C \circ (u_A C).$$

Now, consider the following serially commutative diagram

$$\begin{tikzcd}
AAX \arrow[r, shift left, "m_A X"] \arrow[r, shift right, "A{}^A\mu_X"'] \arrow[d, shift right, "AA_{\mathbb{A}}Uf"'] \arrow[d, shift left, "AA_{\mathbb{A}}Ug"] & AX \arrow[r, "{}^A\mu_X"] \arrow[d, shift right, "A_{\mathbb{A}}Uf"'] \arrow[d, shift left, "A_{\mathbb{A}}Ug"] & X \arrow[d, shift right, "{}_{\mathbb{A}}Uf"'] \arrow[d, shift left, "{}_{\mathbb{A}}Ug"] \\
AAY \arrow[r, shift left, "m_A Y"] \arrow[r, shift right, "AA^{\mu}_Y"'] \arrow[d, "AAc"] & AY \arrow[r, "{}^A\mu_Y"] \arrow[d, "Ac"] & Y \arrow[d, "c"] \\
AAC \arrow[r, shift left, "m_A C"] \arrow[r, shift right, "AA^{\mu}_C"'] & AC \arrow[r, "{}^A\mu_C"] & C.
\end{tikzcd}$$

Since we already observed that the columns are coequalizers and also the first and the second row are coequalizers by Proposition 3.14, in view of Lemma 2.11 also the third row is a coequalizer, so that (C,c) has a left $\mathbb{A}$-module structure, i.e. there exists $(\Gamma,\gamma) \in {}_{\mathbb{A}}\mathcal{A}$ such that $(\Gamma,\gamma) = \mathrm{Coequ}_{{}_{\mathbb{A}}\mathcal{A}}(f,g)$ and ${}_{\mathbb{A}}U(\Gamma,\gamma) = (C,c)$. $\square$

LEMMA 3.21 ([BMV, Lemma 4.3]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ with coequalizers and let $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ be the adjunction associated. The following statements are equivalent:*

(*i*) *$A : \mathcal{A} \to \mathcal{A}$ preserves coequalizers*
(*ii*) *$AA : \mathcal{A} \to \mathcal{A}$ preserves coequalizers*
(*iii*) *${}_{\mathbb{A}}\mathcal{A}$ has coequalizers and they are preserved by ${}_{\mathbb{A}}U : {}_{\mathbb{A}}\mathcal{A} \to \mathcal{A}$*
(*iv*) *${}_{\mathbb{A}}U : {}_{\mathbb{A}}\mathcal{A} \to \mathcal{A}$ preserves coequalizers.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are clear.
$(ii) \Rightarrow (iii)$ follows by Lemma 3.20.
$(iv) \Rightarrow (i)$ Note that ${}_{\mathbb{A}}F$ is a left adjoint, so that in particular it preserves coequalizers. Then ${}_{\mathbb{A}}U {}_{\mathbb{A}}F = A$ also preserves coequalizers. $\square$

LEMMA 3.22. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ and assume that A preserves equalizers. Then ${}_{\mathbb{A}}F$ preserves equalizers where $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ is the adjunction associated to the monad.*

Proof. Let

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

be an equalizer in $\mathcal{A}$. Let us consider the fork obtained by applying the functor ${}_{\mathbb{A}}F$ to the equalizer

$${}_{\mathbb{A}}FE \xrightarrow{{}_{\mathbb{A}}Fe} {}_{\mathbb{A}}FX \underset{{}_{\mathbb{A}}Fg}{\overset{{}_{\mathbb{A}}Ff}{\rightrightarrows}} {}_{\mathbb{A}}FY$$

i.e.

$$(AE, m_AE) \xrightarrow{Ae} (AX, m_AX) \underset{Ag}{\overset{Af}{\rightrightarrows}} (AY, m_AY)$$

Now, let $\left(Z, {}^A\mu_Z\right) \in {}_{\mathbb{A}}\mathcal{A}$ and $z : \left(Z, {}^A\mu_Z\right) \to (AX, m_AX)$ be a morphism in ${}_{\mathbb{A}}\mathcal{A}$ such that $(Af) \circ z = (Ag) \circ z$. Since A preserves equalizers, we know that $(AE, Ae) = \mathrm{Equ}_{\mathcal{A}}(Af, Ag)$. By the universal property of the equalizer (AE, Ae) in $\mathcal{A}$, there exists a unique morphism $z' : Z \to AE$ in $\mathcal{A}$ such that $(Ae) \circ z' = z$. We now want to prove that z' is a morphism in ${}_{\mathbb{A}}\mathcal{A}$, i.e. that $(m_AE) \circ (Az') = z' \circ {}^A\mu_Z$. Since z is a morphism in ${}_{\mathbb{A}}\mathcal{A}$ we have that

$$(m_AX) \circ (Az) = z \circ {}^A\mu_Z$$

and since also Ae is a morphism in ${}_{\mathbb{A}}\mathcal{A}$ we have that

$$(m_AX) \circ (AAe) = (Ae) \circ (m_AE).$$

Then we have

$$\begin{aligned} &(Ae) \circ (m_AE) \circ (Az') \overset{Ae \in {}_{\mathbb{A}}\mathcal{A}}{=} (m_AX) \circ (AAe) \circ (Az') \\ &\overset{\mathrm{prop}\, z}{=} (m_AX) \circ (Az) \overset{z \in {}_{\mathbb{A}}\mathcal{A}}{=} z \circ {}^A\mu_Z \overset{\mathrm{prop}\, z}{=} (Ae) \circ z' \circ {}^A\mu_Z \end{aligned}$$

and since A preserves equalizers, Ae is a monomorphism, so that we get

$$(m_A E) \circ (Az') = z' \circ {}^A\mu_Z.$$

$\square$

LEMMA 3.23. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$, let $L, M : \mathcal{B} \to \mathcal{A}$ be functors and let $\mu : AL \to L$ be an associative and unital functorial morphism, that is (L, μ) is a left $\mathbb{A}$-module functor. Let $h : L \to M$ and let $\varphi : AM \to M$ be functorial morphisms such that*

$$h \circ \mu = \varphi \circ (Ah) . \tag{5}$$

If AAh and h are epimorphisms, then φ is associative and unital, that is (M, φ) is a left $\mathbb{A}$-module functor.

Proof. We calculate

$$\begin{aligned} &\varphi \circ (A\varphi) \circ (AAh) \overset{(5)}{=} \varphi \circ (Ah) \circ (A\mu) \overset{(5)}{=} h \circ \mu \circ (A\mu) \\ &\overset{\mu \text{is ass.}}{=} h \circ \mu \circ (m_A L) \overset{(5)}{=} \varphi \circ (Ah) \circ (m_A L) \overset{m_A}{=} \varphi \circ (m_A M) \circ (AAh) . \end{aligned}$$

Since AAh is an epimorphism, we deduce that φ is associative. Moreover we have

$$\varphi \circ (u_A M) \circ h \overset{u_A}{=} \varphi \circ (Ah) \circ (u_A L) \overset{(5)}{=} h \circ \mu \circ (u_A L) = h.$$

Since h is an epimorphism, we get that φ is unital. $\square$

3.1. **Liftings of module functors.**

PROPOSITION 3.24 ([Ap] and [J]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$ and let $Q : \mathcal{A} \to \mathcal{B}$ be a functor. Then there is a bijection between the following collections of data*

$\mathcal{F}$ *functors $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ that are liftings of Q (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U$)*

$\mathcal{M}$ *functorial morphisms $\Phi : BQ \to QA$ such that*

$$\Phi \circ (m_B Q) = (Qm_A) \circ (\Phi A) \circ (B\Phi) \qquad \textit{and} \qquad \Phi \circ (u_B Q) = Qu_A$$

given by

$a \ : \ \mathcal{F} \to \mathcal{M}$ *where* $a\left(\widetilde{Q}\right) = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A)$

$b \ : \ \mathcal{M} \to \mathcal{F}$ *where* ${}_{\mathbb{B}}Ub(\Phi) = Q_{\mathbb{A}}U$ *and* ${}_{\mathbb{B}}U\lambda_B b(\Phi) = (Q_{\mathbb{A}}U\lambda_A) \circ \Phi$ *i.e.*

$b \ : \ \mathcal{M} \to \mathcal{F}$ *where* $b(\Phi)\left(\left(X, {}^A\mu_X\right)\right) = \left(QX, \left(Q^A\mu_X\right) \circ (\Phi X)\right)$ *and* $b(\Phi)(f) = Q(f)$.

Proof. Let $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ be a lifting of the functor $Q : \mathcal{A} \to \mathcal{B}$ (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U$). Define a functorial morphism $\phi : {}_{\mathbb{B}}FQ \to \widetilde{Q}_{\mathbb{A}}F$ as the composite

$$\phi := \left(\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}FQu_A)$$

where $u_A : \mathcal{A} \to {}_{\mathbb{A}}U_{\mathbb{A}}F = A$ is also the unit of the adjunction $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ and $\lambda_B : {}_{\mathbb{B}}F_{\mathbb{B}}U \to {}_{\mathbb{B}}\mathcal{B}$ is the counit of the adjunction. Let now define

$$\Phi \overset{\text{def}}{=} {}_{\mathbb{B}}U\phi : {}_{\mathbb{B}}U_{\mathbb{B}}FQ = BQ \to {}_{\mathbb{B}}U\widetilde{Q}_{\mathbb{A}}F = Q_{\mathbb{A}}U_{\mathbb{A}}F = QA.$$

We have to prove that such a Φ satisfies $\Phi \circ (m_B Q) = (Q m_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Q u_A$. First, let us compute

$$\begin{aligned}
(Qm_A) \circ (\Phi A) \circ (B\Phi) &= (Qm_A) \circ \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} FA\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}} FQu_A A) \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}} FQu_A) \\
&\overset{{}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F}{=} (Q_{\mathbb{A}}U\lambda_{A\mathbb{A}}F) \circ \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} FA\right) \\
&\quad \circ ({}_{\mathbb{B}}U_{\mathbb{B}} FQu_A A) \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}} FQu_A) \\
&\overset{\widetilde{Q}\text{lifting}}{=} \left({}_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} FA\right) \\
&\quad \circ ({}_{\mathbb{B}}U_{\mathbb{B}} FQu_A A) \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (B_{\mathbb{B}}U_{\mathbb{B}} FQu_A) \\
&= {}_{\mathbb{B}}U \left[\left(\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ \left(\lambda_B \widetilde{Q}_{\mathbb{A}} FA\right) \circ ({}_{\mathbb{B}}FQu_A A)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{\lambda_B}{=} {}_{\mathbb{B}}U \left[\left(\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ \left({}_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_{A\mathbb{A}}F\right) \circ ({}_{\mathbb{B}}FQu_A A)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{\widetilde{Q}\text{lifting}}{=} {}_{\mathbb{B}}U \left[\left(\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ ({}_{\mathbb{B}}FQ_{\mathbb{A}}U\lambda_{A\mathbb{A}}F) \circ ({}_{\mathbb{B}}FQu_A A)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{{}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F}{=} {}_{\mathbb{B}}U \left[\left(\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ ({}_{\mathbb{B}}FQm_A) \circ ({}_{\mathbb{B}}FQu_A A)\right] \\
&\quad \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{A\text{monad}}{=} \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ \left(B_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{\widetilde{Q}\text{is}B\text{mod}}{=} \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ \left(m_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}} F\right) \circ (BBQu_A) \\
&\overset{m_B}{=} \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (BQu_A) \circ (m_B Q) \\
&= \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}} FQu_A) \circ (m_B Q) \\
&= ({}_{\mathbb{B}}U\phi) \circ (m_B Q) = \Phi \circ (m_B Q)\,.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\Phi \circ (u_B Q) = ({}_{\mathbb{B}}U\phi) \circ (u_B Q) &= \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ ({}_{\mathbb{B}}U_{\mathbb{B}} FQu_A) \circ (u_B Q) \\
&\overset{u_B}{=} \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ (u_B Q_{\mathbb{A}}U_{\mathbb{A}}F) \circ (Qu_A) \\
&\overset{\widetilde{Q}\text{lifting}}{=} \left({}_{\mathbb{B}}U\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ \left(u_{B\mathbb{B}}U\widetilde{Q}_{\mathbb{A}} F\right) \circ (Qu_A) \overset{({}_{\mathbb{B}}F,{}_{\mathbb{B}}U)\text{adj}}{=} Qu_A.
\end{aligned}$$

Conversely, let Φ be a functorial morphism satisfying $\Phi \circ (m_B Q) = (Q m_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_B Q) = Q u_A$. We define $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ by setting, for every $\left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left(\left(X, {}^{A}\mu_X\right)\right) = \left(QX, \left(Q^{A}\mu_X\right) \circ (\Phi X)\right).$$

We have to check that $\left(Q(X), \left(Q^{A}\mu_X\right) \circ (\Phi X)\right) \in {}_{\mathbb{B}}\mathcal{B}$, that is

$${}^{B}\mu_{\widetilde{Q}X} \circ \left(B^{B}\mu_{\widetilde{Q}X}\right) = {}^{B}\mu_{\widetilde{Q}X} \circ (m_B QX) \quad \text{and} \quad {}^{B}\mu_{\widetilde{Q}X} \circ (u_B QX) = QX.$$

We compute

$$\begin{aligned}
{}^{B}\mu_{\widetilde{Q}X} \circ \left(B^{B}\mu_{\widetilde{Q}X}\right) &= \left(Q^{A}\mu_X\right) \circ (\Phi X) \circ \left(BQ^{A}\mu_X\right) \circ (B\Phi X) \\
&\overset{\Phi}{=} \left(Q^{A}\mu_X\right) \circ \left(QA^{A}\mu_X\right) \circ (\Phi AX) \circ (B\Phi X) \\
&\overset{X\text{module}}{=} \left(Q^{A}\mu_X\right) \circ (Q m_A X) \circ (\Phi AX) \circ (B\Phi X) \\
&\overset{\text{propertyof}\Phi}{=} \left(Q^{A}\mu_X\right) \circ (\Phi X) \circ (m_B QX) = {}^{B}\mu_{\widetilde{Q}X} \circ (m_B QX).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
{}^{B}\mu_{\widetilde{Q}X} \circ (u_B QX) &= \left(Q^{A}\mu_X\right) \circ (\Phi X) \circ (u_B QX) \\
&\overset{\text{propertyof}\Phi}{=} \left(Q^{A}\mu_X\right) \circ (Q u_A X) \overset{X\text{module}}{=} QX.
\end{aligned}$$

Now, let $f : \left(X, {}^{A}\mu_X\right) \to \left(Y, {}^{A}\mu_Y\right)$ a morphism of left $\mathbb{A}$-modules, that is a morphism $f : X \to Y$ in $\mathcal{A}$ such that

$${}^{A}\mu_Y \circ (Af) = f \circ {}^{A}\mu_X.$$

We have to prove that $\widetilde{Q}(f) : \widetilde{Q}X = \left(QX, {}^{B}\mu_{QX}\right) \to \widetilde{Q}Y = \left(QX, {}^{B}\mu_{QY}\right)$ is a morphism of left $\mathbb{B}$-modules. We set $\widetilde{Q}(f) = Q(f)$ and we compute

$${}^{B}\mu_{\widetilde{Q}Y} \circ \left(B\widetilde{Q}f\right) \overset{?}{=} \left(\widetilde{Q}f\right) \circ {}^{B}\mu_{\widetilde{Q}X}$$

i.e. by definition of the functor $\widetilde{Q}$

$${}^{B}\mu_{QY} \circ (BQf) \overset{?}{=} (Qf) \circ {}^{B}\mu_{QX}$$

in fact

$$\begin{aligned}
{}^{B}\mu_{QY} \circ (BQf) &= \left(Q^{A}\mu_Y\right) \circ (\Phi Y) \circ (BQf) \overset{\Phi}{=} \left(Q^{A}\mu_Y\right) \circ (QAf) \circ (\Phi X) \\
&\overset{f\text{morph}A\text{-mod}}{=} (Qf) \circ \left(Q^{A}\mu_X\right) \circ (\Phi X) = (Qf) \circ {}^{B}\mu_{QX}.
\end{aligned}$$

Let now check that $\widetilde{Q}$ is a lifting of Q. Let $\left(X, {}^{A}\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$ and compute

$${}_{\mathbb{B}}U\widetilde{Q}\left(\left(X, {}^{A}\mu_X\right)\right) = {}_{\mathbb{B}}U\left(QX, {}^{B}\mu_{QX}\right) = QX = Q_{\mathbb{A}}U\left(\left(X, {}^{A}\mu_X\right)\right)$$

and thus on the objects

$${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U.$$

Let $f : \left(X, {}^{A}\mu_X\right) \to \left(Y, {}^{A}\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{A}$ be a morphism, we have

$${}_{\mathbb{B}}U\widetilde{Q}(f) : QX \to QY = Q_{\mathbb{A}}U(f) : QX \to QY.$$

Therefore $\widetilde{Q}$ is a lifting of the functor Q.

We have to prove that it is a bijection. Let us start with $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ a lifting of the functor $Q : \mathcal{A} \to \mathcal{B}$. Then we construct $\Phi : BQ \to QA$ given by

$$\Phi = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right)$$

and using this functorial morphism we define a functor $\overline{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ as follows: for every $\left(X,{}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$

$$\overline{Q}\left(\left(X,{}^A\mu_X\right)\right) = \left(QX, \left(Q^A\mu_X\right) \circ (\Phi X)\right).$$

Since both $\widetilde{Q}$ and $\overline{Q}$ are lifting of Q, we have that ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U = {}_{\mathbb{B}}U\overline{Q}$. We have to prove that ${}_{\mathbb{B}}U\left(\lambda_B\overline{Q}\right) = {}_{\mathbb{B}}U\left(\lambda_B\widetilde{Q}\right)$. Let $Z \in {}_{\mathbb{A}}\mathcal{A}$. We compute

$$\begin{aligned}
{}_{\mathbb{B}}U\left(\lambda_B\overline{Q}Z\right) &= \left(Q_{\mathbb{A}}U\lambda_AZ\right) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}UZ\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\
&\overset{\widetilde{Q}\text{lifting}Q}{=} \left({}_{\mathbb{B}}U\widetilde{Q}\lambda_AZ\right) \circ \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F_{\mathbb{A}}UZ\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\
&\overset{\lambda_B}{=} \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}Z\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U\widetilde{Q}\lambda_AZ\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_{A\mathbb{A}}UZ\right) \\
&= \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}Z\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}F\left[Q_{\mathbb{A}}U\lambda_AZ \circ Qu_{A\mathbb{A}}UZ\right]\right) \overset{(u_A,\lambda_A)\text{adj}}{=} {}_{\mathbb{B}}U\lambda_B\widetilde{Q}Z.
\end{aligned}$$

Conversely, let us start with a functorial morphism $\Phi : BQ \to QA$ satisfying $\Phi \circ (m_BQ) = (Qm_A) \circ (\Phi A) \circ (B\Phi)$ and $\Phi \circ (u_BQ) = Qu_A$. Then we construct a functor $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ by setting, for every $\left(X,{}^A\mu_X\right) \in {}_{\mathbb{A}}\mathcal{A}$,

$$\widetilde{Q}\left(\left(X,{}^A\mu_X\right)\right) = \left(QX, \left(Q^A\mu_X\right) \circ (\Phi X)\right)$$

which lifts $Q : \mathcal{A} \to \mathcal{B}$. Now, we define a functorial morphism $\Psi : BQ \to QA$ given by

$$\Psi = \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right).$$

Then we have

$$\begin{aligned}
\Psi &= \left({}_{\mathbb{B}}U\lambda_B\widetilde{Q}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right) \overset{\text{def}\widetilde{Q}}{=} \left(Q_{\mathbb{A}}U\lambda_{A\mathbb{A}}F\right) \circ \left(\Phi_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FQu_A\right) \\
&= (Qm_A) \circ (\Phi A) \circ (BQu_A) \overset{\Phi}{=} (Qm_A) \circ (QAu_A) \circ \Phi \overset{A\text{monad}}{=} \Phi.
\end{aligned}$$

∎

Corollary 3.25. *Let $\mathcal{X}, \mathcal{A}$ be categories, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $F : \mathcal{X} \to \mathcal{A}$ be a functor. Then there exists a bijective correspondence between the following collections of data:*

$\mathcal{H}$ *Left $\mathbb{A}$-module actions ${}^A\mu_F : AF \to F$*
$\mathcal{G}$ *Functors ${}_AF : \mathcal{X} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U_AF = F$,*

given by

$$\widetilde{a} : \mathcal{H} \to \mathcal{G} \text{ where } {}_{\mathbb{A}}U\widetilde{a}\left({}^A\mu_F\right) = F \text{ and } {}_{\mathbb{A}}U\lambda_A\widetilde{a}\left({}^A\mu_F\right) = {}^A\mu_F \text{ i.e.}$$
$$\widetilde{a}\left({}^A\mu_F\right)(X) = \left(FX, {}^A\mu_FX\right) \text{ and } \widetilde{a}\left({}^A\mu_F\right)(f) = F(f)$$
$$\widetilde{b} : \mathcal{G} \to \mathcal{H} \text{ where } \widetilde{b}\left({}_AF\right) = {}_{\mathbb{A}}U\lambda_{AA}F : AF \to F.$$

Proof. Apply Proposition 3.24 to the case $\mathcal{A} = \mathcal{X}, \mathcal{B} = \mathcal{A}$, $\mathbb{A} = \mathrm{Id}_{\mathcal{X}}$ and $\mathbb{B} = \mathbb{A}$. Then $\widetilde{Q} = {}_A F$ is the lifting of F and $\Phi = {}^A\mu_F$ satisfies ${}^A\mu_F \circ (m_A F) = {}^A\mu_F \circ \left(A^A\mu_F\right)$ and ${}^A\mu_F \circ (u_A F) = F$ that is $\left(F, {}^A\mu_F\right)$ is a left $\mathbb{A}$-module functor. $\square$

COROLLARY 3.26. *Let (L, R) be an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on $\mathcal{B}$. Then there is a bijective correspondence between the following collections of data*

- $\mathfrak{K}$ *Functors $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U \circ K = R$,*
- $\mathfrak{L}$ *functorial morphism $\alpha : AR \to R$ such that (R, α) is a left module functor for the monad $\mathbb{A}$*

given by

$$\begin{aligned} \Phi &: \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi(K) = {}_{\mathbb{A}}U\lambda_A K : AR \to R \\ \Omega &: \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega(\alpha)(X) = (RX, \alpha X) \text{ and } {}_{\mathbb{A}}U\Omega(\alpha)(f) = R(f). \end{aligned}$$

Proof. Apply Corollary 3.25 to the case "F" $= R : \mathcal{A} \to \mathcal{B}$ where (L, R) is an adjunction with $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and $\mathbb{A} = (A, m_A, u_A)$ a monad on $\mathcal{B}$. $\square$

In the following Proposition we give a more precise version of Lemma 3 in [J].

PROPOSITION 3.27. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$. Let $Q : \mathcal{A} \to \mathcal{B}$ be a functor and let $\widetilde{Q} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ be a lifting of Q (i.e. ${}_{\mathbb{B}}U\widetilde{Q} = Q_{\mathbb{A}}U$) and $\Phi : BQ \to QA$ as in Proposition 3.24. Then Φ is an isomorphism if and only if $\phi = \left(\lambda_B \widetilde{Q}_{\mathbb{A}} F\right) \circ ({}_{\mathbb{B}}FQu_A) : {}_{\mathbb{B}}FQ \to \widetilde{Q}_{\mathbb{A}}F$ is an isomorphism.*

Proof. By construction in Proposition 3.24 we have that $\Phi = {}_{\mathbb{B}}U\phi$. Assume that Φ is an isomorphism. Since, by Proposition 3.18, ${}_{\mathbb{B}}U$ reflects isomorphisms, $\phi : {}_{\mathbb{B}}FQ \to \widetilde{Q}_{\mathbb{A}}F$ is an isomorphism. Conversely, assume that $\phi : {}_{\mathbb{B}}FQ \to \widetilde{Q}_{\mathbb{A}}F$ is an isomorphism. Then $\Phi = {}_{\mathbb{B}}U\phi$ is also an isomorphism. $\square$

COROLLARY 3.28. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad on $\mathcal{B}$. Let $K : \mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ be a functor such that ${}_{\mathbb{B}}U \circ K = R$ and let (R, α) be a left $\mathbb{B}$-module functor as in Corollary 3.26. Then α is an isomorphism if and only if $\lambda_B K : {}_{\mathbb{B}}FR \to K$ is an isomorphism.*

Proof. Apply Proposition 3.27 with $Q = R$, $\mathbb{A} = \mathrm{Id}_{\mathcal{A}}$. Then $\widetilde{Q} = K$ is the lifting of R and $\Phi = \alpha : BR \to R$, given by $\alpha = {}_{\mathbb{B}}U\phi = {}_{\mathbb{B}}U\lambda_B K$. $\square$

Some results in the following part of this section can be found in the literature (see e.g. [BM] and [BMV]). To introduce our main tools of investigation, for the reader's sake, we give here a full description.

LEMMA 3.29. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ with coequalizers. Let $Q : \mathcal{B} \to \mathcal{A}$ be a left $\mathbb{A}$-module functor with functorial morphisms ${}^A\mu_Q : AQ \to Q$. Then there exists a unique functor ${}_AQ : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that*

$$ {}_{\mathbb{A}}U \circ {}_AQ = Q \text{ and } {}_{\mathbb{A}}U\lambda_{A\,A}Q = {}^A\mu_Q. $$

Moreover if $\varphi : Q \to T$ is a functorial morphism between left $\mathbb{A}$-module functors and φ satisfies

$$ {}^{A}\mu_T \circ (A\varphi) = \varphi \circ \left({}^{A}\mu_Q\right) $$

then there is a unique functorial morphism ${}_A\varphi : {}_AQ \to {}_AT$ such that

$$ {}_{\mathbb{A}}U_A\varphi = \varphi. $$

Proof. Corollary 3.25 applied to the case where $F = Q$ gives us the first statement. Let $B \in \mathcal{B}$. Then we have

$$ \left({}^{A}\mu_T B\right) \circ (A\varphi B) = (\varphi B) \circ \left({}^{A}\mu_Q B\right) $$

which means that φB yields a morphism ${}_A\varphi B$ in ${}_{\mathbb{A}}\mathcal{A}$. $\square$

PROPOSITION 3.30. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and that A preserves coequalizers. Let $Q : \mathcal{B} \to \mathcal{A}$ be a functor and let ${}^{A}\mu_Q : AQ \to Q$ and $\mu_Q^B : QB \to Q$ be functorial morphisms. Assume that ${}^{A}\mu_Q$ is associative and unital and that ${}^{A}\mu_Q \circ \left(A\mu_Q^B\right) = \mu_Q^B \circ \left({}^{A}\mu_Q B\right)$. Set*

$$ (Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right) \tag{6} $$

Then $Q_B : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ is a left $\mathbb{A}$-module functor where ${}^{A}\mu_{Q_B} : AQ_B \to Q_B$ is uniquely determined by

$$ p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) = {}^{A}\mu_{Q_B} \circ (Ap_Q). \tag{7} $$

Moreover there exists a unique functor ${}_A(Q_B) : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that

$$ {}_{\mathbb{A}}U_A(Q_B) = Q_B \text{ and } {}_{\mathbb{A}}U\lambda_{AA}(Q_B) = {}^{A}\mu_{Q_B}. \tag{8} $$

Proof. By Lemma 2.7 we can consider $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$. Since

$$ {}^{A}\mu_Q \circ \left(A\mu_Q^B\right) = \mu_Q^B \circ \left({}^{A}\mu_Q B\right) $$

we deduce that

$$ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ \left(A\mu^B_{Q\mathbb{B}}U\right) = \left(\mu^B_{Q\mathbb{B}}U\right) \circ \left({}^{A}\mu_Q B_{\mathbb{B}}U\right). \tag{9} $$

Also, in view of the naturality of ${}^{A}\mu_Q$, we have

$$ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ (AQ_{\mathbb{B}}U\lambda_B) = (Q_{\mathbb{B}}U\lambda_B) \circ \left({}^{A}\mu_Q B_{\mathbb{B}}U\right). \tag{10} $$

We compute

$$ \begin{aligned} p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ (AQ_{\mathbb{B}}U\lambda_B) &\overset{(10)}{=} p_Q \circ (Q_{\mathbb{B}}U\lambda_B) \circ \left({}^{A}\mu_Q B_{\mathbb{B}}U\right) \\ \overset{p_Q \text{ coeq}}{=} p_Q \circ \left(\mu^B_{Q\mathbb{B}}U\right) \circ \left({}^{A}\mu_Q B_{\mathbb{B}}U\right) &\overset{(9)}{=} p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ \left(A\mu^B_{Q\mathbb{B}}U\right) \end{aligned} $$

and hence we obtain

$$ p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ (AQ_{\mathbb{B}}U\lambda_B) = p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) \circ \left(A\mu^B_{Q\mathbb{B}}U\right). $$

Since A preserves coequalizers, we get

$$ (AQ_B, Ap_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(A\mu^B_{Q\mathbb{B}}U, AQ_{\mathbb{B}}U\lambda_B\right). $$

Hence there exists a unique functorial morphism ${}^{A}\mu_{Q_B} : AQ_B \to Q_B$ such that

$$ p_Q \circ \left({}^{A}\mu_{Q\mathbb{B}}U\right) = {}^{A}\mu_{Q_B} \circ (Ap_Q). $$

Since Q is a left $\mathbb{A}$-module functor, by Lemma 3.17, also $Q_{\mathbb{B}}U$ is a left $\mathbb{A}$-module functor. Now p_Q is an epimorphism and hence, since A preserves coequalizers, also AAp_Q is an epimorphism. Therefore we can apply Lemma 3.23 to "φ" $= {}^A\mu_{Q_B}$, "h" $= p_Q$ and "μ" $= {}^A\mu_{Q\mathbb{B}}U$ and hence we obtain that $\left(Q_B, {}^A\mu_{Q_B}\right)$ is a left $\mathbb{A}$-module functor that is ${}^A\mu_{Q_B}$ is associative and unital. By Lemma 3.29 applied to $\left(Q_B, {}^A\mu_{Q_B}\right)$ there exists a functor ${}_A(Q_B) : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U_A(Q_B) = Q_B$ and ${}_{\mathbb{A}}U\lambda_{AA}(Q_B) = {}^A\mu_{Q_B}$. Moreover ${}_A(Q_B)$ is unique with respect to these properties. □

Proposition 3.31. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and A preserves them. Let $Q : \mathcal{B} \to \mathcal{A}$ be an $\mathbb{A}$-$\mathbb{B}$-bimodule functor with functorial morphisms ${}^A\mu_Q : AQ \to Q$ and $\mu_Q^B : QB \to Q$. Then the functor ${}_AQ : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ is a right $\mathbb{B}$-module functor via $\mu^B_{{}_AQ} : {}_AQB \to {}_AQ$ where $\mu^B_{{}_AQ}$ is uniquely determined by*

$$
{}_{\mathbb{A}}U\mu^B_{{}_AQ} = \mu^B_Q. \tag{11}
$$

Let $\left(({}_AQ)_B, p_{{}_AQ}\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{{}_AQ\mathbb{B}}U, {}_AQ_{\mathbb{B}}U\lambda_B\right)$*. Then we have*

$$
({}_AQ)_B = {}_A(Q_B) : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}.
$$

Proof. Since Q is endowed with a left $\mathbb{A}$-module structure, by Lemma 3.29 there exists a unique functor ${}_AQ : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U_AQ = Q$ and ${}_{\mathbb{A}}U\lambda_{AA}Q = {}^A\mu_Q$. Note that, since Q is an $\mathbb{A}$-$\mathbb{B}$-bimodule functor, in particular the compatibility condition

$$
{}^A\mu_Q \circ \left(A\mu^B_Q\right) = \mu^B_Q \circ \left({}^A\mu_Q B\right).
$$

holds. Thus, by Lemma 3.29, there exists a functorial morphism $\mu^B_{{}_AQ} : {}_AQB \to {}_AQ$ such that

$$
{}_{\mathbb{A}}U\mu^B_{{}_AQ} = \mu^B_Q.
$$

By the associativity and unitality properties of μ^B_Q and since ${}_{\mathbb{A}}U$ is faithful, we get that also $\mu^B_{{}_AQ}$ is associative and unital, so that $\left({}_AQ, \mu^B_{{}_AQ}\right)$ is a right $\mathbb{B}$-module functor. Thus we can consider the coequalizer

$$
\begin{tikzcd}
{}_AQB_{\mathbb{B}}U \arrow[r, shift left, "\mu^B_{{}_AQ\mathbb{B}}U"] \arrow[r, shift right, "{}_AQ_{\mathbb{B}}U\lambda_B"'] & {}_AQ_{\mathbb{B}}U \arrow[r, "p_{{}_AQ}"] & ({}_AQ)_B
\end{tikzcd} \tag{12}
$$

so that we get a functor $({}_AQ)_B : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$. Since A preserves coequalizers, by Lemma 3.21, also ${}_{\mathbb{A}}U$ preserves coequalizers. Then, by applying the functor ${}_{\mathbb{A}}U$ to 12 we still get a coequalizer

$$
\begin{tikzcd}
{}_{\mathbb{A}}U_AQB_{\mathbb{B}}U \arrow[r, shift left, "{}_{\mathbb{A}}U\mu^B_{{}_AQ\mathbb{B}}U"] \arrow[r, shift right, "{}_{\mathbb{A}}U_AQ_{\mathbb{B}}U\lambda_B"'] & {}_{\mathbb{A}}U_AQ_{\mathbb{B}}U \arrow[r, "{}_{\mathbb{A}}Up_{{}_AQ}"] & {}_{\mathbb{A}}U({}_AQ)_B
\end{tikzcd}
$$

that can be written as

$$
\begin{tikzcd}
QB_{\mathbb{B}}U \arrow[r, shift left, "\mu^B_{Q\mathbb{B}}U"] \arrow[r, shift right, "Q_{\mathbb{B}}U\lambda_B"'] & Q_{\mathbb{B}}U \arrow[r, "{}_{\mathbb{A}}Up_{{}_AQ}"] & {}_{\mathbb{A}}U({}_AQ)_B
\end{tikzcd}
$$

Since, by Proposition 3.30, $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$, we get that

$$ {}_{\mathbb{A}}U\left({}_AQ\right)_B = Q_B \text{ and } {}_{\mathbb{A}}Up_{{}_AQ} = p_Q. $$

Moreover

$$ {}_{\mathbb{A}}U\lambda_A\left({}_AQ\right)_B : A_{\mathbb{A}}U\left({}_AQ\right)_B = AQ_B \to {}_{\mathbb{A}}U\left({}_AQ\right)_B = Q_B. $$

By Proposition 3.30, we know that $\left(Q_B, {}^A\mu_{Q_B}\right)$ is a left $\mathbb{A}$-module functor and ${}_{\mathbb{A}}U\lambda_{AA}\left(Q_B\right) = {}^A\mu_{Q_B}$. Hence we get

$$ {}_{\mathbb{A}}U\lambda_A\left({}_AQ\right)_B = {}^A\mu_{Q_B} \overset{(8)}{=} {}_{\mathbb{A}}U\lambda_{AA}\left(Q_B\right) $$

i.e.

$$ \left({}_AQ\right)_B = {}_A\left(Q_B\right). $$

$\square$

Notation 3.32. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and A preserves them. Let $Q : \mathcal{B} \to \mathcal{A}$ be an $\mathbb{A}$-$\mathbb{B}$-bimodule functor. In view of Proposition 3.31, we set*

$$ {}_AQ_B = \left({}_AQ\right)_B = {}_A\left(Q_B\right). $$

Lemma 3.33. *Let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$ and assume that $\mathcal{B}$ have coequalizers. Let $\left(Q : \mathcal{B} \to \mathcal{A}, \mu^B_Q\right)$ be a right $\mathbb{B}$-module functor. With notations of Proposition 3.30 we have that*

$$ Q_B\lambda_{B\mathbb{B}}F = \mu^B_Q. \tag{13} $$

Furthermore, if we assume that the functors Q, B preserve coequalizers we also have

$$ Q_B\lambda_{BB}P = p_{QB}P. \tag{14} $$

Proof. Let us consider the following diagram

$$
\begin{tikzcd}
QB_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F \arrow[r, shift left, "\mu^B_{Q\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F"] \arrow[r, shift right, "Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F"'] \arrow[d, "QB_{\mathbb{B}}U\lambda_{B\mathbb{B}}F"] & Q_{\mathbb{B}}U_{\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F \arrow[r, "p_{Q\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F"] \arrow[d, "Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F"] & Q_{B\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F \arrow[d, "Q_B\lambda_{B\mathbb{B}}F"] \\
QB_{\mathbb{B}}U_{\mathbb{B}}F \arrow[r, shift left, "\mu^B_{Q\mathbb{B}}U_{\mathbb{B}}F"] \arrow[r, shift right, "Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F"'] & Q_{\mathbb{B}}U_{\mathbb{B}}F \arrow[r, "p_{Q\mathbb{B}}F"] & Q_{B\mathbb{B}}F
\end{tikzcd}
$$

Note that $QB_{\mathbb{B}}U\lambda_{B\mathbb{B}}F = QBm_B$ and $Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F = Qm_B$ so that the left square serially commutes because of the associativity of m_B and of μ^B_Q. Both the rows are coequalizers in view of the dual version of Lemma 2.10 so that, by the universal property of coequalizers, there exists a unique functorial morphism $\zeta : Q_{B\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F \to Q_{B\mathbb{B}}F$ such that $\zeta \circ (p_{Q\mathbb{B}}F_{\mathbb{B}}U_{\mathbb{B}}F) = (p_{Q\mathbb{B}}F) \circ (Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F)$. Since $p_Q : Q_{\mathbb{B}}U \to Q_B$ is a functorial morphism, we know that $Q_B\lambda_{B\mathbb{B}}F$ makes the right square be commutative, but since by (15) we have $p_{Q\mathbb{B}}F = \mu^B_Q$ we also have that μ^B_Q makes the right square commute. Therefore, we deduce that $\zeta = Q_B\lambda_{B\mathbb{B}}F = \mu^B_Q$. Assuming that Q and B preserve coequalizers, by Lemma 3.21, we get that ${}_{\mathbb{B}}U$ also preserves coequalizers so that, in view of Corollary 2.12 we also have that

$(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$ preserves them. Hence, using that $(Q_B, Q_B\lambda_B) = \mathrm{Coequ}_{\mathrm{Fun}}(Q_B\lambda_{B\mathbb{B}}F_{\mathbb{B}}U, Q_{B\mathbb{B}}F_{\mathbb{B}}U\lambda_B)$, in view of Lemmas 2.10 and 3.29, we have

$$\begin{aligned}(Q_{BB}P, Q_B\lambda_{BB}P) &= \mathrm{Coequ}_{\mathrm{Fun}}(Q_B\lambda_{B\mathbb{B}}F_{\mathbb{B}}U_BP, Q_{B\mathbb{B}}F_{\mathbb{B}}U\lambda_{BB}P) \\ &= \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U_BP, Q^B\mu_P\right) \\ &= (Q_{BB}P, p_{QB}P)\end{aligned}$$

so that we get $Q_B\lambda_{BB}P = p_{QB}P$. □

PROPOSITION 3.34. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ and let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and let $Q : \mathcal{B} \to \mathcal{A}$ be an $\mathbb{A}$-$\mathbb{B}$-bimodule functor. Then, with notations of Proposition 3.30, we can consider the functor Q_B where $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$. Then*

$$Q_{B\mathbb{B}}F = Q \text{ and } p_{Q\mathbb{B}}F = \mu^B_Q. \tag{15}$$

Proof. By construction we have that $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$. By applying it to the functor ${}_{\mathbb{B}}F$ we get that

$$\begin{aligned}(Q_{B\mathbb{B}}F, p_{Q\mathbb{B}}F) &= \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U_{\mathbb{B}}F, Q_{\mathbb{B}}U\lambda_{B\mathbb{B}}F\right) \\ &= \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_QB, Qm_B\right).\end{aligned}$$

Since Q is a right $\mathbb{B}$-module functor, by Proposition 3.16 we have that

$$\left(Q, \mu^B_Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_QB, Qm_B\right)$$

so that we get

$$(Q_{B\mathbb{B}}F, p_{Q\mathbb{B}}F) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_QB, Qm_B\right) = \left(Q, \mu^B_Q\right).$$

□

PROPOSITION 3.35. *Let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$ with coequalizers such that B preserves coequalizers. Let $G : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ be a functor preserving coequalizers. Set*

$$Q = G \circ {}_{\mathbb{B}}F \text{ and let } \mu^B_Q = G\lambda_{B\mathbb{B}}F$$

Then $\left(Q, \mu^B_Q\right)$ is a right $\mathbb{B}$-module functor and

$$Q_B = \left(G \circ {}_{\mathbb{B}}F\right)_B = G. \tag{16}$$

Proof. We compute

$$\begin{aligned}\mu^B_Q \circ \left(\mu^B_QB\right) = (G\lambda_{B\mathbb{B}}F) \circ (G\lambda_{B\mathbb{B}}FB) &\overset{\lambda_B}{=} (G\lambda_{B\mathbb{B}}F) \circ (G_{\mathbb{B}}F_{\mathbb{B}}U\lambda_{B\mathbb{B}}F) \\ &= (G\lambda_{B\mathbb{B}}F) \circ (G \circ {}_{\mathbb{B}}Fm_B) = \mu^B_Q \circ (Qm_B)\end{aligned}$$

and

$$\mu^B_Q \circ (Qu_B) = (G\lambda_{B\mathbb{B}}F) \circ (G_{\mathbb{B}}Fu_B) \overset{\text{adj}}{=} G \circ {}_{\mathbb{B}}F = Q.$$

Thus $\left(Q, \mu^B_Q\right)$ is a right $\mathbb{B}$-module functor. Recall that (see Proposition 3.30)

$$(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$$

and by Proposition 3.34 we have $Q_{B\mathbb{B}}F = Q$ and $p_{Q\mathbb{B}}F = \mu_Q^B$. In particular we get

$$Q_{B\mathbb{B}}F = Q = G_{\mathbb{B}}F.$$

In order to prove that $Q_B = G$ it suffices to prove that $(G, G\lambda_B) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$. In fact, by Corollary 3.15, $({}_{\mathbb{B}}U, {}_{\mathbb{B}}U\lambda_B) = \mathrm{Coequ}_{\mathrm{Fun}}\left(B_{\mathbb{B}}U\lambda_B, m_{B\mathbb{B}}U\right)$ and, since by Lemma 3.20 ${}_{\mathbb{B}}U$ reflects coequalizers, we have

$$(\mathrm{Id}_{{}_{\mathbb{B}}\mathcal{B}}, \lambda_B) = \mathrm{Coequ}_{\mathrm{Fun}}\left({}_{\mathbb{B}}F_{\mathbb{B}}U\lambda_B, \lambda_{B\mathbb{B}}F_{\mathbb{B}}U\right).$$

Since G preserves coequalizers, we get that

$$\begin{aligned} (G, G\lambda_B) &= \mathrm{Coequ}_{\mathrm{Fun}}\left(G_{\mathbb{B}}F_{\mathbb{B}}U\lambda_B, G\lambda_{B\mathbb{B}}F_{\mathbb{B}}U\right) \\ &= \mathrm{Coequ}_{\mathrm{Fun}}\left(Q_{\mathbb{B}}U\lambda_B, \mu^B_{Q\mathbb{B}}U\right) = (Q_B, p_Q). \end{aligned}$$

$\square$

PROPOSITION 3.36. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ with coequalizers such that A preserves coequalizers. Let $H : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ be a functor preserving coequalizers. Set*

$$Q = {}_{\mathbb{A}}U \circ H \text{ and let } {}^A\mu_Q = {}_{\mathbb{A}}U\lambda_A H$$

Then $\left(Q, {}^A\mu_Q\right)$ is a left $\mathbb{A}$-module functor and

$$_AQ = {}_A\left({}_{\mathbb{A}}U \circ H\right) = H. \tag{17}$$

Proof. First we want to prove that ${}^A\mu_Q = {}_{\mathbb{A}}U\lambda_A H$ is associative. We have

$$\begin{aligned} {}^A\mu_Q \circ \left(A\,{}^A\mu_Q\right) &= ({}_{\mathbb{A}}U\lambda_A H) \circ (A_{\mathbb{A}}U\lambda_A H) \overset{\lambda_A}{=} ({}_{\mathbb{A}}U\lambda_A H) \circ ({}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F_{\mathbb{A}}UH) \\ &= ({}_{\mathbb{A}}U\lambda_A H) \circ (m_{A\mathbb{A}}UH) = {}^A\mu_Q \circ (m_A Q) \end{aligned}$$

so that we get

$${}^A\mu_Q \circ \left(A\,{}^A\mu_Q\right) = {}^A\mu_Q \circ (m_A Q).$$

Now we prove that ${}^A\mu_Q = {}_{\mathbb{A}}U\lambda_A H$ is unital. We compute

$${}^A\mu_Q \circ (u_A Q) = ({}_{\mathbb{A}}U\lambda_A H) \circ (u_{A\mathbb{A}}UH) \overset{\mathrm{adj}}{=} {}_{\mathbb{A}}UH = Q$$

so that we get

$${}^A\mu_Q \circ (u_A Q) = Q.$$

Thus $\left(Q, {}^A\mu_Q\right)$ is a left $\mathbb{A}$-module functor. Recall that (see Lemma 3.29) there exists a unique functor ${}_AQ : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that

$${}_{\mathbb{A}}U \circ {}_AQ = Q \text{ and } {}_{\mathbb{A}}U\lambda_{AA}Q = {}^A\mu_Q.$$

Thus we have

$${}_{\mathbb{A}}U \circ {}_AQ = Q = {}_{\mathbb{A}}U \circ H$$

and

$${}_{\mathbb{A}}U\lambda_{AA}Q = {}^A\mu_Q = {}_{\mathbb{A}}U\lambda_A H$$

so that, by Proposition 3.12, we obtain that

$${}_AQ = H.$$

$\square$

THEOREM 3.37. *Let $\mathbb{B} = (B, m_B, u_B)$ be a monad on a category $\mathcal{B}$ with coequalizers such that B preserves coequalizers. Then there exists a bijective correspondence between the following collections of data:*

$\mathcal{F}_B$ *right $\mathbb{B}$-module functors $Q : \mathcal{B} \to \mathcal{A}$ such that QB preserves coequalizers*
$(\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B})$ *functors $G : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ preserving coequalizers*

given by

$$\begin{aligned} \nu_B &: \mathcal{F}_B \to (\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B}) \text{ where } \nu_B\left(\left(Q, \mu_Q^B\right)\right) = Q_B \\ \kappa_B &: (\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B}) \to \mathcal{F}_B \text{ where } \kappa_B(G) = (G_{\mathbb{B}}F, G\lambda_{B\mathbb{B}}F) \end{aligned}$$

where Q_B is uniquely determined by $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$.

Proof. Let $Q : \mathcal{B} \to \mathcal{A}$ be a right $\mathbb{B}$-module functor. Then we can consider $Q_B : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ defined by (6) as

$$(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right).$$

Since by assumption QB preserves coequalizers, by Lemma 3.19 also Q preserves coequalizers. Moreover, since B preserves coequalizers, by Lemma 3.21 also the functor ${}_{\mathbb{B}}U$ preserves coequalizers. Thus both $QB_{\mathbb{B}}U$ and $Q_{\mathbb{B}}U$ preserve coequalizers. By Corollary 2.12 we get that also $Q_B : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ preserves coequalizers.

Conversely, let us consider a functor $G : {}_{\mathbb{B}}\mathcal{B} \to \mathcal{A}$ that preserves coequalizers. By Proposition 3.35 we can consider the right $\mathbb{B}$-module functor defined as follows

$$Q = G \circ {}_{\mathbb{B}}F \text{ and let } \mu_Q^B = G\lambda_{B\mathbb{B}}F.$$

Since ${}_{\mathbb{B}}F$ is left adjoint to ${}_{\mathbb{B}}U$, in particular ${}_{\mathbb{B}}F$ preserves coequalizers and since by assumption G preserves coequalizers, we get that also $Q = G \circ {}_{\mathbb{B}}F$ preserves coequalizers and so does QB.

Now, we want to prove that ν_B and κ_B determine a bijective correspondence between $\mathcal{F}_B$ and $(\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B})$. Let us start with a right $\mathbb{B}$-module functor $\left(Q : \mathcal{B} \to \mathcal{A}, \mu_Q^B\right)$. Then we have

$$\begin{aligned} (\kappa_B \circ \nu_B)\left(\left(Q, \mu_Q^B\right)\right) &= \kappa_B(Q_B) = (Q_{B\mathbb{B}}F, Q_B\lambda_{B\mathbb{B}}F) \\ &= \left(Q_{B\mathbb{B}}F, \mu^B_{Q_{B\mathbb{B}}F}\right) \overset{(15)}{=} \left(Q, \mu_Q^B\right). \end{aligned}$$

Moreover we have

$$(\nu_B \circ \kappa_B)(G) = \nu_B\left((G_{\mathbb{B}}F, G\lambda_{B\mathbb{B}}F)\right) = (G_{\mathbb{B}}F)_B \overset{(16)}{=} G.$$

□

THEOREM 3.38. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ with coequalizers such that A preserves coequalizers. Then there exists a bijective correspondence between the following collections of data:*

${}_A\mathcal{F}$ *left $\mathbb{A}$-module functors $Q : \mathcal{B} \to \mathcal{A}$ such that AQ preserves coequalizers*
$({}_{\mathbb{A}}\mathcal{A} \leftarrow \mathcal{B})$ *functors $H : \mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ preserving coequalizers*

given by

$$\begin{aligned} {}_A\nu &: {}_A\mathcal{F} \to ({}_{\mathbb{A}}\mathcal{A} \leftarrow \mathcal{B}) \text{ where } {}_A\nu\left(\left(Q, {}^A\mu_Q\right)\right) = {}_AQ \\ {}_A\kappa &: ({}_{\mathbb{A}}\mathcal{A} \leftarrow \mathcal{B}) \to {}_A\mathcal{F} \text{ where } {}_A\kappa(H) = ({}_{\mathbb{A}}U \circ H, {}_{\mathbb{A}}U\lambda_A H) \end{aligned}$$

where ${}_AQ:\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ *is the functor defined in Lemma 3.29.*

Proof. Let $\left(Q:\mathcal{B}\to\mathcal{A},{}^A\mu_Q\right)$ be a left $\mathbb{A}$-module functor. Then, by Lemma 3.29, there exists a unique functor ${}_AQ:\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ such that

$$ {}_{\mathbb{A}}U\circ{}_AQ=Q \text{ and } {}_{\mathbb{A}}U\lambda_{A\,A}Q={}^A\mu_Q. $$

Note that, since AQ preserves coequalizers, by Lemma 3.19, $Q={}_{\mathbb{A}}U\circ{}_AQ$ preserves coequalizers. Then, by Lemma 3.20, also ${}_AQ$ preserves coequalizers. Conversely, if $H:\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ is a functor preserving coequalizers, we get that ${}_{\mathbb{A}}U\circ H:\mathcal{B}\to\mathcal{A}$. Moreover, by Lemma 3.21, ${}_{\mathbb{A}}U$ preserves coequalizers and thus also ${}_{\mathbb{A}}U\circ H$ preserves coequalizers. Now, let us prove that ${}_A\nu$ and ${}_A\kappa$ determine a bijective correspondence between ${}_A\mathcal{F}$ and $({}_{\mathbb{A}}\mathcal{A}\leftarrow\mathcal{B})$. We compute

$$ \left({}_A\kappa\circ{}_A\nu\right)\left(\left(Q,{}^A\mu_Q\right)\right)={}_A\kappa\left({}_AQ\right)=\left({}_{\mathbb{A}}U{}_AQ,{}_{\mathbb{A}}U\lambda_{A\,A}Q\right)=\left(Q,{}^A\mu_Q\right). $$

On the other hand we have

$$ \left({}_A\nu\circ{}_A\kappa\right)(H)={}_A\nu\left(\left({}_{\mathbb{A}}U\circ H,{}_{\mathbb{A}}U\lambda_AH\right)\right)={}_A\left({}_{\mathbb{A}}U\circ H\right)\overset{(17)}{=}H. $$

$\square$

THEOREM 3.39. *Let* $\mathbb{A}=(A,m_A,u_A)$ *be a monad on a category* $\mathcal{A}$ *with coequalizers such that* A *preserves coequalizers. Let* $\mathbb{B}=(B,m_B,u_B)$ *be a monad on a category* $\mathcal{B}$ *with coequalizers such that* B *preserves coequalizers. Then there exists a bijective correspondence between the following collections of data:*

${}_A\mathcal{F}_B$ $\mathbb{A}$-$\mathbb{B}$-*bimodule functors* $Q:\mathcal{B}\to\mathcal{A}$ *such that* AQ *and* QB *preserve coequalizers*

$({}_{\mathbb{A}}\mathcal{A}\leftarrow{}_{\mathbb{B}}\mathcal{B})$ *functors* $G:{}_{\mathbb{B}}\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ *preserving coequalizers*

given by

$$ \begin{aligned} {}_A\nu_B &: {}_A\mathcal{F}_B\to({}_{\mathbb{A}}\mathcal{A}\leftarrow{}_{\mathbb{B}}\mathcal{B}) \text{ where } {}_A\nu_B\left(\left(Q,{}^A\mu_Q,\mu_Q^B\right)\right)={}_AQ_B \\ {}_A\kappa_B &: ({}_{\mathbb{A}}\mathcal{A}\leftarrow{}_{\mathbb{B}}\mathcal{B})\to{}_A\mathcal{F}_B \text{ where } {}_A\kappa_B(G)=\left({}_{\mathbb{A}}U\circ G\circ{}_{\mathbb{B}}F,{}_{\mathbb{A}}U\lambda_AG{}_{\mathbb{B}}F,{}_{\mathbb{A}}UG\lambda_{B\mathbb{B}}F\right). \end{aligned} $$

Proof. Let us consider an $\mathbb{A}$-$\mathbb{B}$-bimodule functor $\left(Q:\mathcal{B}\to\mathcal{A},{}^A\mu_Q,\mu_Q^B\right)$ such that AQ and QB preserve coequalizers. In particular, $\left(Q,\mu_Q^B\right)$ is a right $\mathbb{B}$-module functor, so that we can apply the map $\nu_B:\mathcal{F}_B\to(\mathcal{A}\leftarrow{}_{\mathbb{B}}\mathcal{B})$ of Theorem 3.37 and we get a functor $\nu_B\left(\left(Q,\mu_Q^B\right)\right)=Q_B:{}_{\mathbb{B}}\mathcal{B}\to\mathcal{A}$ which preserves coequalizers. By Proposition 3.30, $\left(Q_B,{}^A\mu_{Q_B}\right)$ is a left $\mathbb{A}$-module functor so that we can also apply the map ${}_A\nu:{}_A\mathcal{F}\to({}_{\mathbb{A}}\mathcal{A}\leftarrow\mathcal{B})$ of Theorem 3.38 where the category $\mathcal{B}$ is ${}_{\mathbb{B}}\mathcal{B}$. The map ${}_A\nu$ is defined by ${}_A\nu\left(\left(Q_B,{}^A\mu_{Q_B}\right)\right)={}_A\left(Q_B\right)={}_AQ_B:{}_{\mathbb{B}}\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ and ${}_AQ_B$ preserves coequalizers. Conversely, let us consider a functor $G:{}_{\mathbb{B}}\mathcal{B}\to{}_{\mathbb{A}}\mathcal{A}$ which preserves coequalizers. By Theorem 3.38, we get a left $\mathbb{A}$-module functor given by

$$ {}_A\kappa(G)=\left({}_{\mathbb{A}}U\circ G,{}_{\mathbb{A}}U\lambda_AG\right) $$

where ${}_{\mathbb{A}}U\circ G:{}_{\mathbb{B}}\mathcal{B}\to\mathcal{A}$ and $A{}_{\mathbb{A}}UG$ preserves coequalizers. By Lemma 3.19, also ${}_{\mathbb{A}}U\circ G:{}_{\mathbb{B}}\mathcal{B}\to\mathcal{A}$ preserves coequalizers. Thus, we can apply Theorem 3.37 and we get a right $\mathbb{B}$-module functor

$$ \kappa_B\left({}_{\mathbb{A}}UG\right)=\left({}_{\mathbb{A}}UG{}_{\mathbb{B}}F,{}_{\mathbb{A}}UG\lambda_{B\mathbb{B}}F\right) $$

where ${}_{\mathbb{A}}UG_{\mathbb{B}}F : \mathcal{B} \to \mathcal{A}$ is such that ${}_{\mathbb{A}}UG_{\mathbb{B}}FB$ preserves coequalizers. Clearly, since ${}_{\mathbb{A}}UG$ preserves coequalizers, ${}_{\mathbb{B}}F$ is a left adjoint and A preserves coequalizers by assumption, we deduce that also $A{}_{\mathbb{A}}UG_{\mathbb{B}}F$ preserves coequalizers. Now, we want to prove that ${}_{A}\nu_{B} : {}_{A}\mathcal{F}_{B} \to ({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B})$ and ${}_{A}\kappa_{B} : ({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{B}}\mathcal{B}) \to {}_{A}\mathcal{F}_{B}$ determine a bijection. We have

$$\begin{aligned}
\left({}_{A}\kappa_{B} \circ {}_{A}\nu_{B}\right)\left(\left(Q, {}^{A}\mu_{Q}, \mu_{Q}^{B}\right)\right) &= {}_{A}\kappa_{B}\left({}_{A}Q_{B}\right) \\
&= \left({}_{\mathbb{A}}U \circ {}_{A}Q_{B} \circ {}_{\mathbb{B}}F, {}_{\mathbb{A}}U\lambda_{A\,A}Q_{B\mathbb{B}}F, {}_{\mathbb{A}}U\,{}_{A}Q_{B}\lambda_{B\mathbb{B}}F\right) \\
&= \left(Q, {}_{\mathbb{A}}U\lambda_{A\,A}Q, Q_{B}\lambda_{B\mathbb{B}}F\right) = \left(Q, {}^{A}\mu_{Q}, \mu^{B}_{Q_{B\mathbb{B}}F}\right) = \left(Q, {}^{A}\mu_{Q}, \mu_{Q}^{B}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left({}_{A}\nu_{B} \circ {}_{A}\kappa_{B}\right)(G) &= {}_{A}\nu_{B}\left(\left({}_{\mathbb{A}}U \circ G \circ {}_{\mathbb{B}}F, {}_{\mathbb{A}}U\lambda_{A}G_{\mathbb{B}}F, {}_{\mathbb{A}}UG\lambda_{B\mathbb{B}}F\right)\right) \\
&= {}_{A}\left({}_{\mathbb{A}}U \circ G \circ {}_{\mathbb{B}}F\right)_{B} = {}_{A}\left(\left({}_{\mathbb{A}}U \circ G \circ {}_{\mathbb{B}}F\right)_{B}\right) \\
&\overset{(16)}{=} {}_{A}\left({}_{\mathbb{A}}U \circ G\right) \overset{(17)}{=} G.
\end{aligned}$$

□

Proposition 3.40. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ with coequalizers and assume that A preserves coequalizers. Let $Q : \mathcal{A} \to \mathcal{A}$ be an $\mathbb{A}$-bimodule functor. Then there exists a unique lifted functor ${}_{A}Q_{A} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ such that*

$$ {}_{\mathbb{A}}U\,{}_{A}Q_{A\mathbb{A}}F = Q. $$

Proof. By Proposition 3.31 there exists a unique functor ${}_{A}Q_{A} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U\,{}_{A}Q_{A} = Q_{A}$. Now, by Proposition 3.34 we also get that $Q_{A\mathbb{A}}F = Q$ so that we obtain

$$ {}_{\mathbb{A}}U\,{}_{A}Q_{A\mathbb{A}}F = Q. $$

□

Proposition 3.41. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ with coequalizers and assume that A preserves coequalizers. Let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$ with coequalizers and let $Q : \mathcal{B} \to \mathcal{A}$ be an $\mathbb{A}$-$\mathbb{B}$-bimodule functor. Then there exists a unique lifted functor ${}_{A}Q_{B} : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that*

$$ {}_{\mathbb{A}}U\,{}_{A}Q_{B\mathbb{B}}F = Q. $$

Proof. By Proposition 3.31 there exists a unique functor ${}_{A}Q_{B} : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ such that ${}_{\mathbb{A}}U\,{}_{A}Q_{B} = Q_{B}$. Now, by Proposition 3.34 we also get that $Q_{B\mathbb{B}}F = Q$ so that we obtain

$$ {}_{\mathbb{A}}U\,{}_{A}Q_{B\mathbb{B}}F = Q. $$

□

Proposition 3.42. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ with coequalizers and assume that A preserves coequalizers. Let $\mathbb{B} = (B, m_B, u_B)$ be a monad over a category $\mathcal{B}$ with coequalizers and let $P, Q : \mathcal{B} \to \mathcal{A}$ be $\mathbb{A}$-$\mathbb{B}$-bimodule functors. Let $f : P \to Q$ be a functorial morphism of left $\mathbb{A}$-module functors and of right $\mathbb{B}$-module functors. Then there exists a unique functorial morphism of left $\mathbb{A}$-module functors*

$$ f_{B} : P_{B} \to Q_{B} $$

satisfying

$$f_B \circ p_P = p_Q \circ (f_{\mathbb{B}} U).$$

Then we can consider

$$_A f_B : {}_A P_B \to {}_A Q_B$$

such that

$$_{\mathbb{A}} U_A f_B = f_B.$$

Proof. Consider the following diagram

$$\begin{tikzcd} PB_{\mathbb{B}}U \arrow[r, shift left, "\mu^B_{P\mathbb{B}}U"] \arrow[r, shift right, "P_{\mathbb{B}}U\lambda_B"'] \arrow[d, "fB_{\mathbb{B}}U"'] & P_{\mathbb{B}}U \arrow[r, "p_P"] \arrow[d, "f_{\mathbb{B}}U"'] & P_B \arrow[d, dotted, "f_B"] \\ QB_{\mathbb{B}}U \arrow[r, shift left, "\mu^B_{Q\mathbb{B}}U"] \arrow[r, shift right, "Q_{\mathbb{B}}U\lambda_B"'] & Q_{\mathbb{B}}U \arrow[r, "p_Q"] & Q_B \end{tikzcd}$$

Since f is a functorial morphism and it is a functorial morphism of right $\mathbb{B}$-module functors, the left square serially commutes. Note that

$$p_Q \circ (f_{\mathbb{B}} U) \circ \left(\mu^B_{P\mathbb{B}} U\right) = p_Q \circ (f_{\mathbb{B}} U) \circ (P_{\mathbb{B}} U \lambda_B)$$

so that, by the universal property of the coequalizer, there exists a unique morphism $f_B : P_B \to Q_B$ such that

$$f_B \circ p_P = p_Q \circ (f_{\mathbb{B}} U). \tag{18}$$

We now want to prove that f_B is a functorial morphism of left $\mathbb{A}$-module functor. In fact we have

$$\begin{aligned} f_B \circ {}^A\mu_{P_B} \circ (Ap_P) &\overset{(7)}{=} f_B \circ p_P \circ \left({}^A\mu_{P\mathbb{B}} U\right) \\ \overset{(18)}{=} p_Q \circ (f_{\mathbb{B}} U) \circ \left({}^A\mu_{P\mathbb{B}} U\right) &\overset{f\text{leftAlin}}{=} p_Q \circ \left({}^A\mu_{Q\mathbb{B}} U\right) \circ (Af_{\mathbb{B}} U) \\ \overset{(7)}{=} {}^A\mu_{Q_B} \circ (Ap_Q) \circ (Af_{\mathbb{B}} U) &\overset{(18)}{=} {}^A\mu_{Q_B} \circ (Af_B) \circ (Ap_P) \end{aligned}$$

and since A preserves coequalizers Ap_Q is an epimorphism so that we get

$$f_B \circ {}^A\mu_{P_B} = {}^A\mu_{Q_B} \circ (Af_B).$$

Then there exists a functorial morphism $_A f_B : {}_A P_B \to {}_A Q_B$ such that

$$_{\mathbb{A}} U_A f_B = f_B.$$

□

3.2. **The category of balanced bimodule functors.** We will construct here the monoidal category of balanced bimodule functors with respect to a monad.

Definition 3.43. Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ such that$\mathcal{A}$ has coequalizers and the underlying functor A preserves coequalizers. Let us define the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$ of *balanced bimodule functors* as follows

- *Ob* *Objects* are functors $_A Q_A : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{A}}\mathcal{A}$ where $Q : \mathcal{A} \to \mathcal{A}$ is an $\mathbb{A}$-$\mathbb{A}$-bimodule functor such that Q_A preserves coequalizers.
- *M* *Morphisms* are functorial morphisms $_A f_A : {}_A P_A \to {}_A Q_A$ where $f : P \to Q$ is a functorial morphism of $\mathbb{A}$-$\mathbb{A}$-bimodule functors.

Proposition 3.44. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ such that the underlying functor A preserves coequalizers and let ${}_AP_A, {}_AQ_A \in \mathrm{Ob}\left(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})\right)$. Then the functor ${}_AP_{AA}Q_A \in \mathrm{Ob}\left(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})\right)$.*

Proof. We will prove that ${}_AP_{AA}Q_A = {}_A\left(P_{AA}Q\right)_A$. Let us consider the functor $P_{AA}Q : \mathcal{A} \to \mathcal{A}$. Since ${}_AQ$ is a right $\mathbb{A}$-module functor by Proposition 3.31, then also $P_{AA}Q$ is a right $\mathbb{A}$-module functor by Lemma 3.17. Thus, we can consider

$$\begin{aligned}
\left(\left(P_{AA}Q\right)_A, p_{P_{AA}Q}\right) &= \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^A_{P_{AA}Q\mathbb{A}}U, P_{AA}Q_{\mathbb{A}}U\lambda_A\right) \\
&\overset{\text{Lem3.17}}{=} \mathrm{Coequ}_{\mathrm{Fun}}\left(P_A\mu^A_{AQ\mathbb{A}}U, P_{AA}Q_{\mathbb{A}}U\lambda_A\right) \\
&\overset{P_A\text{preserves coequ}}{=} \left(P_{AA}Q_A, P_Ap_{AQ}\right)
\end{aligned}$$

i.e.

$$\left(\left(P_{AA}Q\right)_A, p_{P_{AA}Q}\right) = \left(P_{AA}Q_A, P_Ap_{AQ}\right). \tag{19}$$

Now, observe that $\left(P_{AA}Q\right)_A$ is a left $\mathbb{A}$-module functor by Proposition 3.30 and $P_{AA}Q_A$ is a left $\mathbb{A}$-module functor by Lemma 3.17. So we can consider both lifting functors : ${}_A\left(P_{AA}Q\right)_A$ and ${}_A\left(P_A\right)_AQ_A$ and we have

$$\begin{aligned}
{}_{\mathbb{A}}U_A\left(\left(P_{AA}Q\right)_A\right) &\overset{\text{Pro3.31}}{=} \left(P_{AA}Q\right)_A \overset{(19)}{=} P_{AA}Q_A \\
&\overset{\text{Pro3.31}}{=} {}_{\mathbb{A}}U_A\left(P_A\right)_AQ_A
\end{aligned}$$

and

$$\begin{aligned}
{}_{\mathbb{A}}U\lambda_{AA}\left(\left(P_{AA}Q\right)_A\right) &\overset{\text{Pro3.30}}{=} {}^A\mu^A_{\left(P_{AA}Q\right)_A} \overset{(19)}{=} \mu_{P_{AA}Q_A} \\
&\overset{\text{Lem3.17}}{=} {}^A\mu_{P_AA}Q_A \overset{\text{Pro3.30}}{=} {}_{\mathbb{A}}U\lambda_{AA}\left(P_A\right)_AQ_A.
\end{aligned}$$

Hence

$$\begin{aligned}
{}_A\left(\left(P_{AA}Q\right)_A\right) &\overset{\text{Pro3.31}}{=} {}_A\left(P_{AA}Q\right)_A = {}_A\left(P_A\right)_AQ_A \\
&\overset{\text{Pro3.31}}{=} {}_AP_{AA}Q_A.
\end{aligned}$$

Thus

$${}_AP_{AA}Q_A = {}_A\left(P_{AA}Q\right)_A$$

where $P_{AA}Q : \mathcal{A} \to \mathcal{A}$ is an $\mathbb{A}$-bimodule functor satisfying the required conditions. □

Proposition 3.45. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ such that the underlying functor A preserves coequalizers. Then ${}_AA_A \in \mathrm{Ob}\left(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})\right)$ and it is the unit element for the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$.*

Proof. Since A is a monad, in particular an $\mathbb{A}$-bimodule functor. Then we can consider ${}_AA_A \in Ob\left(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})\right)$ as the object coming from the endofunctor $A : \mathcal{A} \to \mathcal{A}$. By definition we have

$$\left(A_A, p_A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(m_{A\mathbb{A}}U, A_{\mathbb{A}}U\lambda_A\right) = \left({}_{\mathbb{A}}U, {}_{\mathbb{A}}U\lambda_A\right)$$

and it is a left $\mathbb{A}$-module functor by Proposition 3.13. By Lemma 3.29, we can consider

$${}_AA_A = {}_A\left({}_{\mathbb{A}}U\right) = \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$$

as the unique functor which satisfies

$$ {}_{\mathbb{A}}U_A A_A = {}_{\mathbb{A}}U \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}} = {}_{\mathbb{A}}U = A_A $$

and

$$ {}_{\mathbb{A}}U \lambda_{A A} A_A = {}_{\mathbb{A}}U \lambda_A \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}} = {}_{\mathbb{A}}U \lambda_A = {}^{A}\mu_{{}_{\mathbb{A}}U} = {}^{A}\mu_{A_A}. $$

Clearly ${}_A A_A = \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ is the identity element for the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. □

Corollary 3.46. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ such that the underlying functor A preserves coequalizers. Then we have*

$$ {}_A A_A \circ F = F \text{ and } F \circ {}_A A_A = F $$

for every $F \in \mathrm{Ob}\,(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A}))$.

Proof. By Proposition 3.45 we have that ${}_A A_A = \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ is the identity element for the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. Therefore, in particular, we have that

$$ {}_A A_A \circ F = F \text{ and } F \circ {}_A A_A = F $$

for every $F \in Ob\,(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A}))$. □

Proposition 3.47. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ such that the underlying functor A preserves coequalizers, let ${}_A P_A, {}_A Q_A, {}_A T_A \in \mathrm{Ob}\,(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A}))$ and let ${}_A f_A : {}_A P_A \to {}_A Q_A$, ${}_A g_A : {}_A Q_A \to {}_A T_A$ be morphisms in $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. Then ${}_A g_A \circ {}_A f_A$ is still a morphism in the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$ and*

$$ {}_A (g \circ f)_A = {}_A g_A \circ {}_A f_A. \tag{20} $$

Proof. We will prove that ${}_A g_A \circ {}_A f_A = {}_A (g \circ f)_A$ where $g \circ f$ is an A-bilinear functorial morphism as composite of A-bilinear functorial morphisms. By assumption, using notations of Proposition 3.42 we have the following serially commutative diagram

$$ \begin{tikzcd} PA_{\mathbb{A}}U \arrow[r, shift left, "\mu^A_{P_{\mathbb{A}}U}"] \arrow[r, shift right, "P_{\mathbb{A}}U\lambda_A"'] \arrow[d, "fA_{\mathbb{A}}U"'] & P_{\mathbb{A}}U \arrow[r, "p_P"] \arrow[d, "f_{\mathbb{A}}U"'] & P_A \arrow[d, "f_A"] \\ QA_{\mathbb{A}}U \arrow[r, shift left, "\mu^A_{Q_{\mathbb{A}}U}"] \arrow[r, shift right, "Q_{\mathbb{A}}U\lambda_A"'] \arrow[d, "gA_{\mathbb{A}}U"'] & Q_{\mathbb{A}}U \arrow[r, "p_Q"] \arrow[d, "g_{\mathbb{A}}U"'] & Q_A \arrow[d, "g_A"] \\ TA_{\mathbb{A}}U \arrow[r, shift left, "\mu^A_{T_{\mathbb{A}}U}"] \arrow[r, shift right, "T_{\mathbb{A}}U\lambda_A"'] & T_{\mathbb{A}}U \arrow[r, "p_T"] & T_A \end{tikzcd} $$

Then ${}_A f_A$ is the unique morphism such that

$$ {}_{\mathbb{A}}U_A f_A = f_A $$

where

$$ f_A \circ p_P = p_Q \circ (f_{\mathbb{A}}U) \tag{21} $$

and ${}_A g_A$ is the unique morphism such that

$$ {}_{\mathbb{A}}U_A g_A = g_A $$

where

$$ g_A \circ p_Q = p_T \circ (g_{\mathbb{A}}U)\,. \tag{22} $$

Note that, since f and g are A-bilinear morphism, $g \circ f$ is still an A-bilinear morphism, so that we can also consider $(g \circ f)_A$ such that

$$(g \circ f)_A \circ p_P = p_T \circ [(g \circ f)\, {}_{\mathbb{A}}U] = p_T \circ (g_{\mathbb{A}}U) \circ (f_{\mathbb{A}}U)\,. \tag{23}$$

First we prove that $(g \circ f)_A = g_A \circ f_A$. In fact we have

$$\begin{aligned}(g \circ f)_A \circ p_P &\overset{(23)}{=} p_T \circ (g_{\mathbb{A}}U) \circ (f_{\mathbb{A}}U) \\ &\overset{(22)}{=} g_A \circ p_Q \circ (f_{\mathbb{A}}U) \overset{(21)}{=} g_A \circ f_A \circ p_P\end{aligned}$$

and since p_P is an epimorphism we obtain

$$(g \circ f)_A = g_A \circ f_A.$$

The, we can both consider ${}_A(g \circ f)_A = {}_A((g \circ f)_A)$ such that

$${}_{\mathbb{A}}U_A\, {}_A(g \circ f)_A = {}_{\mathbb{A}}U_A\, ({}_A(g \circ f)_A) = (g \circ f)_A$$

and the composite of the liftings ${}_A g_A \circ {}_A f_A$ such that

$${}_{\mathbb{A}}U\, [{}_A g_A \circ {}_A f_A] = ({}_{\mathbb{A}}U_A\, {}_A g_A) \circ ({}_{\mathbb{A}}U_A\, {}_A f_A) = g_A \circ f_A.$$

We have

$$\begin{aligned}{}_{\mathbb{A}}U_A\, {}_A(g \circ f)_A \circ p_P &= {}_{\mathbb{A}}U_A\, ({}_A(g \circ f)_A) \circ p_P = (g \circ f)_A \circ p_P \\ &\overset{(23)}{=} p_T \circ (g_{\mathbb{A}}U) \circ (f_{\mathbb{A}}U) = g_A \circ p_Q \circ (f_{\mathbb{A}}U) = g_A \circ f_A \circ p_P\end{aligned}$$

and since p_P is an epimorphism we deduce that

$${}_{\mathbb{A}}U_A\, {}_A(g \circ f)_A = g_A \circ f_A = {}_{\mathbb{A}}U_A\, {}_A g_A \circ {}_{\mathbb{A}}U_A\, {}_A f_A.$$

Since ${}_{\mathbb{A}}U$ reflects we conclude that

$${}_A(g \circ f)_A = {}_A g_A \circ {}_A f_A$$

where ${}_{\mathbb{A}}U_A\, {}_A(g \circ f)_A = (g \circ f)_A$ and $(g \circ f)_A \circ p_P = p_T \circ [(g \circ f)\, {}_{\mathbb{A}}U]$. □

PROPOSITION 3.48. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad on a category $\mathcal{A}$ such that the underlying functor A preserves coequalizers, let ${}_A P_A, {}_A Q_A, {}_A T_A \in \mathrm{Ob}\,(({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A}))$ and let ${}_A f_A : {}_A P_A \to {}_A Q_A$, ${}_A g_A : {}_A Q_A \to {}_A T_A$, ${}_A h_A : {}_A T_A \to {}_A W_A$ be morphisms in $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. Then*

$${}_A h_A \circ ({}_A g_A \circ {}_A f_A) = ({}_A h_A \circ {}_A g_A) \circ {}_A f_A.$$

Proof. By Proposition 3.47 we have that, for every morphisms ${}_A f_A : {}_A P_A \to {}_A Q_A$, ${}_A g_A : {}_A Q_A \to {}_A T_A$ in $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$, also the morphism ${}_A g_A \circ {}_A f_A$ is in $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$ and ${}_A(g \circ f)_A = {}_A g_A \circ {}_A f_A$. Hence we have that

$$\begin{aligned}{}_A h_A \circ ({}_A g_A \circ {}_A f_A) &\overset{(20)}{=} {}_A h_A \circ ({}_A(g \circ f)_A) \overset{(20)}{=} ({}_A(h \circ (g \circ f))_A) \\ &\overset{\mathcal{A}\text{strictly monoidal}}{=} ({}_A((h \circ g) \circ f)_A) \overset{(20)}{=} {}_A(h \circ g)_A \circ {}_A f_A \\ &\overset{(20)}{=} ({}_A h_A \circ {}_A g_A) \circ {}_A f_A.\end{aligned}$$

□

Theorem 3.49. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad over a category $\mathcal{A}$ such that$\mathcal{A}$ has coequalizers and the underlying functor A preserves coequalizers. The category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$ of balanced bimodule functors is a strict monoidal category.*

Proof. By Proposition 3.44, we defined a composition of the objects of the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. Moreover, by Proposition 3.45, ${}_AA_A$ is the unit for the category $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$. Since the composition of functors is associative and by Corollary 3.46, it is easy to prove that $({}_{\mathbb{A}}\mathcal{A} \leftarrow {}_{\mathbb{A}}\mathcal{A})$ is a strict monoidal category. □

3.3. **The comparison functor for monads.**

Proposition 3.50. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ with unit η and counit ϵ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$. There exists a bijective correspondence between the following collections of data:*

- $\mathfrak{M}$ *monad morphisms $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$*
- $\mathfrak{R}$ *functorial morphism $r : LA \to L$ such that (L, r) is a right module functor for the monad $\mathbb{A}$*
- $\mathfrak{L}$ *functorial morphism $l : AR \to R$ such that (R, l) is a left module functor for the monad $\mathbb{A}$*

given by

$$\begin{aligned}
\Theta &: \mathfrak{M} \to \mathfrak{R} \text{ where } \Theta(\psi) = (\epsilon L) \circ (L\psi) \\
\Xi &: \mathfrak{R} \to \mathfrak{M} \text{ where } \Xi(r) = (Rr) \circ (\eta A) \\
\Gamma &: \mathfrak{M} \to \mathfrak{L} \text{ where } \Gamma(\psi) = (R\epsilon) \circ (\psi R) \\
\Lambda &: \mathfrak{L} \to \mathfrak{M} \text{ where } \Lambda(l) = (lL) \circ (A\eta).
\end{aligned}$$

Theorem 3.51. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$. There exists a bijective correspondence between the following collections of data:*

- $\mathfrak{K}$ *Functors $K : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$ such that ${}_{\mathbb{A}}U \circ K = R$*
- $\mathfrak{M}$ *monad morphisms $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$*

given by

$$\begin{aligned}
\Psi &: \mathfrak{K} \to \mathfrak{M} \text{ where } \Psi(K) = ([{}_{\mathbb{A}}U\lambda_A K]\, L) \circ (A\eta) \\
\Upsilon &: \mathfrak{M} \to \mathfrak{K} \text{ where } \Upsilon(\psi)(X) = (RX, (R\epsilon X) \circ (\psi RX)) \text{ and } \Upsilon(\psi)(f) = Rf.
\end{aligned}$$

Remark 3.52. When $\mathbb{A} = \mathbb{RL} = (RL, R\epsilon L, \eta)$ and $\psi = \mathrm{Id}_{\mathbb{RL}}$ the functor $K = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{RL}}\mathcal{B}$ such that ${}_{\mathbb{RL}}U \circ K = R$ is called the *Eilenberg-Moore comparison functor.*

Corollary 3.53. *Let $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ be monads on a category $\mathcal{B}$. There exists a bijective correspondence between the following collections of data:*

- $\mathcal{K}$ *Functors $K : {}_{\mathbb{A}}\mathcal{B} \to {}_{\mathbb{B}}\mathcal{B}$ such that ${}_{\mathbb{B}}U \circ K = {}_{\mathbb{A}}U$,*
- $\mathcal{M}$ *monad morphisms $\psi : \mathbb{A} \to \mathbb{B}$*

given by

$$\begin{aligned}
\Psi &: \mathcal{K} \to \mathcal{M} \text{ where } \Psi(K) = ([{}_{\mathbb{A}}U\lambda_A K]\, {}_{\mathbb{A}}F) \circ (Au_A) \\
\Upsilon &: \mathcal{M} \to \mathcal{K} \text{ where } \Upsilon(\psi)(X) = ({}_{\mathbb{A}}UX, ({}_{\mathbb{A}}U\lambda_A X) \circ (\psi_{\mathbb{A}}UX)) \text{ and } \Upsilon(\psi)(f) = {}_{\mathbb{A}}U(f).
\end{aligned}$$

Proposition 3.54. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monad morphism. Let $r=\Theta(\psi)=(\epsilon L)\circ(L\psi)$. Then the functor $K_\psi=\Upsilon(\psi):\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ has a left adjoint $D_\psi:{}_{\mathbb{A}}\mathcal{B}\to\mathcal{A}$ if and only if, for every $\left(Y,{}^{A}\mu_Y\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_Y\right)$. In this case, there exists a functorial morphism $d_\psi:L_{\mathbb{A}}U\to D_\psi$ such that*

$$(D_\psi,d_\psi)=\mathrm{Coequ}_{\mathrm{Fun}}\left(r_{\mathbb{A}}U,L_{\mathbb{A}}U\lambda_A\right)$$

and thus

$$\left[D_\psi\left(\left(Y,{}^{A}\mu_Y\right)\right),d_\psi\left(Y,{}^{A}\mu_Y\right)\right]=\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_Y\right).$$

Corollary 3.55. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Let $r=\Theta(\mathrm{Id}_{\mathbb{RL}})=\epsilon L$. Then the functor $K=\Upsilon(\mathrm{Id}_{\mathbb{RL}}):\mathcal{A}\to{}_{\mathbb{RL}}\mathcal{B}$ has a left adjoint $D:{}_{\mathbb{RL}}\mathcal{B}\to\mathcal{A}$ if and only, for every $\left(Y,{}^{RL}\mu_Y\right)\in{}_{\mathbb{RL}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY,L^{RL}\mu_Y\right)$. In this case, there exists a functorial morphism $d:L_{\mathbb{RL}}U\to D$ such that*

$$(D,d)=\mathrm{Coequ}_{\mathrm{Fun}}\left(\epsilon L_{\mathbb{RL}}U,L_{\mathbb{RL}}U\lambda_{RL}\right)$$

and thus

$$\left[D\left(\left(Y,{}^{RL}\mu_Y\right)\right),d\left(Y,{}^{RL}\mu_Y\right)\right]=\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY,L^{RL}\mu_Y\right).$$

Remark 3.56. In the setting of Proposition 3.54, for every $X\in\mathcal{A}$, we note that the counit of the adjunction (D_ψ,K_ψ) is given by

$$\widetilde{\epsilon}X=\widetilde{a}_{X,K_\psi X}^{-1}\left(\mathrm{Id}_{K_\psi X}\right):D_\psi K_\psi(X)\to X.$$

We will consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(D_\psi\left(\left(Y,{}^{A}\mu_Y\right)\right),X\right) \arrow[r, "\tilde{a}_{X,Y}"] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}\left(d_\psi\left(\left(Y,{}^{A}\mu_Y\right)\right),X\right)"'] & \mathrm{Hom}_{{}_{\mathbb{A}}\mathcal{B}}\left(\left(Y,{}^{A}\mu_Y\right),K_\psi X\right) \arrow[d] \\
\mathrm{Hom}_{\mathcal{A}}(LY,X) \arrow[r, "a_{X,Y}"] \arrow[d, shift right, "\mathrm{Hom}_{\mathcal{A}}(rY,X)"'] \arrow[d, shift left, "\mathrm{Hom}_{\mathcal{A}}\left(L^{A}\mu_Y,X\right)"] & \mathrm{Hom}_{\mathcal{B}}(Y,RX) \arrow[d, shift right, "(\Gamma(\psi)X)\circ(A-)"'] \arrow[d, shift left, "\mathrm{Hom}_{\mathcal{B}}\left({}^{A}\mu_Y,RX\right)"] \\
\mathrm{Hom}_{\mathcal{A}}(LAY,X) \arrow[r, "a_{X,AY}"] & \mathrm{Hom}_{\mathcal{B}}(AY,RX)
\end{tikzcd} \tag{24}$$

defining $\widetilde{a}_{-,Y}$ in the particular case of $\left(Y,{}^{A}\mu_Y\right)=K_\psi X$. Note that, since $K_\psi X=(RX,(R\epsilon X)\circ(\psi RX))=(RX,lX)$, we have

$$\begin{aligned}
\left(D_\psi K_\psi(X),d_\psi K_\psi(X)\right) &= \left(D_\psi(RX,lX),d_\psi K_\psi(X)\right)=\mathrm{Coequ}_{\mathcal{B}}(rRX,LlX)\\
&= \mathrm{Coequ}_{\mathcal{B}}\left((\epsilon LRX)\circ(L\psi RX),(LR\epsilon X)\circ(L\psi RX)\right)
\end{aligned}$$

i.e.

$$\left(D_\psi K_\psi(X),d_\psi K_\psi(X)\right)=\mathrm{Coequ}_{\mathcal{B}}(rRX,LlX) \tag{25}$$

where $l=\Gamma(\psi)=(R\epsilon)\circ(\psi R)$. We compute

$$\begin{aligned}
(\widetilde{\epsilon}X)\circ(d_\psi K_\psi X) &= \mathrm{Hom}_{\mathcal{A}}(d_\psi K_\psi X,X)\left((\widetilde{\epsilon}X)\right)\\
&= \mathrm{Hom}_{\mathcal{A}}(d_\psi K_\psi X,X)\left(\widetilde{a}_{X,K_\psi X}^{-1}\left(\mathrm{Id}_{K_\psi X}\right)\right)
\end{aligned}$$

$$= \left[\mathrm{Hom}_{\mathcal{A}}\left(d_\psi K_\psi X, X\right) \circ \widetilde{a}^{-1}_{X,K_\psi X}\right]\left(\mathrm{Id}_{K_\psi X}\right)$$
$$\overset{(24)}{=} a^{-1}_{X,K_\psi X\,\mathbb{A}}U\left(\mathrm{Id}_{K_\psi X}\right) = a^{-1}_{X,K_\psi X}\left(\mathrm{Id}_{\mathbb{A}UK_\psi X}\right)$$
$$= a^{-1}_{X,K_\psi X}\left(\mathrm{Id}_{RX}\right) = \epsilon X$$

so that

$$\left(\widetilde{\epsilon}X\right) \circ \left(d_\psi K_\psi X\right) = \epsilon X.$$

Since $\widetilde{\epsilon}X = \widetilde{a}^{-1}_{X,K_\psi X}\left(\mathrm{Id}_{K_\psi X}\right)$ and $\widetilde{a}^{-1}_{X,K_\psi X}$ is an isomorphism, we deduce that $\widetilde{\epsilon}X : D_\psi K_\psi(X) \to X$ is defined as the unique morphism such that

$$\left(\widetilde{\epsilon}X\right) \circ \left(d_\psi K_\psi X\right) = \epsilon X. \tag{26}$$

On the other hand, for every $\left(Y, {}^A\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, the unit of the adjunction $\left(D_\psi, K_\psi\right)$, $\widetilde{\eta} : {}_{\mathbb{A}}\mathcal{B} \to K_\psi D_\psi$, is given by

$$\widetilde{\eta}\left(Y, {}^A\mu_Y\right) = \widetilde{a}_{D_\psi(Y,{}^A\mu_Y),Y}\left(\mathrm{Id}_{D_\psi(Y,{}^A\mu_Y)}\right) : \left(Y, {}^A\mu_Y\right) \to K_\psi D_\psi\left(\left(Y, {}^A\mu_Y\right)\right).$$

Then by commutativity of the diagram (24), we deduce that

$${}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^A\mu_Y\right) = {}_{\mathbb{A}}U\widetilde{a}_{D_\psi(Y,{}^A\mu_Y),Y}\left(\mathrm{Id}_{D_\psi(Y,{}^A\mu_Y)}\right)$$
$$= a_{D_\psi(Y,{}^A\mu_Y),Y} \circ \mathrm{Hom}_{\mathcal{A}}\left(d_\psi\left(\left(Y, {}^A\mu_Y\right)\right), D_\psi\left(Y, {}^A\mu_Y\right)\right)\left(\mathrm{Id}_{D_\psi(Y,{}^A\mu_Y)}\right)$$
$$= a_{D_\psi(Y,{}^A\mu_Y),Y}\left(d_\psi\left(\left(Y, {}^A\mu_Y\right)\right)\right) = \left(Rd_\psi\left(Y, {}^A\mu_Y\right)\right) \circ (\eta Y).$$

Thus we obtain that

$${}_{\mathbb{A}}U\widetilde{\eta}\left(Y, {}^A\mu_Y\right) = \left(Rd_\psi\left(Y, {}^A\mu_Y\right)\right) \circ (\eta Y). \tag{27}$$

Observe that, for every $Y \in \mathcal{B}$ we have that ${}_{\mathbb{A}}F(Y) = (AY, m_AY)$. Moreover

$$\left(D_{\psi\mathbb{A}}F(Y), d_{\psi\mathbb{A}}F(Y)\right) = \left(D_\psi(AY, m_AY), d_\psi(AY, m_AY)\right)$$
$$= \mathrm{Coequ}_{\mathcal{A}}\left(rAY, Lm_AY\right) \overset{(2)}{=} (LY, rY)$$

so that we get

$$\left(D_{\psi\mathbb{A}}F, d_{\psi\mathbb{A}}F\right) = (L, r). \tag{28}$$

In particular

$$d_\psi(AY, m_AY) = rY. \tag{29}$$

Theorem 3.57. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{A} = (A, m_A, u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi : \mathbb{A} = (A, m_A, u_A) \to \mathbb{RL} = (RL, R\epsilon L, \eta)$ be a monad morphism. Let $r = \Theta(\psi) = (\epsilon L) \circ (L\psi)$. Assume that, for every $\left(Y, {}^A\mu_Y\right) \in {}_{\mathbb{A}}\mathcal{B}$, there exists $\mathrm{Coequ}_{\mathcal{A}}\left(rY, L{}^A\mu_Y\right)$. Then we can consider the functor $K_\psi = \Upsilon(\psi) : \mathcal{A} \to {}_{\mathbb{A}}\mathcal{B}$. Its left adjoint $D_\psi : {}_{\mathbb{A}}\mathcal{B} \to \mathcal{A}$ is full and faithful if and only if*

1) *R preserves the coequalizer*

$$\left(D_\psi, d_\psi\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(r_{\mathbb{A}}U, L_{\mathbb{A}}U\lambda_A\right)$$

2) *$\psi : \mathbb{A} \to \mathbb{RL}$ is a monad isomorphism.*

Corollary 3.58. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Let $r=\Theta(\mathrm{Id}_{\mathbb{RL}})=\epsilon L$. Assume that, for every $\left(Y,{}^{RL}\mu_Y\right)\in{}_{\mathbb{RL}}\mathcal{B}$, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY,L^{RL}\mu_Y\right)$. *Then we can consider the functor $K=\Upsilon(\mathrm{Id}_{\mathbb{RL}}):\mathcal{A}\to{}_{\mathbb{RL}}\mathcal{B}$. Its left adjoint $D:{}_{\mathbb{RL}}\mathcal{B}\to\mathcal{A}$ is full and faithful if and only if R preserves the coequalizer*

$$(D,d)=\mathrm{Coequ}_{\mathrm{Fun}}\left(\epsilon L_{\mathbb{RL}}U,L_{\mathbb{RL}}U\lambda_{RL}\right).$$

Theorem 3.59. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monad morphism. Let $r=\Theta(\psi)=(\epsilon L)\circ(L\psi)$ and $l=\Gamma(\psi)=(R\epsilon)\circ(\psi R)$. Assume that, for every $\left(Y,{}^{A}\mu_Y\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_Y\right)$. *Then we can consider the functor $K_\psi=\Upsilon(\psi):\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ and its left adjoint $D_\psi:{}_{\mathbb{A}}\mathcal{B}\to\mathcal{A}$. The functor K_ψ is an equivalence of categories if and only if*

1) *R preserves the coequalizer*

$$(D_\psi,d_\psi)=\mathrm{Coequ}_{\mathrm{Fun}}\left(r_{\mathbb{A}}U,L_{\mathbb{A}}U\lambda_A\right)$$

2) *R reflects isomorphisms and*
3) *$\psi:\mathbb{A}\to\mathbb{RL}$ is a monad isomorphism.*

Definition 3.60. Let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\left(R,{}^{A}\mu_R\right)$ be a left $\mathbb{A}$-module functor. We say that $\left(R,{}^{A}\mu_R\right)$ is a left $\mathbb{A}$-*coGalois functor* if R has a left adjoint L and if the canonical morphism

$$cocan:=\left({}^{A}\mu_R L\right)\circ(A\eta):\mathbb{A}\to\mathbb{RL}$$

is a monad isomorphism, where η denotes the unit of the adjunction (L,R).

Corollary 3.61. *Let $\left(R,{}^{A}\mu_R\right)$ be a left A-coGalois functor where $R:\mathcal{A}\to\mathcal{B}$ preserves coequalizers, R reflects isomorphisms and $\mathbb{A}=(A,m_A,u_A)$ is a monad on $\mathcal{B}$. Assume that, for every $\left(Y,{}^{A}\mu_Y\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_Y\right)$ *where $r=(\epsilon L)\circ(Lcocan)$ where L is the left adjoint of R and ϵ is the counit of the adjunction (L,R). Then we can consider the functor $K_{cocan}:\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ and its left adjoint $D_{cocan}:{}_{\mathbb{A}}\mathcal{B}\to\mathcal{A}$. Then the functor K_{cocan} is an equivalence of categories.*

Theorem 3.62 (Beck's Theorem for monads). *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Let $r=\Theta(\mathrm{Id}_{\mathbb{RL}})=\epsilon L$ and assume that, for every $\left(Y,{}^{RL}\mu_Y\right)\in{}_{\mathbb{RL}}\mathcal{B}$, there exists* $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY,L^{RL}\mu_Y\right)$. *Then we can consider the functor $K=\Upsilon(\mathrm{Id}_{\mathbb{RL}}):\mathcal{A}\to{}_{\mathbb{RL}}\mathcal{B}$ and its left adjoint $D:{}_{\mathbb{RL}}\mathcal{B}\to\mathcal{A}$. The functor K is an equivalence of categories if and only if*

1) *R preserves the coequalizer*

$$(D,d)=\mathrm{Coequ}_{\mathrm{Fun}}\left(\epsilon L_{\mathbb{RL}}U,L_{\mathbb{RL}}U\lambda_{RL}\right).$$

2) *R reflects isomorphisms.*

Definition 3.63. Let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $R:\mathcal{A}\to\mathcal{B}$ be a functor. The functor R is called ψ-*monadic* if it has a left adjoint $L:\mathcal{B}\to\mathcal{A}$ for which there exists $\psi:\mathbb{A}\to\mathbb{RL}$ a monad morphism such that the functor $K_\psi=\Upsilon(\psi):\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ is an equivalence of categories.

Definition 3.64. Let $R:\mathcal{A}\to\mathcal{B}$ be a functor. The functor R is called *monadic* if it has a left adjoint $L:\mathcal{B}\to\mathcal{A}$ for which the functor $K=\Upsilon\left(\mathrm{Id}_{\mathbb{RL}}\right):\mathcal{A}\to{}_{\mathbb{RL}}\mathcal{B}$ is an equivalence of categories.

The following is a slightly improved version of Theorem 3.14 p. 101 [BW].

Theorem 3.65 (Generalized Beck's Precise Tripleability Theorem). *Let $R:\mathcal{A}\to\mathcal{B}$ be a functor and let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$. Then R is ψ-monadic if and only if*

1) *R has a left adjoint $L:\mathcal{B}\to\mathcal{A}$,*
2) *$\psi:\mathbb{A}\to\mathbb{RL}$ is a monads isomorphism where $\mathbb{RL}=(RL,R\epsilon L,\eta)$ with η and ϵ unit and counit of (L,R),*
3) *for every $\left(Y,{}^{A}\mu_Y\right)\in{}_{\mathbb{A}}\mathcal{B}$, there exist $\mathrm{Coequ}_{\mathcal{A}}\left(rY,L^{A}\mu_Y\right)$, where $r=\Theta(\psi)=(\epsilon L)\circ(L\psi)$, and R preserves the coequalizer*
$$\mathrm{Coequ}_{\mathrm{Fun}}\left(r_{\mathbb{A}}U,L_{\mathbb{A}}U\lambda_A\right),$$
4) *R reflects isomorphisms.*

In this case in $\mathcal{A}$ there exist coequalizers of R-contractible coequalizer pairs and R preserves them.

Corollary 3.66 (Beck's Precise Tripleability Theorem). *Let $R:\mathcal{A}\to\mathcal{B}$ be a functor. Then R is monadic if and only if*

1) *R has a left adjoint $L:\mathcal{B}\to\mathcal{A}$,*
2) *for every $\left(Y,{}^{RL}\mu_Y\right)\in{}_{\mathbb{RL}}\mathcal{B}$, there exist $\mathrm{Coequ}_{\mathcal{A}}\left(\epsilon LY,L^{RL}\mu_Y\right)$ and R preserves the coequalizer*
$$\mathrm{Coequ}_{\mathrm{Fun}}\left(\epsilon L_{\mathbb{RL}}U,L_{\mathbb{RL}}U\lambda_{RL}\right),$$
3) *R reflects isomorphisms.*

In this case in $\mathcal{A}$ there exist coequalizers of R-contractible coequalizer pairs and R preserves them.

Theorem 3.67 (Generalized Beck's Theorem for Monads). *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{A}=(A,m_A,u_A)$ be a monad on the category $\mathcal{B}$ and let $\psi:\mathbb{A}=(A,m_A,u_A)\to\mathbb{RL}=(RL,R\epsilon L,\eta)$ be a monads morphism such that ψY is an epimorphism for every $Y\in\mathcal{B}$. Let $K_\psi=\Upsilon(\psi)=(R,(R\epsilon)\circ(\psi R))$ and ${}_{\mathbb{A}}UK_\psi(f)={}_{\mathbb{A}}U\Upsilon(\psi)(f)=R(f)$ for every morphism f in $\mathcal{A}$. Then $K_\psi:\mathcal{A}\to{}_{\mathbb{A}}\mathcal{B}$ is full and faithful if and only if for every $X\in\mathcal{A}$ we have that $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$.*

Corollary 3.68 (Beck's Theorem for Monads). *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Then $K=\Upsilon\left(\mathrm{Id}_{RL}\right):\mathcal{A}\to{}_{\mathbb{RL}}\mathcal{B}$ is full and faithful if and only if for every $X\in\mathcal{A}$ we have that $(X,\epsilon X)=\mathrm{Coequ}_{\mathcal{A}}(LR\epsilon X,\epsilon LRX)$.*

4. Comonads

Definition 4.1. A *comonad* on a category $\mathcal{A}$ is a triple $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$, where $C:\mathcal{A}\to\mathcal{A}$ is a functor, $\Delta^C:C\to CC$ and $\varepsilon^C:C\to\mathcal{A}$ are functorial morphisms satisfying the coassociativity and the counitality conditions
$$\left(\Delta^C C\right)\circ\Delta^C=\left(C\Delta^C\right)\circ\Delta^C\quad\text{and}\quad\left(C\varepsilon^C\right)\circ\Delta^C=C=\left(\varepsilon^C C\right)\circ\Delta^C.$$

DEFINITION 4.2. A *morphism between two comonads* $\mathbb{C} = \big(C, \Delta^C, \varepsilon^C\big)$ and $\mathbb{D} = \big(D, \Delta^D, \varepsilon^D\big)$ on a category $\mathcal{A}$ is a functorial morphism $\varphi : C \to D$ such that

$$\Delta^C \circ \varphi = (\varphi\varphi) \circ \Delta^D \quad \text{and} \quad \varepsilon^C \circ \varphi = \varepsilon^D.$$

EXAMPLE 4.3. Let $\big(\mathcal{C}, \Delta^{\mathcal{C}}, \varepsilon^{\mathcal{C}}\big)$ an A-coring where A is a ring. Then

- $\mathcal{C}$ is an A-A-bimodule
- $\Delta^{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ is a morphism of A-A-bimodules
- $\varepsilon^{\mathcal{C}} : \mathcal{C} \to A$ is a morphism of A-A-bimodules satisfying the following

$$\big(\Delta^{\mathcal{C}} \otimes_A \mathcal{C}\big) \circ \Delta^{\mathcal{C}} = \big(\mathcal{C} \otimes_A \Delta^{\mathcal{C}}\big) \circ \Delta^{\mathcal{C}}, \big(\mathcal{C} \otimes_A \varepsilon^{\mathcal{C}}\big) \circ \Delta^{\mathcal{C}} = r_{\mathcal{C}}^{-1} \quad \text{and} \quad \big(\varepsilon^{\mathcal{C}} \otimes_A \mathcal{C}\big) \circ \Delta^{\mathcal{C}} = l_{\mathcal{C}}^{-1}$$

where $r_{\mathcal{C}} : \mathcal{C} \otimes_A A \to \mathcal{C}$ and $l_{\mathcal{C}} : A \otimes_A \mathcal{C} \to \mathcal{C}$ are the right and left constraints. Let

$$\begin{aligned} C &= - \otimes_A \mathcal{C} : Mod\text{-}A \to Mod\text{-}A \\ \Delta^C &= - \otimes_A \Delta^{\mathcal{C}} : - \otimes_A \mathcal{C} \to - \otimes_A \mathcal{C} \otimes_A \mathcal{C} \\ \varepsilon^C &= r_- \circ \big(- \otimes_A \varepsilon^{\mathcal{C}}\big) : - \otimes_A \mathcal{C} \to - \otimes_A A \to - \end{aligned}$$

Then, dually to the case of the R-ring, $\mathbb{C} = \big(C, \Delta^C, \varepsilon^C\big)$ is a comonad on the category $Mod\text{-}A$.

PROPOSITION 4.4 ([H]). *Let (L, R) be an adjunction with unit η and counit ϵ where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $\mathbb{LR} = (LR, L\eta R, \epsilon)$ is a comonad on the category $\mathcal{A}$.*

Proof. Dual to the proof of Proposition 3.4. □

DEFINITION 4.5. A *left comodule functor* for a comonad $\mathbb{C} = \big(C, \Delta^C, \varepsilon^C\big)$ on a category $\mathcal{A}$ is a pair $\big(Q, {}^C\rho_Q\big)$ where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and ${}^C\rho_Q : Q \to CQ$ is a functorial morphism such that

$$\big(C^C\rho_Q\big) \circ {}^C\rho_Q = \big(\Delta^C Q\big) \circ {}^C\rho_Q \quad \text{and} \quad Q = \big(\varepsilon^C Q\big) \circ {}^C\rho_Q.$$

DEFINITION 4.6. A *right comodule functor* for a comonad $\mathbb{C} = \big(C, \Delta^C, \varepsilon^C\big)$ on a category $\mathcal{A}$ is a pair $\big(P, \rho_P^C\big)$ where $P : \mathcal{A} \to \mathcal{B}$ is a functor and $\rho_P^C : P \to PC$ is a functorial morphism such that

$$\big(\rho_P^C C\big) \circ \rho_P^C = \big(P\Delta^C\big) \circ \rho_P^C \quad \text{and} \quad P = \big(P\varepsilon^C\big) \circ \rho_P^C.$$

DEFINITION 4.7. For two comonads $\mathbb{C} = \big(C, \Delta^C, \varepsilon^C\big)$ on a category $\mathcal{A}$ and $\mathbb{D} = \big(D, \Delta^D, \varepsilon^D\big)$ on a category $\mathcal{B}$, a $\mathbb{C}$-$\mathbb{D}$-*bicomodule functor* is a triple $\big(Q, {}^C\rho_Q, \rho_Q^D\big)$, where $Q : \mathcal{B} \to \mathcal{A}$ is a functor and $\big(Q, {}^C\rho_Q\big)$ is a left $\mathbb{C}$-comodule, $\big(Q, \rho_Q^D\big)$ is a right $\mathbb{D}$-comodule such that in addition

$$\big(C\rho_Q^D\big) \circ {}^C\rho_Q = \big({}^C\rho_Q D\big) \circ \rho_Q^D.$$

DEFINITION 4.8. A *morphism between two left $\mathbb{C}$-comodule functors* $\big(Q, {}^C\rho_Q\big)$ and $\big(Q', {}^C\rho_Q\big)$ is a morphism $f : Q \to Q'$ in $\mathcal{A}$ such that

$${}^C\rho_Q \circ f = (Cf) \circ {}^C\rho_Q.$$

DEFINITION 4.9. A *comodule* for a comonad $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ on a category $\mathcal{A}$ is a pair $\left(X, {}^C\rho_X\right)$ where $X \in \mathcal{A}$ and ${}^C\rho_X : X \to CX$ is a morphism in $\mathcal{A}$ such that

$$\left(C{}^C\rho_X\right) \circ {}^C\rho_X = \left(\Delta^C X\right) \circ {}^C\rho_X \quad \text{and} \quad X = \left(\varepsilon^C X\right) \circ {}^C\rho_X.$$

A *morphism between two* $\mathbb{C}$*-comodules* $\left(X, {}^C\rho_X\right)$ and $\left(X', {}^C\rho_{X'}\right)$ is a morphism $f : X \to X'$ in $\mathcal{A}$ such that

$${}^C\rho_X \circ f = (Cf) \circ {}^C\rho_X.$$

We denote by ${}^{\mathbb{C}}\mathcal{A}$ the category of $\mathbb{C}$-comodule and their morphisms.

DEFINITION 4.10. Corresponding to a comonad $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ on $\mathcal{A}$, there is an adjunction $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ where ${}^{\mathbb{C}}U$ is the forgetful functor and ${}^{\mathbb{C}}F$ is the free functor

$$\begin{array}{rccc} {}^{\mathbb{C}}U: & {}^{\mathbb{C}}\mathcal{A} & \to & \mathcal{A} \\ & \left(X, {}^C\rho_X\right) & \to & X \\ & f & \to & f \end{array} \qquad \begin{array}{rccc} {}^{\mathbb{C}}F: & \mathcal{A} & \to & {}^{\mathbb{C}}\mathcal{A} \\ & X & \to & \left(CX, \Delta^C X\right) \\ & f & \to & Cf \end{array}$$

Note that ${}^{\mathbb{C}}U{}^{\mathbb{C}}F = C$. The counit of the adjunction is given by the counit ε^C of the comonad $\mathbb{C}$

$$\varepsilon^C : C = {}^{\mathbb{C}}U{}^{\mathbb{C}}F \to \mathcal{A}.$$

The unit $\gamma^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{C}}F{}^{\mathbb{C}}U$ of this adjunction is defined by setting

$${}^{\mathbb{C}}U\left(\gamma^C\left(X, {}^C\rho_X\right)\right) = {}^C\rho_X \text{ for every } \left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}.$$

Therefore we have

$$\left(\varepsilon^C{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) = {}^{\mathbb{C}}U \qquad \text{and} \qquad \left({}^{\mathbb{C}}F\varepsilon^C\right) \circ \left(\gamma^C{}^{\mathbb{C}}F\right) = {}^{\mathbb{C}}F.$$

PROPOSITION 4.11. *Let* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ *be a comonad on a category* $\mathcal{A}$ *and let* $Z, W \in {}^{\mathbb{C}}\mathcal{A}$. *Then* $Z = W$ *if and only if* ${}^{\mathbb{C}}U(Z) = {}^{\mathbb{C}}U(W)$ *and* ${}^{\mathbb{C}}U\left(\gamma^C Z\right) = {}^{\mathbb{C}}U\left(\gamma^C W\right)$. *In particular, if* $F, G : \mathcal{X} \to {}^{\mathbb{C}}\mathcal{A}$ *are functors, we have*

$$F = G \text{ if and only if } {}^{\mathbb{C}}UF = {}^{\mathbb{C}}UG \text{ and } {}^{\mathbb{C}}U\left(\gamma^C F\right) = {}^{\mathbb{C}}U\left(\gamma^C G\right).$$

PROPOSITION 4.12. *Let* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ *be a comonad on a category* $\mathcal{A}$. *Then* $\left({}^{\mathbb{C}}U, \left({}^{\mathbb{C}}U\gamma^C\right)\right)$ *is a left* $\mathbb{C}$*-comodule functor.*

Proof. We have to prove these two equalities

$$\begin{aligned} \left(C{}^{\mathbb{C}}U\gamma^C\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) &= \left(\Delta^C{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) \\ \left(\varepsilon^C{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) &= {}^{\mathbb{C}}U \end{aligned}$$

Let us consider $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, we have to show that

$$\left(C{}^{\mathbb{C}}U\gamma^C\right)\left(X, {}^C\rho_X\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)\left(X, {}^C\rho_X\right) = \left(\Delta^C{}^{\mathbb{C}}U\right)\left(X, {}^C\rho_X\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)\left(X, {}^C\rho_X\right)$$

and that

$$\left(\varepsilon^C{}^{\mathbb{C}}U\right)\left(X, {}^C\rho_X\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)\left(X, {}^C\rho_X\right) = {}^{\mathbb{C}}U\left(X, {}^C\rho_X\right)$$

i.e.

$$\left(C{}^C\rho_X\right) \circ {}^C\rho_X = \left(\Delta^C X\right) \circ {}^C\rho_X \qquad \text{and} \qquad \left(\varepsilon^C X\right) \circ {}^C\rho_X = X$$

which both hold in view of the definition of $\mathbb{C}$-comodule. $\square$

Proposition 4.13. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\left(X, {}^C\rho_X\right)$ be a comodule for $\mathbb{C}$. Then we have*

$$\left(X, {}^C\rho_X\right) = \mathrm{Equ}_{\mathcal{A}}\left(C{}^C\rho_X, \Delta^C X\right).$$

In particular if $\left(Q, {}^C\rho_Q\right)$ is a left $\mathbb{C}$-comodule functor, then

$$\left(Q, {}^C\rho_Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(C{}^C\rho_Q, \Delta^C Q\right).$$

Corollary 4.14. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ be the associated adjunction. Then $\left({}^{\mathbb{C}}U, \left({}^{\mathbb{C}}U\gamma^C\right)\right)$ is a left $\mathbb{C}$-comodule functor and*

$$\left({}^{\mathbb{C}}U, \left({}^{\mathbb{C}}U\gamma^C\right)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(C{}^{\mathbb{C}}U\gamma^C, \Delta^C{}^{\mathbb{C}}U\right).$$

Proof. By Proposition 4.12 $\left({}^{\mathbb{C}}U, \left({}^{\mathbb{C}}U\gamma^C\right)\right)$ is a left $\mathbb{C}$-comodule functor. By Proposition 4.13 we get that $\left({}^{\mathbb{C}}U, \left({}^{\mathbb{C}}U\gamma^C\right)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(C{}^{\mathbb{C}}U\gamma^C, \Delta^C{}^{\mathbb{C}}U\right)$. □

Proposition 4.15. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\left(P, \rho_P^C\right)$ where $P : \mathcal{A} \to \mathcal{B}$ a right $\mathbb{C}$-comodule functor. Then we have*

$$\left(P, \rho_P^C\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_P^C C, P\Delta^C\right).$$

Proof. By definition we have that

$$\left(\rho_P^C C\right) \circ \rho_P^C = \left(P\Delta^C\right) \circ \rho_P^C.$$

Now, let $\zeta : Z \to PC$ be a functorial morphism such that $\left(\rho_P^C C\right) \circ \zeta = \left(P\Delta^C\right) \circ \zeta$ and consider $\overline{\zeta} := \left(P\varepsilon^C\right) \circ \zeta : Z \to P$. Then we have

$$\begin{aligned}\rho_P^C \circ \overline{\zeta} = \rho_P^C \circ \left(P\varepsilon^C\right) \circ \zeta &\overset{\rho_P^C}{=} \left(PC\varepsilon^C\right) \circ \left(\rho_P^C C\right) \circ \zeta = \\ &= \left(PC\varepsilon^C\right) \circ \left(P\Delta^C\right) \circ \zeta \overset{C\text{comonad}}{=} \zeta.\end{aligned}$$

Moreover, let $\zeta' : Z \to PC$ be another functorial morphism such that $\left(\rho_P^C C\right) \circ \zeta' = \zeta$. Then

$$\zeta' = \left(P\varepsilon^C\right) \circ \rho_P^C \circ \zeta' = \left(P\varepsilon^C\right) \circ \zeta = \overline{\zeta}$$

so that $\overline{\zeta}$ is the unique functorial morphism such that $\left(\rho_P^C C\right) \circ \overline{\zeta} = \zeta$. □

Lemma 4.16. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\left(Q, {}^C\rho_Q\right)$ be a left and $\left(P, \rho_P^C\right)$ be a right $\mathbb{C}$-comodule functors where $Q : \mathcal{Q} \to \mathcal{A}$ and $P : \mathcal{A} \to \mathcal{P}$. Let $F : \mathcal{X} \to \mathcal{Q}$ and $G : \mathcal{P} \to \mathcal{B}$ be functors. Then*

(1) *$\left(QF, {}^C\rho_Q F\right)$ is a left $\mathbb{C}$-comodule functor and*
(2) *$\left(GP, G\rho_P^C\right)$ is a right $\mathbb{C}$-comodule functor.*

Proposition 4.17. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on $\mathcal{A}$ and let $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ be the adjunction associated. Then ${}^{\mathbb{C}}U$ reflects isomorphisms.*

Proof. Let $f : \left(X, {}^C\rho_X\right) \to \left(Y, {}^C\rho_Y\right)$ be a morphism in ${}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}Uf$ has a two-sided inverse f^{-1} in $\mathcal{A}$. Since

$${}^C\rho_Y \circ f = (Cf) \circ {}^C\rho_X$$

we get that

$$\left(Cf^{-1}\right) \circ {}^{C}\rho_{Y} = {}^{C}\rho_{X} \circ f^{-1}.$$

□

LEMMA 4.18. *Let* $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ *be a comonad on a category* $\mathcal{A}$, *let* $\left(P, \rho_{P}^{C}\right)$ *be a right* $\mathbb{C}$*-comodule functor where* $P : \mathcal{A} \to \mathcal{B}$ *and let* $\left(Q, {}^{C}\rho_{Q}\right)$ *be a left* $\mathbb{C}$*-comodule functor. Then any equalizer preserved by* PC *is also preserved by* P *and any equalizer preserved by* CQ *is also preserved by* Q.

Proof. Consider the following equalizer

$$X \xrightarrow{x} Y \overset{f}{\underset{g}{\rightrightarrows}} Z$$

in the category $\mathcal{A}$ and assume that PC preserves it. Applying to it functors PC and P we get the following diagrams in $\mathcal{B}$

$$\begin{tikzcd}
PX \arrow[r, "Px"] \arrow[d, shift left, "\rho_{P}^{C}X"] & PY \arrow[r, shift left, "Pf"] \arrow[r, shift right, "Pg"'] \arrow[d, shift left, "\rho_{P}^{C}Y"] & PZ \arrow[d, shift left, "\rho_{P}^{C}Z"] \\
PCX \arrow[r, "PCx"] \arrow[u, shift left, "P\varepsilon^{C}X"] & PCY \arrow[r, shift left, "PCf"] \arrow[r, shift right, "PCg"'] \arrow[u, shift left, "P\varepsilon^{C}Y"] & PCZ \arrow[u, shift left, "P\varepsilon^{C}Z"]
\end{tikzcd}$$

By assumption, the second row is an equalizer. Assume that there exists a morphism $h : H \to PY$ such that

$$(Pf) \circ h = (Pg) \circ h.$$

Then, by composing with $\rho_{P}^{C}Z$ we get

$$\left(\rho_{P}^{C}Z\right) \circ (Pf) \circ h = \left(\rho_{P}^{C}Z\right) \circ (Pg) \circ h$$

and since ρ_{P}^{C} is a functorial morphism we obtain

$$(PCf) \circ \left(\rho_{P}^{C}Y\right) \circ h = (PCg) \circ \left(\rho_{P}^{C}Y\right) \circ h.$$

Since $(PCX, PCx) = \mathrm{Equ}_{\mathcal{B}}(PCf, PCg)$, there exists a unique morphism $k : H \to PCX$ such that

$$(PCx) \circ k = \left(\rho_{P}^{C}Y\right) \circ h. \tag{30}$$

By composing with $P\varepsilon^{C}Y$ we get

$$\left(P\varepsilon^{C}Y\right) \circ (PCx) \circ k = \left(P\varepsilon^{C}Y\right) \circ \left(\rho_{P}^{C}Y\right) \circ h$$

and thus

$$(Px) \circ \left(P\varepsilon^{C}X\right) \circ k = h.$$

Let $l := \left(P\varepsilon^{C}X\right) \circ k : H \to PX$. Then we have

$$\begin{aligned}
(Px) \circ l = (Px) \circ \left(P\varepsilon^{C}X\right) \circ k &\overset{\varepsilon^{C}}{=} \left(P\varepsilon^{C}Y\right) \circ (PCx) \circ k \\
&\overset{(30)}{=} \left(P\varepsilon^{C}Y\right) \circ \left(\rho_{P}^{C}Y\right) \circ h = h.
\end{aligned}$$

Let $l' : H \to PX$ be another morphism such that

$$(Px) \circ l' = h.$$

Then we have

$$\begin{aligned}(PCx)\circ\left(\rho_P^C X\right)\circ l' &= \left(\rho_P^C Y\right)\circ(Px)\circ l' = \left(\rho_P^C Y\right)\circ h\\ &= \left(\rho_P^C Y\right)\circ(Px)\circ l = (PCx)\circ\left(\rho_P^C X\right)\circ l.\end{aligned}$$

Since PC preserves equalizers, we have that PCx is a monomorphism. Since $\rho_P^C X$ is also a monomorphism, we deduce that $l = l'$. Therefore we obtain that $(PX, Px) = \mathrm{Equ}_{\mathcal{B}}(Pf, Pg)$. The second statement can be proved similarly. $\square$

Lemma 4.19. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $f, g : \left(X, {}^C\rho_X\right) \to \left(Y, {}^C\rho_Y\right)$ be morphisms in ${}^{\mathbb{C}}\mathcal{A}$. Assume that there exists $(E, e) = \mathrm{Equ}_{\mathcal{A}}\left({}^{\mathbb{C}}Uf, {}^{\mathbb{C}}Ug\right)$ and assume that CC preserves equalizers. Then there exists $(\Xi, \xi) = \mathrm{Equ}_{{}^{\mathbb{C}}\mathcal{A}}(f, g)$ and ${}^{\mathbb{C}}U(\Xi, \xi) = (E, e)$.*

Proof. Since CC preserves equalizers and $\left(C, \Delta^C\right)$ is a right $\mathbb{C}$-comodule functor, also C preserves equalizers by Lemma 4.18, in particular, C preserves (E, e). Since

$$\begin{aligned}&\left(C{}^{\mathbb{C}}Uf\right)\circ{}^C\rho_X\circ e \overset{f\in{}^{\mathbb{C}}\mathcal{A}}{=} {}^C\rho_Y\circ\left({}^{\mathbb{C}}Uf\right)\circ e\\ &\overset{\text{eequ}}{=} {}^C\rho_Y\circ\left({}^{\mathbb{C}}Ug\right)\circ e \overset{g\in{}^{\mathbb{C}}\mathcal{A}}{=} \left(C{}^{\mathbb{C}}Ug\right)\circ{}^C\rho_X\circ e\end{aligned}$$

by the universal property of the equalizer (CE, Ce) there exists a unique morphism ${}^C\rho_E : E \to CE$ such that

$$(Ce)\circ{}^C\rho_E = {}^C\rho_X\circ e.$$

Moreover, by composing with $\varepsilon^C X$ the first term of this equality we get

$$\left(\varepsilon^C X\right)\circ(Ce)\circ{}^C\rho_E \overset{\varepsilon^C}{=} e\circ\left(\varepsilon^C E\right)\circ{}^C\rho_E$$

whereas the second term becomes

$$\left(\varepsilon^C X\right)\circ{}^C\rho_X\circ e = e$$

so that we obtain the following equality

$$e\circ\left(\varepsilon^C E\right)\circ{}^C\rho_E = e.$$

Since e is a monomorphism we deduce that

$$\left(\varepsilon^C E\right)\circ{}^C\rho_E = E.$$

Now, consider the following serially commutative diagram

$$\begin{tikzcd}
E \arrow[r, "{}^C\rho_E"] \arrow[d, "e"] & CE \arrow[r, shift left, "\Delta^C E"] \arrow[r, shift right, "C{}^C\rho_E"'] \arrow[d, "Ce"] & CCE \arrow[d, "CCe"] \\
X \arrow[r, "{}^C\rho_X"] \arrow[d, shift right, "{}^{\mathbb{C}}Uf"'] \arrow[d, shift left, "{}^{\mathbb{C}}Ug"] & CX \arrow[r, shift left, "\Delta^C X"] \arrow[r, shift right, "C{}^C\rho_X"'] \arrow[d, shift right, "C{}^{\mathbb{C}}Uf"'] \arrow[d, shift left, "C{}^{\mathbb{C}}Ug"] & CCX \arrow[d, shift right, "CC{}^{\mathbb{C}}Uf"'] \arrow[d, shift left, "CC{}^{\mathbb{C}}Ug"] \\
Y \arrow[r, "{}^C\rho_Y"] & CY \arrow[r, shift left, "\Delta^C Y"] \arrow[r, shift right, "C{}^C\rho_Y"'] & CCY
\end{tikzcd}$$

Since we already observed that the columns are equalizers and also the second and the third row are equalizers by Proposition 4.13, in view of Lemma 2.13 also the first row is an equalizer, so that (E, e) has a left $\mathbb{C}$-comodule structure, i.e. there exists $(\Xi, \xi) \in {}^{\mathbb{C}}\mathcal{A}$ such that $(\Xi, \xi) = \mathrm{Equ}_{{}^{\mathbb{C}}\mathcal{A}}(f, g)$ and ${}^{\mathbb{C}}U(\Xi, \xi) = (E, e)$. □

Lemma 4.20. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ with equalizers and let $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ be the adjunction associated. The following statements are equivalent:*

(i) $C : \mathcal{A} \to \mathcal{A}$ preserves equalizers
(ii) $CC : \mathcal{A} \to \mathcal{A}$ preserves equalizers
(iii) ${}^{\mathbb{C}}\mathcal{A}$ has equalizers and they are preserved by ${}^{\mathbb{C}}U : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{A}$
(iv) ${}^{\mathbb{C}}U : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{A}$ preserves equalizers.

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are clear.
$(ii) \Rightarrow (iii)$ follows by Lemma 4.19.
$(iv) \Rightarrow (i)$ Note that ${}^{\mathbb{C}}F$ is a right adjoint, so that in particular it preserves equalizers. Then ${}^{\mathbb{C}}U{}^{\mathbb{C}}F = C$ also preserves equalizers. □

Lemma 4.21. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ and assume that C preserves coequalizers. Then ${}^{\mathbb{C}}F$ preserves coequalizers where $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ is the adjunction associated to the comonad.*

Proof. Dual to proof of Lemma 3.22. Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{k} K$$

be a coequalizer in $\mathcal{A}$. Let us consider the fork obtained by applying the functor ${}^{\mathbb{C}}F$ to the coequalizer

$${}^{\mathbb{C}}FX \underset{{}^{\mathbb{C}}Fg}{\overset{{}^{\mathbb{C}}Ff}{\rightrightarrows}} {}^{\mathbb{C}}FY \xrightarrow{{}^{\mathbb{C}}Fk} {}^{\mathbb{C}}FK$$

i.e.

$$\left(CX, \Delta^C X\right) \underset{Cg}{\overset{Cf}{\rightrightarrows}} \left(CY, \Delta^C Y\right) \xrightarrow{Ck} \left(CK, \Delta^C K\right)$$

Now, let $\left(Z, {}^C\rho_Z\right) \in {}^{\mathbb{C}}\mathcal{A}$ and $z : \left(CY, \Delta^C Y\right) \to \left(Z, {}^C\rho_Z\right)$ be a morphism in ${}^{\mathbb{C}}\mathcal{A}$ such that $z \circ (Cf) = z \circ (Cg)$. Since C preserves coequalizers, we know that $(CK, Ck) = \mathrm{Coequ}_{\mathcal{A}}(Cf, Cg)$. By the universal property of the coequalizer (CK, Ck) in $\mathcal{A}$, there exists a unique morphism $z' : CK \to Z$ in $\mathcal{A}$ such that $z' \circ (Ck) = z$. We now want to prove that z' is a morphism in ${}^{\mathbb{C}}\mathcal{A}$, i.e. that $(Cz') \circ \left(\Delta^C K\right) = {}^C\rho_Z \circ z'$. Since z is a morphism in ${}^{\mathbb{C}}\mathcal{A}$ we have that

$$(Cz) \circ \left(\Delta^C Y\right) = {}^C\rho_Z \circ z$$

and since also Ck is a morphism in ${}^{\mathbb{C}}\mathcal{A}$ we have that

$$(CCk) \circ \left(\Delta^C Y\right) = \left(\Delta^C K\right) \circ (Ck).$$

Then we have

$$\begin{aligned} (Cz') \circ \left(\Delta^C K\right) \circ (Ck) &= (Cz') \circ (CCk) \circ \left(\Delta^C Y\right) \\ \overset{\text{prop}z}{=} (Cz) \circ \left(\Delta^C Y\right) &= {}^C\rho_Z \circ z \overset{\text{prop}z}{=} {}^C\rho_Z \circ z' \circ (Ck) \end{aligned}$$

and since C preserves coequalizers, Ck is an epimorphism, so that we get

$$(Cz') \circ \left(\Delta^C K\right) = {}^C\rho_Z \circ z'.$$

$\square$

Lemma 4.22. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$, let $L, N : \mathcal{B} \to \mathcal{A}$ be functors and let $\rho : L \to CL$ be a coassociative and counital functorial morphism, that is (L, ρ) is a left $\mathbb{C}$-comodule functor. Let $u : N \to L$ and let $\phi : N \to CN$ be functorial morphisms such that*

$$\rho \circ u = (Cu) \circ \phi. \tag{31}$$

If CCu and u are monomorphisms, then ϕ is coassociative and counital, that is (N, ϕ) is a left $\mathbb{C}$-comodule functor.

Proof. Let us prove that ϕ is coassociative

$$(CCu) \circ (C\phi) \circ \phi \overset{(31)}{=} (C\rho) \circ (Cu) \circ \phi \overset{(31)}{=} (C\rho) \circ \rho \circ u$$
$$\overset{\rho\text{coass}}{=} \left(\Delta^C L\right) \circ \rho \circ u \overset{(31)}{=} \left(\Delta^C L\right) \circ (Cu) \circ \phi \overset{\Delta^C}{=} (CCu) \circ \left(\Delta^C N\right) \circ \phi.$$

Since CCu is a monomorphism we get that

$$(C\phi) \circ \phi = \left(\Delta^C N\right) \circ \phi.$$

Let us prove that ϕ is counital

$$u \circ \left(\varepsilon^C N\right) \circ \phi \overset{\varepsilon^C}{=} \left(\varepsilon^C L\right) \circ (Cu) \circ \phi \overset{(31)}{=} \left(\varepsilon^C L\right) \circ \rho \circ u \overset{\rho\text{counit}}{=} u.$$

Since u is a monomorphism we conclude. $\square$

4.1. **Lifting of comodule functors.** This subsection collects the dual results for liftings of module functors so that one can skip reading all the proofs we keep here in order to give details of the results we use in the following.

Proposition 4.23 ([W] 3.5). *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$, let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad on a category $\mathcal{B}$ and let $T : \mathcal{A} \to \mathcal{B}$ be a functor. Then there is a bijection between the following collections of data*

$\mathcal{F}$ functors $\widetilde{T} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ that are liftings of T (i.e. ${}^{\mathbb{D}}U\widetilde{T} = T{}^{\mathbb{C}}U$)

$\mathcal{M}$ functorial morphisms $\Xi : TC \to DT$ such that

$$\left(\Delta^D T\right) \circ \Xi = (D\Xi) \circ (\Xi C) \circ \left(T\Delta^C\right) \qquad \text{and} \qquad \left(\varepsilon^D T\right) \circ \Xi = T\varepsilon^C$$

given by

$$a : \mathcal{F} \to \mathcal{M} \text{ where } a\left(\widetilde{T}\right) = \left({}^{\mathbb{D}}U{}^{\mathbb{D}}FT\varepsilon^C\right) \circ \left({}^{\mathbb{D}}U\gamma^D\widetilde{T}{}^{\mathbb{C}}F\right)$$
$$b : \mathcal{M} \to \mathcal{F} \text{ where } {}^{\mathbb{D}}Ub\left(\Xi\right) = T{}^{\mathbb{C}}U \text{ and } {}^{\mathbb{D}}U\gamma^D b\left(\Xi\right) = \Xi \circ \left(T{}^{\mathbb{C}}U\gamma^C\right) \text{ i.e.}$$
$$b\left(\Xi\right)\left(\left(X, {}^C\rho_X\right)\right) = \left(TX, (\Xi X) \circ \left(T{}^C\rho_X\right)\right) \text{ and } b\left(\Xi\right)(f) = T(f).$$

Proof. Let $\widetilde{T} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ be a lifting of the functor $T : \mathcal{A} \to \mathcal{B}$ (i.e. ${}^{\mathbb{D}}U\widetilde{T} = T{}^{\mathbb{C}}U$). Define a functorial morphism $\xi : \widetilde{T}{}^{\mathbb{C}}F \to {}^{\mathbb{D}}FT$ as the composite

$$\xi := \left({}^{\mathbb{D}}FT\varepsilon^C\right) \circ \left(\gamma^D\widetilde{T}{}^{\mathbb{C}}F\right)$$

where $\varepsilon^C : C = {}^{\mathbb{C}}U^{\mathbb{C}}F \to \mathcal{A}$ is also the counit of the adjunction $\left({}^{\mathbb{C}}U, {}^{\mathbb{C}}F\right)$ and $\gamma^D : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{D}}F^{\mathbb{D}}U$ is the unit of the adjunction $\left({}^{\mathbb{D}}U, {}^{\mathbb{D}}F\right)$. Let now define

$$\Xi \overset{def}{=} {}^{\mathbb{D}}U\xi : {}^{\mathbb{D}}U\widetilde{T}^{\mathbb{C}}F = T^{\mathbb{C}}U^{\mathbb{C}}F = TC \to {}^{\mathbb{D}}U^{\mathbb{D}}FT = DT$$

that is

$$\Xi = {}^{\mathbb{D}}U\xi = \left({}^{\mathbb{D}}U^{\mathbb{D}}FT\varepsilon^C\right) \circ \left({}^{\mathbb{D}}U\gamma^D\widetilde{T}^{\mathbb{C}}F\right).$$

Dually to Proposition 3.24 you can prove that Ξ is a functorial morphism satisfying

$$\left(\Delta^D T\right) \circ \Xi = (D\Xi) \circ (\Xi C) \circ \left(T\Delta^C\right) \text{ and } \left(\varepsilon^D T\right) \circ \Xi = T\varepsilon^C.$$

Conversely, let Ξ be a functorial morphism satisfying $\left(\Delta^D T\right) \circ \Xi = (D\Xi) \circ (\Xi C) \circ \left(T\Delta^C\right)$ and $\left(\varepsilon^D T\right) \circ \Xi = T\varepsilon^C$. We define $\widetilde{T} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ by setting, for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$,

$$\widetilde{T}\left(\left(X, {}^C\rho_X\right)\right) = \left(TX, (\Xi X) \circ \left(T^C\rho_X\right)\right)$$

and for every $f : \left(X, {}^C\rho_X\right) \to \left(Y, {}^C\rho_Y\right) \in {}^{\mathbb{C}}\mathcal{A}$,

$$\widetilde{T}(f) = T(f).$$

Dually to Proposition 3.24 you can prove that $\widetilde{T}$ is a functor between ${}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ which lifts T and that $a : \mathcal{F} \to \mathcal{M}$ and $b : \mathcal{M} \to \mathcal{F}$ define a bijective correspondence. $\square$

Corollary 4.24. *Let $\mathcal{X}, \mathcal{A}$ be categories and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $F : \mathcal{X} \to \mathcal{A}$ be a functor. Then there is a bijection between the following collections of data:*

- $\mathcal{F}$ *Functors ${}^CF : \mathcal{X} \to {}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U^C F = F$,*
- $\mathcal{G}$ *Left $\mathbb{C}$-comodule coactions ${}^C\rho_F : F \to CF$*

given by

$$\begin{aligned} \alpha &: \mathcal{F} \to \mathcal{G} \text{ where } \alpha\left({}^CF\right) = {}^{\mathbb{C}}U\gamma^{CC}F : F \to CF \\ \beta &: \mathcal{G} \to \mathcal{F} \text{ where } {}^{\mathbb{C}}U\beta\left({}^C\rho_F\right) = F \text{ and } {}^{\mathbb{C}}U\gamma^C\beta\left({}^C\rho_F\right) = {}^C\rho_F \text{ i.e.} \\ \beta &: \mathcal{G} \to \mathcal{F} \text{ where } \beta\left({}^C\rho_F\right)(X) = \left(FX, {}^C\rho_F X\right) \text{ and } \beta\left({}^C\rho_F\right)(f) = F(f). \end{aligned}$$

Proof. Apply Proposition 4.23 to the case $\mathcal{A} = \mathcal{X}, \mathcal{B} = \mathcal{A}, \mathbb{C} = \mathrm{Id}_{\mathcal{X}}, \mathbb{D} = \mathbb{C}$. Then $\widetilde{T} = {}^CF$ is the lifting of F and $\Xi = {}^C\rho_F : F \to CF$ satisfies $\left(\Delta^C F\right) \circ {}^C\rho_F = \left(C^C\rho_F\right) \circ {}^C\rho_F$ and $\left(\varepsilon^C F\right) \circ {}^C\rho_F = F$ that is $\left(F, {}^C\rho_F\right)$ is a left $\mathbb{C}$-comodule functor. $\square$

Corollary 4.25. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. Then there exists a bijective correspondence between the following collections of data:*

- $\mathfrak{K}$ *Functors $K : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U \circ K = L$,*
- $\mathfrak{L}$ *Functorial morphism $\beta : L \to CL$ such that (L, β) is a left comodule functor for the comonad $\mathbb{C}$*

given by

$$\begin{aligned} \Phi &: \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi(K) = {}^{\mathbb{C}}U\left(\gamma^C K\right) : L \to CL \\ \Omega &: \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega(\beta)(Y) = (LY, \beta Y) \text{ and } {}^{\mathbb{C}}U\Omega(\beta)(f) = L(f). \end{aligned}$$

Proof. Apply Corollary 4.24 to the case "F" $= L : \mathcal{B} \to \mathcal{A}$ where (L, R) is an adjunction and $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ a comonad on $\mathcal{A}$. □

PROPOSITION 4.26. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad on a category $\mathcal{B}$. Let $T : \mathcal{A} \to \mathcal{B}$ be a functor, let $\widetilde{T} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ be a lifting of T (i.e. ${}^{\mathbb{D}}U\widetilde{T} = T{}^{\mathbb{C}}U$) and let $\Xi : TC \to DT$ as in Proposition 4.23. Then Ξ is an isomorphism if and only if $\xi = \left({}^{\mathbb{D}}FT\varepsilon^C\right) \circ \left(\gamma^D \widetilde{T}{}^{\mathbb{C}}F\right) : \widetilde{T}{}^{\mathbb{C}}F \to {}^{\mathbb{D}}FT$ is an isomorphism.*

Proof. By construction in Proposition 4.23 we have that $\Xi = {}^{\mathbb{D}}U\xi$. Assume that Ξ is an isomorphism. Since, by Proposition 4.17, ${}^{\mathbb{D}}U$ reflects isomorphisms, $\xi : \widetilde{T}{}^{\mathbb{C}}F \to {}^{\mathbb{D}}FT$ is an isomorphism. Conversely, assume that $\xi : \widetilde{T}{}^{\mathbb{C}}F \to {}^{\mathbb{D}}FT$ is an isomorphism. Then ${}^{\mathbb{D}}U\xi$ is also an isomorphism. □

COROLLARY 4.27. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on $\mathcal{B}$. Let $K : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ be a functor such that ${}^{\mathbb{C}}U \circ K = L$ and let (L, β) be a left $\mathbb{C}$-comodule functor as in Corollary 4.25. Then β is an isomorphism if and only if $\gamma^C K : K \to {}^{\mathbb{C}}FL$ is an isomorphism.*

Proof. Apply Proposition 4.26 with $T = L$ so that the categories $\mathcal{A}$ and $\mathcal{B}$ are interchanged, $\mathbb{C} = \mathrm{Id}_{\mathcal{B}}$ and $\mathbb{D} = \mathbb{C}$. Then $\widetilde{T} = K$ is the lifting of L and $\Xi = \beta : L \to CL$, given by $\beta = {}^{\mathbb{C}}U\xi = {}^{\mathbb{C}}U\gamma^C K$. □

LEMMA 4.28. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ with equalizers. Let $Q : \mathcal{B} \to \mathcal{A}$ be a left $\mathbb{C}$-comodule functor with functorial morphisms ${}^C\rho_Q : Q \to CQ$. Then there exists a unique functor ${}^CQ : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that*

$$ {}^{\mathbb{C}}U{}^CQ = Q \text{ and } {}^{\mathbb{C}}U\gamma^{C\,C}Q = {}^C\rho_Q. $$

Moreover if $\psi : Q \to T$ is a functorial morphism between left $\mathbb{C}$-module functors and ψ satisfies

$$ {}^C\rho_Q \circ (C\psi) = \psi \circ \left({}^C\rho_T\right) $$

then there is a unique functorial morphism ${}^C\psi : {}^CQ \to {}^CT$ such that

$$ {}^{\mathbb{C}}U{}^C\psi = \psi. $$

Proof. Corollary 4.24 applied to the case where $F = Q$ and ${}^C\rho_F = {}^C\rho_Q$ gives us the first statement. Let $B \in \mathcal{B}$. Then we have

$$ \left({}^C\rho_Q B\right) \circ (C\psi B) = (\psi B) \circ \left({}^C\rho_T B\right) $$

which means that ψB yields a morphism ${}^C\psi B$ in ${}^{\mathbb{C}}\mathcal{A}$. □

PROPOSITION 4.29. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ and let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have equalizers and that C preserves equalizers. Let $Q : \mathcal{B} \to \mathcal{A}$ be a functor and let ${}^C\rho_Q : Q \to CQ$ and $\rho_Q^D : Q \to QD$ be functorial morphisms. Assume that ${}^C\rho_Q$ is coassociative and counital and that $\left(C\rho_Q^D\right) \circ {}^C\rho_Q = \left({}^C\rho_Q D\right) \circ \rho_Q^D$. Set*

$$ \left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right). \tag{32} $$

Then $Q^D : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ is a left $\mathbb{C}$-comodule functor where ${}^C\rho_{Q^D} : Q^D \to CQ^D$ is uniquely determined by

$$\left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q = \left(C\iota^Q\right) \circ {}^C\rho_{Q^D}. \tag{33}$$

Moreover there exists a unique functor ${}^C\left(Q^D\right) : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that

$$^{\mathbb{C}}U^C\left(Q^D\right) = Q^D \text{ and } {}^{\mathbb{C}}U\gamma^{CC}\left(Q^D\right) = {}^C\rho_{Q^D}. \tag{34}$$

Proof. By Lemma 2.8 we can consider $\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$. Since

$$\left(C\rho_Q^D\right) \circ {}^C\rho_Q = \left({}^C\rho_Q D\right) \circ \rho_Q^D$$

we deduce that

$$\left(C\rho_Q^{D\mathbb{D}}U\right) \circ \left({}^C\rho_Q{}^{\mathbb{D}}U\right) = \left({}^C\rho_Q D^{\mathbb{D}}U\right) \circ \left(\rho_Q^{D\mathbb{D}}U\right). \tag{35}$$

Also, in view of the naturality of ${}^C\rho_Q$, we have

$$\left(CQ^{\mathbb{D}}U\gamma^{D}\right) \circ \left({}^C\rho_Q{}^{\mathbb{D}}U\right) = \left({}^C\rho_Q D^{\mathbb{D}}U\right) \circ \left(Q^{\mathbb{D}}U\gamma^{D}\right). \tag{36}$$

We compute

$$\begin{aligned}
&\left(CQ^{\mathbb{D}}U\gamma^{D}\right) \circ \left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q \overset{(36)}{=} \left({}^C\rho_Q D^{\mathbb{D}}U\right) \circ \left(Q^{\mathbb{D}}U\gamma^{D}\right) \circ \iota^Q \\
&\overset{\iota^Q \mathrm{equ}}{=} \left({}^C\rho_Q D^{\mathbb{D}}U\right) \circ \left(\rho_Q^{D\mathbb{D}}U\right) \circ \iota^Q \overset{(35)}{=} \left(C\rho_Q^{D\mathbb{D}}U\right) \circ \left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q.
\end{aligned}$$

Since C preserves equalizers, we have

$$\left(CQ^D, C\iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(C\rho_Q^{D\mathbb{D}}U, CQ^{\mathbb{D}}U\gamma^D\right)$$

hence there exists a unique functorial morphism ${}^C\rho_{Q^D} : Q^D \to CQ^D$ such that

$$\left(C\iota^Q\right) \circ {}^C\rho_{Q^D} = \left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q.$$

Since Q is a left $\mathbb{C}$-comodule functor, by Lemma 4.16, also $Q^{\mathbb{D}}U$ is a left $\mathbb{C}$-comodule functor. Now ι^Q is a monomorphism and hence, since C preserves equalizers, also $CC\iota^Q$ is a monomorphism. Therefore we can apply Lemma 4.22 to "ϕ" $= {}^C\rho_{Q^D}$, "u" $= \iota^Q$ and "ρ" $= {}^C\rho_Q{}^{\mathbb{D}}U$ and hence we obtain that $\left(Q^D, {}^C\rho_{Q^D}\right)$ is a left $\mathbb{C}$-comodule functor that is ${}^C\rho_{Q^D}$ is coassociative and counital. By Lemma 4.28 applied to $\left(Q^D, {}^C\rho_{Q^D}\right)$ there exists a functor ${}^C\left(Q^D\right) : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U^C\left(Q^D\right) = Q^D$ and ${}^C\rho_{Q^D} = {}^{\mathbb{C}}U\gamma^{CC}\left(Q^D\right)$. Moreover ${}^C\left(Q^D\right)$ is unique with respect to these properties. □

Proposition 4.30. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ and let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have equalizers and C preserves them. Let $Q : \mathcal{B} \to \mathcal{A}$ be a $\mathbb{C}$-$\mathbb{D}$-bicomodule functor with functorial morphisms ${}^C\rho_Q : Q \to CQ$ and $\rho_Q^D : Q \to QD$. Then the functor ${}^CQ : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is a right $\mathbb{D}$-comodule functor via $\rho^D_{{}^CQ} : {}^CQ \to {}^CQD$ where $\rho^D_{{}^CQ}$ is uniquely determined by*

$$^{\mathbb{C}}U\rho^D_{{}^CQ} = \rho_Q^D. \tag{37}$$

Let $\left(\left({}^{C}Q\right)^{D},\iota^{{}^{C}Q}\right)=\mathrm{Equ}_{\mathrm{Fun}}\left(\rho^{D}_{{}^{C}Q}{}^{\mathbb{C}}U,{}^{C}Q^{\mathbb{C}}U\gamma^{D}\right)$. *Then we have*

$$\left({}^{C}Q\right)^{D}={}^{C}\left(Q^{D}\right):{}^{\mathbb{D}}\mathcal{B}\to{}^{\mathbb{C}}\mathcal{A}.$$

Proof. Since Q is endowed with a left $\mathbb{C}$-comodule structure, by Lemma 4.28 there exists a unique functor ${}^{C}Q:\mathcal{B}\to{}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U^{C}Q=Q$ and ${}^{\mathbb{C}}U\gamma^{C\,C}Q={}^{C}\rho_{Q}$. Note that, since Q is a $\mathbb{C}$-$\mathbb{D}$-bicomodule functor, in particular the compatibility condition

$$(C\rho^{D}_{Q})\circ{}^{C}\rho_{Q}=\left({}^{C}\rho_{Q}D\right)\circ\rho^{D}_{Q}$$

holds, that is $\rho^{D}_{Q}:Q={}^{\mathbb{C}}U^{C}Q\to QD={}^{\mathbb{C}}U^{C}QD$ is a morphism in ${}^{\mathbb{C}}\mathcal{A}$. Thus, there exists a functorial morphism $\rho^{D}_{{}^{C}Q}:{}^{C}Q\to{}^{C}QD$ such that

$${}^{\mathbb{C}}U\rho^{D}_{{}^{C}Q}=\rho^{D}_{Q}.$$

By the coassociativity and counitality properties of ρ^{D}_{Q} we get that also $\rho^{D}_{{}^{C}Q}$ is coassociative and counital, so that $\left({}^{C}Q,\rho^{D}_{{}^{C}Q}\right)$ is a right $\mathbb{D}$-comodule functor. Thus we can consider the equalizer

$$\left({}^{C}Q\right)^{D}\xrightarrow{\iota^{{}^{C}Q}}{}^{C}Q^{\mathbb{D}}U\underset{{}^{C}Q^{\mathbb{D}}U\gamma^{D}}{\overset{\rho^{D}_{{}^{C}Q}{}^{\mathbb{D}}U}{\rightrightarrows}}{}^{C}QD^{\mathbb{D}}U\tag{38}$$

so that we get a functor $\left({}^{C}Q\right)^{D}:{}^{\mathbb{D}}\mathcal{B}\to{}^{\mathbb{C}}\mathcal{A}$. Since C preserves equalizers, by Lemma 4.20 also ${}^{\mathbb{C}}U$ preserves equalizers. Then, by applying the functor ${}^{\mathbb{C}}U$ to (38) we still get an equalizer

$${}^{\mathbb{C}}U\left({}^{C}Q\right)^{D}\xrightarrow{{}^{\mathbb{C}}U\iota^{{}^{C}Q}}{}^{\mathbb{C}}U^{C}Q^{\mathbb{D}}U\underset{{}^{\mathbb{C}}U^{C}Q^{\mathbb{D}}U\gamma^{D}}{\overset{{}^{\mathbb{C}}U\rho^{D}_{{}^{C}Q}{}^{\mathbb{D}}U}{\rightrightarrows}}{}^{\mathbb{C}}U^{C}QD^{\mathbb{D}}U$$

that is

$${}^{\mathbb{C}}U\left({}^{C}Q\right)^{D}\xrightarrow{{}^{\mathbb{C}}U\iota^{{}^{C}Q}}Q^{\mathbb{D}}U\underset{Q^{\mathbb{D}}U\gamma^{D}}{\overset{\rho^{D\mathbb{D}}_{Q}U}{\rightrightarrows}}QD^{\mathbb{D}}U$$

By Proposition 4.29 $\left(Q^{D},\iota^{Q}\right)=\mathrm{Equ}_{\mathrm{Fun}}\left(\rho^{D\mathbb{D}}_{Q}U,Q^{\mathbb{D}}U\gamma^{D}\right)$, then we have

$${}^{\mathbb{C}}U\left({}^{C}Q\right)^{D}=Q^{D}\text{ and }{}^{\mathbb{C}}U\iota^{{}^{C}Q}=\iota^{Q}.$$

Moreover

$${}^{\mathbb{C}}U\gamma^{C}\left({}^{C}Q\right)^{D}:{}^{\mathbb{C}}U\left({}^{C}Q\right)^{D}=Q^{D}\to C^{\mathbb{C}}U\left({}^{C}Q\right)^{D}=CQ^{D}$$

so that, using Proposition 4.29 where we prove that $\left(Q^{D},{}^{C}\rho_{Q^{D}}\right)$ is a left $\mathbb{C}$-comodule functor and that ${}^{\mathbb{C}}U\gamma^{C\,C}\left(Q^{D}\right)={}^{C}\rho_{Q^{D}}$, we get

$${}^{\mathbb{C}}U\gamma^{C}\left({}^{C}Q\right)^{D}={}^{C}\rho_{Q^{D}}={}^{\mathbb{C}}U\gamma^{C\,C}\left(Q^{D}\right)$$

i.e.

$$\left({}^{C}Q\right)^{D}={}^{C}\left(Q^{D}\right).$$

∎

Notation 4.31. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ and let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have equalizers and A preserves them. Let $Q : \mathcal{B} \to \mathcal{A}$ be a $\mathbb{C}$-$\mathbb{D}$-bicomodule functor. In view of Proposition 4.30, we set*

$$^{C}Q^{D} = \left({}^{C}Q\right)^{D} = {}^{C}\left(Q^{D}\right).$$

Proposition 4.32. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ and let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad over a category $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have equalizers and let $Q : \mathcal{B} \to \mathcal{A}$ be an $\mathbb{C}$-$\mathbb{D}$-bicomodule functor. Then, with notations of Proposition 4.29, we can consider the functor Q^D where $\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$. Then*

$$Q^{D\mathbb{D}}F = Q \text{ and } \iota^{Q\mathbb{D}}F = \rho_Q^D. \tag{39}$$

Proof. By construction we have that $\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$. By applying it to the functor ${}^{\mathbb{D}}F$ we get that

$$\begin{aligned}\left(Q^{D\mathbb{D}}F, \iota^{Q\mathbb{D}}F\right) &= \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U^{\mathbb{D}}F, Q^{\mathbb{D}}U\gamma^{D\mathbb{D}}F\right) \\ &= \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D}D, Q\Delta^D\right).\end{aligned}$$

Since Q is a right $\mathbb{D}$-comodule functor, by Proposition 4.15 we have that

$$\left(Q, \rho_Q^D\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D}D, Q\Delta^D\right)$$

so that we get

$$\left(Q^{D\mathbb{D}}F, \iota^{Q\mathbb{D}}F\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D}D, Q\Delta^D\right) = \left(Q, \rho_Q^D\right).$$

$\square$

Proposition 4.33. *Let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad over a category $\mathcal{B}$ with equalizers such that D preserves equalizers. Let $G : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ be a functor preserving equalizers. Set*

$$Q = G \circ {}^{\mathbb{D}}F \text{ and let } \rho_Q^D = G\gamma^{D\mathbb{D}}F$$

Then $\left(Q, \rho_Q^D\right)$ is a right $\mathbb{D}$-comodule functor and

$$Q^D = \left(G \circ {}^{\mathbb{D}}F\right)^D = G. \tag{40}$$

Proof. We compute

$$\begin{aligned}\left(\rho_Q^D D\right) \circ \rho_Q^D &= \left(G\gamma^{D\mathbb{D}}FD\right) \circ \left(G\gamma^{D\mathbb{D}}F\right) \overset{\gamma^D}{=} \left(G^{\mathbb{D}}F^{\mathbb{D}}U\gamma^{D\mathbb{D}}F\right) \circ \left(G\gamma^{D\mathbb{D}}F\right) \\ &= \left(G^{\mathbb{D}}F\Delta^D\right) \circ \left(G\gamma^{D\mathbb{D}}F\right) = \left(Q\Delta^D\right) \circ \rho_Q^D\end{aligned}$$

and

$$\left(Q\varepsilon^D\right) \circ \rho_Q^D = \left(G^{\mathbb{D}}F\varepsilon^D\right) \circ \left(G\gamma^{D\mathbb{D}}F\right) \overset{\text{adj}}{=} G^{\mathbb{D}}F = Q.$$

Thus $\left(Q, \rho_Q^D\right)$ is a right $\mathbb{D}$-comodule functor. Recall that (see Proposition 4.29)

$$\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$$

and by Proposition 4.32 we have $Q^{D\mathbb{D}}F = Q$ and $\iota^{Q\mathbb{D}}F = \rho_Q^D$. In particular we get

$$Q^{D\mathbb{D}}F = Q = G^{\mathbb{D}}F.$$

In order to prove that $Q^D = G$ it suffices to prove that $\left(G, G\gamma^D\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$. In fact, by Corollary 4.14, $\left({}^{\mathbb{D}}U, \left({}^{\mathbb{D}}U\gamma^D\right)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(D^{\mathbb{D}}U\gamma^D, \Delta^{D\mathbb{D}}U\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(D^{\mathbb{D}}U\gamma^D, {}^{\mathbb{D}}U\gamma^{D\mathbb{D}}F^{\mathbb{D}}U\right)$ and, since by Lemma 4.19 ${}^{\mathbb{D}}U$ reflects equalizers, we have

$$\left(\mathrm{Id}_{{}^{\mathbb{D}}\mathcal{B}}, \gamma^D\right) = \mathrm{Equ}_{\mathrm{Fun}}\left({}^{\mathbb{D}}F^{\mathbb{D}}U\gamma^D, \gamma^{D\mathbb{D}}F^{\mathbb{D}}U\right).$$

Since G preserves equalizers, we get that

$$\begin{aligned}\left(G, G\gamma^D\right) &= \mathrm{Equ}_{\mathrm{Fun}}\left(G^{\mathbb{D}}F^{\mathbb{D}}U\gamma^D, G\gamma^{D\mathbb{D}}F^{\mathbb{D}}U\right)\\ &= \mathrm{Equ}_{\mathrm{Fun}}\left(Q^{\mathbb{D}}U\gamma^D, \rho_Q^{D\mathbb{D}}U\right) = \left(Q^D, \iota^Q\right).\end{aligned}$$

□

PROPOSITION 4.34. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ with equalizers such that C preserves equalizers. Let $H : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ be a functor preserving equalizers. Set*

$$Q = {}^{\mathbb{C}}U \circ H \text{ and let } {}^{C}\rho_Q = {}^{\mathbb{C}}U\gamma^C H.$$

Then $\left(Q, {}^{C}\rho_Q\right)$ is a left $\mathbb{C}$-comodule functor and

$$ {}^{C}Q = {}^{C}\left({}^{\mathbb{C}}U \circ H\right) = H. \tag{41}$$

Proof. First we want to prove that ${}^{C}\rho_Q = {}^{\mathbb{C}}U\gamma^C H$ is coassociative. We have

$$\begin{aligned}\left(C^{C}\rho_Q\right) \circ {}^{C}\rho_Q = \left(C^{\mathbb{C}}U\gamma^C H\right) \circ \left({}^{\mathbb{C}}U\gamma^C H\right) &\overset{\gamma^C}{=} \left({}^{\mathbb{C}}U\gamma^{C\mathbb{C}}F^{\mathbb{C}}UH\right) \circ \left({}^{\mathbb{C}}U\gamma^C H\right)\\ = \left(\Delta^{C\mathbb{C}}UH\right) \circ \left({}^{\mathbb{C}}U\gamma^C H\right) &= \left(\Delta^C Q\right) \circ {}^{C}\rho_Q\end{aligned}$$

so that we get

$$\left(C^{C}\rho_Q\right) \circ {}^{C}\rho_Q = \left(\Delta^C Q\right) \circ {}^{C}\rho_Q.$$

Now we prove that ${}^{C}\rho_Q = {}^{\mathbb{C}}U\gamma^C H$ is counital. We compute

$$\left(\varepsilon^C Q\right) \circ {}^{C}\rho_Q = \left(\varepsilon^{C\mathbb{C}}UH\right) \circ {}^{\mathbb{C}}U\gamma^C H \overset{\mathrm{adj}}{=} {}^{\mathbb{C}}UH = Q$$

so that we get

$$\left(\varepsilon^C Q\right) \circ {}^{C}\rho_Q = Q.$$

Thus $\left(Q, {}^{C}\rho_Q\right)$ is a left $\mathbb{C}$-comodule functor. Recall that (see Lemma 4.28) there exists a unique functor ${}^{C}Q : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that

$${}^{\mathbb{C}}U \circ {}^{C}Q = Q \text{ and } {}^{\mathbb{C}}U\gamma^{CC}Q = {}^{C}\rho_Q.$$

Thus we have

$${}^{\mathbb{C}}U \circ {}^{C}Q = Q = {}^{\mathbb{C}}U \circ H$$

and

$${}^{\mathbb{C}}U\gamma^{CC}Q = {}^{C}\rho_Q = {}^{\mathbb{C}}U\gamma^C H$$

so that, by Proposition 4.11, we obtain that

$${}^{C}Q = H.$$

□

THEOREM 4.35. *Let $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be a comonad on a category $\mathcal{B}$ with equalizers such that D preserves equalizers. Then there exists a bijective correspondence between the following collections of data:*

$\mathcal{F}^D$ *right* $\mathbb{D}$*-comodule functors* $Q : \mathcal{B} \to \mathcal{A}$ *such that* QD *preserves equalizers.*
$\left(\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right)$ *functors* $G : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ *preserving equalizers*

given by

$$\begin{aligned} \nu^D &: \mathcal{F}^D \to \left(\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right) \text{ where } \nu^D\left(\left(Q, \rho_Q^D\right)\right) = Q^D \\ \kappa^D &: \left(\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right) \to \mathcal{F}^D \text{ where } \kappa^D(G) = \left(G^{\mathbb{D}}F, G\gamma^{D\mathbb{D}}F\right) \end{aligned}$$

where Q^D *is uniquely determined by* $\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$.

Proof. Let $Q : \mathcal{B} \to \mathcal{A}$ be a right $\mathbb{D}$-comodule functor. Then we can consider $Q^D : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ defined by (32) as

$$\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right).$$

Since by assumption QD preserves equalizers, by Lemma 4.18 also Q preserves equalizers. Moreover, since D preserves equalizers, by Lemma 4.20 also the functor ${}^{\mathbb{D}}U$ preserves equalizers. Thus both $QD^{\mathbb{D}}U$ and $Q^{\mathbb{D}}U$ preserve equalizers. By Corollary 2.14 we get that also $Q^D : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ preserves equalizers.

Conversely, let us consider a functor $G : {}^{\mathbb{D}}\mathcal{B} \to \mathcal{A}$ that preserves equalizers. By Proposition 4.33 we can consider the right $\mathbb{D}$-comodule functor defined as follows

$$Q = G \circ {}^{\mathbb{D}}F \text{ and let } \rho_Q^D = G\gamma^{D\mathbb{D}}F.$$

Since ${}^{\mathbb{D}}F$ is right adjoint to ${}^{\mathbb{D}}U$ in particular ${}^{\mathbb{D}}F$ preserves equalizers and since by assumption G preserves equalizers, we get that also $Q = G \circ {}^{\mathbb{D}}F$ preserves equalizers and so does QD.

Now, we want to prove that ν^D and κ^D determine a bijective correspondence between $\mathcal{F}^D$ and $\left(\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right)$. Let us start with a right $\mathbb{D}$-comodule functor $\left(Q : \mathcal{B} \to \mathcal{A}, \rho_Q^D\right)$. Then we have

$$\begin{aligned} \left(\kappa^D \circ \nu^D\right)\left(\left(Q, \rho_Q^D\right)\right) &= \kappa^D\left(Q^D\right) = \left(Q^{D\mathbb{D}}F, Q^D\gamma^{D\mathbb{D}}F\right) \\ &= \left(Q^{D\mathbb{D}}F, \rho^D_{Q^{D\mathbb{D}}F}\right) \overset{(39)}{=} \left(Q, \rho_Q^D\right). \end{aligned}$$

Moreover we have

$$\left(\nu^D \circ \kappa^D\right)(G) = \nu^D\left(\left(G^{\mathbb{D}}F, G\gamma^{D\mathbb{D}}F\right)\right) = \left(G^{\mathbb{D}}F\right)^D \overset{(40)}{=} G.$$

∎

THEOREM 4.36. *Let* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ *be a comonad on a category* $\mathcal{A}$ *with equalizers such that* C *preserves equalizers. Then there exists a bijective correspondence between the following collections of data:*

${}^C\mathcal{F}$ *left* $\mathbb{C}$*-comodule functors* $Q : \mathcal{B} \to \mathcal{A}$ *such that* CQ *preserves equalizers*
$\left({}^{\mathbb{C}}\mathcal{A} \leftarrow \mathcal{B}\right)$ *functors* $H : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ *preserving equalizers*

given by

$$\begin{aligned} {}^C\nu &: {}^C\mathcal{F} \to \left({}^{\mathbb{C}}\mathcal{A} \leftarrow \mathcal{B}\right) \text{ where } {}^C\nu\left(\left(Q, {}^C\rho_Q\right)\right) = {}^CQ \\ {}^C\kappa &: \left({}^{\mathbb{C}}\mathcal{A} \leftarrow \mathcal{B}\right) \to {}^C\mathcal{F} \text{ where } {}^C\kappa(H) = \left({}^{\mathbb{C}}U \circ H, {}^{\mathbb{C}}U\gamma^C H\right) \end{aligned}$$

where ${}^CQ : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ *is the functor defined in Lemma 4.28.*

Proof. Let $\left(Q:\mathcal{B}\rightarrow\mathcal{A},{}^{C}\rho_{Q}\right)$ be a left $\mathbb{C}$-comodule functor. Then, by Lemma 4.28, there exists a unique functor ${}^{C}Q:\mathcal{B}\rightarrow{}^{\mathbb{C}}\mathcal{A}$ such that

$$^{\mathbb{C}}U\circ{}^{C}Q=Q\text{ and }{}^{\mathbb{C}}U\gamma^{CC}Q={}^{C}\rho_{Q}.$$

Note that, since CQ preserves equalizers, by Lemma 4.18, $Q={}^{\mathbb{C}}U\circ{}^{C}Q$ preserves equalizers. Then, by Lemma 4.19, also ${}^{C}Q$ preserves equalizers. Conversely, if $H:\mathcal{B}\rightarrow{}^{\mathbb{C}}\mathcal{A}$ is a functor preserving equalizers, we get that ${}^{\mathbb{C}}U\circ H:\mathcal{B}\rightarrow\mathcal{A}$. Moreover, by Lemma 4.20, ${}^{\mathbb{C}}U$ preserves equalizers and thus also ${}^{\mathbb{C}}U\circ H$ preserves equalizers. Now, let us prove that ${}^{C}\nu$ and ${}^{C}\kappa$ determine a bijective correspondence between ${}^{C}\mathcal{F}$ and $\left({}^{\mathbb{C}}\mathcal{A}\leftarrow\mathcal{B}\right)$. We compute

$$\left({}^{C}\kappa\circ{}^{C}\nu\right)\left(\left(Q,{}^{C}\rho_{Q}\right)\right)={}^{C}\kappa\left({}^{C}Q\right)=\left({}^{\mathbb{C}}U^{C}Q,{}^{\mathbb{C}}U\gamma^{CC}Q\right)=\left(Q,{}^{C}\rho_{Q}\right).$$

On the other hand we have

$$\left({}^{C}\nu\circ{}^{C}\kappa\right)(H)={}^{C}\nu\left(\left({}^{\mathbb{C}}U\circ H,{}^{\mathbb{C}}U\gamma^{C}H\right)\right)={}^{C}\left({}^{\mathbb{C}}U\circ H\right)\overset{(41)}{=}H.$$

□

THEOREM 4.37. *Let $\mathbb{C}=\left(C,\Delta^{C},\varepsilon^{C}\right)$ be a comonad on a category $\mathcal{A}$ with equalizers such that C preserves equalizers. Let $\mathbb{D}=\left(D,\Delta^{D},\varepsilon^{D}\right)$ be a comonad on a category $\mathcal{B}$ with equalizers such that D preserves equalizers. Then there exists a bijective correspondence between the following collections of data:*

${}^{C}\mathcal{F}^{D}$ *$\mathbb{C}$-$\mathbb{D}$-bimodule functors $Q:\mathcal{B}\rightarrow\mathcal{A}$ such that CQ and QD preserve equalizers*
$\left({}^{\mathbb{C}}\mathcal{A}\leftarrow{}^{\mathbb{D}}\mathcal{B}\right)$ *functors $G:{}^{\mathbb{D}}\mathcal{B}\rightarrow{}^{\mathbb{C}}\mathcal{A}$ preserving equalizers*

given by

$${}^{C}\nu^{D}\;:\;{}^{C}\mathcal{F}^{D}\rightarrow\left({}^{\mathbb{C}}\mathcal{A}\leftarrow{}^{\mathbb{D}}\mathcal{B}\right)\text{ where }{}^{C}\nu^{D}\left(\left(Q,{}^{C}\rho_{Q},\rho_{Q}^{D}\right)\right)={}^{C}Q^{D}$$
$${}^{C}\kappa^{D}\;:\;\left({}^{\mathbb{C}}\mathcal{A}\leftarrow{}^{\mathbb{D}}\mathcal{B}\right)\rightarrow{}^{C}\mathcal{F}^{D}\text{ where }{}^{C}\kappa^{D}(G)=\left({}^{\mathbb{C}}U\circ G\circ{}^{\mathbb{D}}F,{}^{\mathbb{C}}U\gamma^{C}G^{\mathbb{D}}F,{}^{\mathbb{C}}UG\gamma^{D\mathbb{D}}F\right).$$

Proof. Let us consider a $\mathbb{C}$-$\mathbb{D}$-bicomodule functor $\left(Q:\mathcal{B}\rightarrow\mathcal{A},{}^{C}\rho_{Q},\rho_{Q}^{D}\right)$ such that CQ and QD preserve equalizers. In particular, $\left(Q,\rho_{Q}^{D}\right)$ is a right $\mathbb{D}$-comodule functor, so that we can apply the map $\nu^{D}:\mathcal{F}^{D}\rightarrow\left(\mathcal{A}\leftarrow{}^{\mathbb{D}}\mathcal{B}\right)$ of Theorem 4.35 and we get a functor $\nu^{D}\left(\left(Q,\rho_{Q}^{D}\right)\right)=Q^{D}:{}^{\mathbb{D}}\mathcal{B}\rightarrow\mathcal{A}$ which preserves equalizers. By Proposition 4.29, $\left(Q^{D},{}^{C}\rho_{Q^{D}}\right)$ is a left $\mathbb{C}$-comodule functor so that we can also apply the map ${}^{C}\nu:{}^{C}\mathcal{F}\rightarrow\left({}^{\mathbb{C}}\mathcal{A}\leftarrow\mathcal{B}\right)$ of Theorem 4.36 where the category $\mathcal{B}$ is ${}^{\mathbb{D}}\mathcal{B}$. The map ${}^{C}\nu$ is defined by ${}^{C}\nu\left(\left(Q^{D},{}^{C}\rho_{Q^{D}}\right)\right)={}^{C}\left(Q^{D}\right)={}^{C}Q^{D}:{}^{\mathbb{D}}\mathcal{B}\rightarrow{}^{\mathbb{C}}\mathcal{A}$ and ${}^{C}Q^{D}$ preserves equalizers. Conversely, let us consider a functor $G:{}^{\mathbb{D}}\mathcal{B}\rightarrow{}^{\mathbb{C}}\mathcal{A}$ which preserves equalizers. By Theorem 4.36, we get a left $\mathbb{C}$-comodule functor given by

$${}^{C}\kappa(G)=\left({}^{\mathbb{C}}U\circ G,{}^{\mathbb{C}}U\gamma^{C}G\right)$$

where ${}^{\mathbb{C}}U\circ G:{}^{\mathbb{D}}\mathcal{B}\rightarrow\mathcal{A}$ and $C^{\mathbb{C}}UG$ preserves equalizers. By Lemma 4.18, also ${}^{\mathbb{C}}U\circ G:{}^{\mathbb{D}}\mathcal{B}\rightarrow\mathcal{A}$ preserves equalizers. Thus, we can apply Theorem 4.35 and we get a right $\mathbb{D}$-comodule functor

$$\kappa^{D}\left({}^{\mathbb{C}}UG\right)=\left({}^{\mathbb{C}}UG^{\mathbb{D}}F,{}^{\mathbb{C}}UG\gamma^{D\mathbb{D}}F\right)$$

where ${}^{\mathbb{C}}UG^{\mathbb{D}}F:\mathcal{B}\rightarrow\mathcal{A}$ is such that ${}^{\mathbb{C}}UG^{\mathbb{D}}FD$ preserves equalizers. Clearly, since ${}^{\mathbb{C}}UG$ preserves equalizers, ${}^{\mathbb{D}}F$ is a right adjoint and C preserves equalizers by assumption, we deduce that also $C^{\mathbb{C}}UG^{\mathbb{D}}F$ preserves equalizers. Now, we want to prove that ${}^{C}\nu^{D} : {}^{C}\mathcal{F}^{D} \to \left({}^{\mathbb{C}}\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right)$ and ${}^{C}\kappa^{D} : \left({}^{\mathbb{C}}\mathcal{A} \leftarrow {}^{\mathbb{D}}\mathcal{B}\right) \to {}^{C}\mathcal{F}^{D}$ determine a bijection. We have

$$\begin{aligned}&\left({}^{C}\kappa^{D} \circ {}^{C}\nu^{D}\right)\left(\left(Q, {}^{C}\rho_{Q}, \rho_{Q}^{D}\right)\right) = {}^{C}\kappa^{D}\left({}^{C}Q^{D}\right)\\ &= \left({}^{\mathbb{C}}U \circ {}^{C}Q^{D} \circ {}^{\mathbb{D}}F, {}^{\mathbb{C}}U\gamma^{CC}Q^{D\mathbb{D}}F, {}^{\mathbb{C}}U^{C}Q^{D}\gamma^{D\mathbb{D}}F\right) = \left(Q, {}^{\mathbb{C}}U\gamma^{CC}Q, Q^{D}\gamma^{D\mathbb{D}}F\right)\\ &= \left(Q, {}^{C}\rho_{Q}, \rho_{Q^{D\mathbb{D}}F}^{D}\right) = \left(Q, {}^{C}\rho_{Q}, \rho_{Q}^{D}\right)\end{aligned}$$

and

$$\begin{aligned}\left({}^{C}\nu^{D} \circ {}^{C}\kappa^{D}\right)(G) &= {}^{C}\nu^{D}\left(\left({}^{\mathbb{C}}U \circ G \circ {}^{\mathbb{D}}F, {}^{\mathbb{C}}U\gamma^{C}G^{\mathbb{D}}F, {}^{\mathbb{C}}UG\gamma^{D\mathbb{D}}F\right)\right)\\ &= {}^{C}\left({}^{\mathbb{C}}U \circ G \circ {}^{\mathbb{D}}F\right)^{D} = {}^{C}\left(\left({}^{\mathbb{C}}U \circ G \circ {}^{\mathbb{D}}F\right)^{D}\right)\\ &\overset{(40)}{=} {}^{C}\left({}^{\mathbb{C}}U \circ G\right) \overset{(41)}{=} G.\end{aligned}$$

□

PROPOSITION 4.38. *Let* $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ *be a comonad over a category* $\mathcal{A}$ *with equalizers and assume that* C *preserves equalizers. Let* $\mathbb{D} = \left(D, \Delta^{D}, \varepsilon^{D}\right)$ *be a comonad over a category* $\mathcal{B}$ *with equalizers and let* $Q : \mathcal{B} \to \mathcal{A}$ *be a* $\mathbb{C}$*-*$\mathbb{D}$*-bicomodule functor. Then there exists a unique lifted functor* ${}^{C}Q^{D} : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ *such that*

$${}^{\mathbb{C}}U^{C}Q^{D\mathbb{D}}F = Q.$$

Proof. By Proposition 4.30 there exists a unique functor ${}^{C}Q^{D} : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U^{C}Q^{D} = Q^{D}$. Now, by Proposition 4.32 we also get that $Q^{D\mathbb{D}}F = Q$ so that we obtain

$${}^{\mathbb{C}}U^{C}Q^{D\mathbb{D}}F = Q.$$

□

COROLLARY 4.39. *Let* $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ *be a comonad over a category* $\mathcal{A}$ *with equalizers and assume that* C *preserves equalizers and let* $Q : \mathcal{A} \to \mathcal{A}$ *be a* $\mathbb{C}$*-bicomodule functor. Then there exists a unique lifted functor* ${}^{C}Q^{C} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{C}}\mathcal{A}$ *such that*

$${}^{\mathbb{C}}U^{C}Q^{C\mathbb{C}}F = Q.$$

Proof. We can apply Proposition 4.38 to the case $\mathbb{D} = \mathbb{C}$ and $\mathcal{B} = \mathcal{A}$. □

PROPOSITION 4.40. *Let* $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ *be a comonad over a category* $\mathcal{A}$ *with equalizers and assume that* C *preserves equalizers. Let* $\mathbb{D} = \left(D, \Delta^{D}, \varepsilon^{D}\right)$ *be a comonad over a category* $\mathcal{B}$ *with equalizers and let* $P, Q : \mathcal{B} \to \mathcal{A}$ *be* $\mathbb{C}$*-*$\mathbb{D}$*-bicomodule functors. Let* $f : P \to Q$ *be a functorial morphism of left* $\mathbb{C}$*-comodule functors and of right* $\mathbb{D}$*-comodule functors. Then there exists a unique functorial morphism of left* $\mathbb{C}$*-comodule functors*

$$f^{D} : P^{D} \to Q^{D}$$

satisfying

$$\iota^{Q} \circ f^{D} = \left(f^{\mathbb{D}}U\right) \circ \iota^{P}.$$

Then we can consider

$${}^{C}f^{D} : {}^{C}P^{D} \to {}^{C}Q^{D}$$

such that

$${}^{\mathbb{D}}U^{C}f^{D} = f^{D}.$$

Proof. Consider the following diagram

$$\begin{tikzcd} P^D \arrow[r, "\iota^P"] \arrow[d, dotted, "f^D"'] & P^{\mathbb{D}}U \arrow[r, shift left, "\rho_P^{D\mathbb{D}}U"] \arrow[r, shift right, "P^{\mathbb{D}}U\gamma^D"'] \arrow[d, "f^{\mathbb{D}}U"'] & PD_{\mathbb{D}}U \arrow[d, "fD^{\mathbb{D}}U"] \\ Q^D \arrow[r, "\iota^Q"] & Q^{\mathbb{D}}U \arrow[r, shift left, "\rho_Q^{D\mathbb{D}}U"] \arrow[r, shift right, "Q^{\mathbb{D}}U\gamma^D"'] & QD_{\mathbb{D}}U \end{tikzcd}$$

Since f is a functorial morphism and it is a functorial morphism of right $\mathbb{D}$-comodule functors, the right square serially commutes. Note that

$$\left(\rho_Q^{D\mathbb{D}}U\right) \circ \left(f^{\mathbb{D}}U\right) \circ \iota^P = \left(Q^{\mathbb{D}}U\gamma^D\right) \circ \left(f^{\mathbb{D}}U\right) \circ \iota^P$$

so that, by the universal property of the equalizer, there exists a unique morphism $f^D : P^D \to Q^D$ such that

$$\left(f^{\mathbb{D}}U\right) \circ \iota^P = \iota^Q \circ f^D. \tag{42}$$

We now want to prove that f^D is a functorial morphism of left $\mathbb{C}$-comodule functor. In fact we have

$$\begin{aligned} \left(C\iota^Q\right) \circ {}^C\rho_{Q^D} \circ f^D &\overset{(33)}{=} \left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q \circ f^D \\ &\overset{(42)}{=} \left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \left(f^{\mathbb{D}}U\right) \circ \iota^P \\ &\overset{f\text{left}C\text{colin}}{=} \left(Cf^{\mathbb{D}}U\right) \circ \left({}^C\rho_P{}^{\mathbb{D}}U\right) \circ \iota^P \\ &\overset{(33)}{=} \left(Cf^{\mathbb{D}}U\right) \circ \left(C\iota^P\right) \circ {}^C\rho_{P^D} \\ &\overset{(42)}{=} \left(C\iota^Q\right) \circ \left(Cf^D\right) \circ {}^C\rho_{P^D} \end{aligned}$$

and since C preserves equalizers $C\iota^Q$ is a monomorphism so that we get

$${}^C\rho_{Q^D} \circ f^D = \left(Cf^D\right) \circ {}^C\rho_{P^D}.$$

Then there exists a functorial morphism ${}^Cf^D : {}^CP^D \to {}^CQ^D$ such that

$${}^{\mathbb{C}}U{}^Cf^D = f^D.$$

□

COROLLARY 4.41. *Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad over a category $\mathcal{A}$ with equalizers and assume that C preserves equalizers and let $P, Q : \mathcal{B} \to \mathcal{A}$ be $\mathbb{C}$-bicomodule functors. Let $f : P \to Q$ be a functorial morphism of $\mathbb{C}$-bicomodule functors. Then there exists a unique functorial morphism of left $\mathbb{C}$-comodule functors*

$$f^C : P^C \to Q^C$$

satisfying

$$\iota^Q \circ f^C = \left(f^{\mathbb{C}}U\right) \circ \iota^P.$$

Then we can consider

$${}^Cf^C : {}^CP^C \to {}^CQ^C$$

such that

$${}^{\mathbb{C}}U{}^Cf^C = f^C.$$

Proof. We can apply Proposition 4.40 to the case $\mathbb{D} = \mathbb{C}$ and $\mathcal{B} = \mathcal{A}$. □

4.2. **The comparison functor for comonads.**

PROPOSITION 4.42 ([GT, Proposition 2.1]). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. There exists a bijective correspondence between the following collections of data:*

- $\mathfrak{M}$ *comonad morphisms $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$*
- $\mathfrak{R}$ *functorial morphism $\alpha : R \to RC$ such that (R, α) is a right comodule functor for the comonad $\mathbb{C}$*
- $\mathfrak{L}$ *functorial morphism $\beta : L \to CL$ such that (L, β) is a left comodule functor for the comonad $\mathbb{C}$*

given by

$$\begin{aligned}
\Theta &: \mathfrak{M} \to \mathfrak{R} \text{ where } \Theta(\varphi) = (R\varphi) \circ (\eta R) \\
\Xi &: \mathfrak{R} \to \mathfrak{M} \text{ where } \Xi(\alpha) = (\epsilon C) \circ (L\alpha) \\
\Gamma &: \mathfrak{M} \to \mathfrak{L} \text{ where } \Gamma(\varphi) = (\varphi L) \circ (L\eta) \\
\Lambda &: \mathfrak{L} \to \mathfrak{M} \text{ where } \Lambda(\beta) = (C\epsilon) \circ (\beta R).
\end{aligned}$$

Proof. For a given $\varphi \in \mathfrak{M}$, we compute

$$\begin{aligned}
(\Theta(\varphi) C) \circ \Theta(\varphi) &= (R\varphi C) \circ (\eta RC) \circ (R\varphi) \circ (\eta R) \\
&\overset{\eta}{=} (R\varphi C) \circ (RLR\varphi) \circ (\eta RLR) \circ (\eta R)
\end{aligned}$$

$$\overset{\eta,\varphi}{=} (R\varphi\varphi) \circ (RL\eta R) \circ (\eta R) \overset{\varphi\text{morphcom}}{=} \left(R\Delta^C\right) \circ (R\varphi) \circ (\eta R) = \left(R\Delta^C\right) \circ \Theta(\varphi)$$

and

$$\left(R\varepsilon^C\right) \circ \Theta(\varphi) = \left(R\varepsilon^C\right) \circ (R\varphi) \circ (\eta R) \overset{\varphi\text{morphcom}}{=} (R\epsilon) \circ (\eta R) = R.$$

Therefore we deduce that $\Theta(\varphi) \in \mathfrak{R}$. For a given $\alpha \in \mathfrak{R}$, we compute

$$\begin{aligned}
(\Xi(\alpha)\Xi(\alpha)) \circ (L\eta R) &\overset{\Xi(\alpha)}{=} (\Xi(\alpha) C) \circ (LR\Xi(\alpha)) \circ (L\eta R) \\
&= (\epsilon CC) \circ (L\alpha C) \circ (LR\epsilon C) \circ (LRL\alpha) \circ (L\eta R) \\
&\overset{\eta}{=} (\epsilon CC) \circ (L\alpha C) \circ (L\alpha) \overset{(R,\alpha)}{=} (\epsilon CC) \circ \left(LR\Delta^C\right) \circ (L\alpha) \\
&\overset{\epsilon}{=} \Delta^C \circ (\epsilon C) \circ (L\alpha) = \Delta^C \circ \Xi(\alpha)
\end{aligned}$$

and

$$\varepsilon^C \circ \Xi(\alpha) = \varepsilon^C \circ (\epsilon C) \circ (L\alpha) \overset{\epsilon}{=} \epsilon \circ \left(LR\varepsilon^C\right) \circ (L\alpha) \overset{(R,\alpha)}{=} \epsilon.$$

Therefore we deduce that $\Xi(\alpha) \in \mathfrak{M}$. For a given $\varphi \in \mathfrak{M}$, we compute

$$[C\Gamma(\varphi)] \circ \Gamma(\varphi) = (C\varphi L) \circ (CL\eta) \circ (\varphi L) \circ (L\eta)$$

$$\overset{\varphi}{=} (\varphi CL) \circ (LR\varphi L) \circ (LRL\eta) \circ (L\eta) \overset{\eta}{=} (\varphi CL) \circ (LR\varphi L) \circ (L\eta RL) \circ (L\eta)$$

$$= (\varphi\varphi L) \circ (L\eta RL) \circ (L\eta) \overset{\varphi\text{morphcom}}{=} \left(\Delta^C L\right) \circ (\varphi L) \circ (L\eta) = \left(\Delta^C L\right) \circ \Gamma(\varphi)$$

and

$$\left(\varepsilon^C L\right) \circ \Gamma(\varphi) = \left(\varepsilon^C L\right) \circ (\varphi L) \circ (L\eta) \overset{\varphi\text{morphcom}}{=} (\epsilon L) \circ (L\eta) = L.$$

Therefore we deduce that $\Gamma(\varphi) \in \mathfrak{L}$. For a given $\beta \in \mathfrak{L}$, we compute

$$(\Lambda(\beta)\Lambda(\beta)) \circ (L\eta R) \overset{\Lambda(\beta)}{=} (C\Lambda(\beta)) \circ (\Lambda(\beta) LR) \circ (L\eta R)$$

$$= (CC\epsilon) \circ (C\beta R) \circ (C\epsilon LR) \circ (\beta RLR) \circ (L\eta R)$$

$$\overset{\beta}{=} (CC\epsilon) \circ (C\beta R) \circ (C\epsilon LR) \circ (CL\eta R) \circ (\beta R) = (CC\epsilon) \circ (C\beta R) \circ (\beta R)$$

$$\overset{(L,\beta)}{=} (CC\epsilon) \circ \left(\Delta^C LR\right) \circ (\beta R) =$$

$$\overset{\Delta^C}{=} \Delta^C \circ (C\epsilon) \circ (\beta R) = \Delta^C \circ \Lambda (\beta)$$

and

$$\varepsilon^C \circ \Lambda (\beta) = \varepsilon^C \circ (C\epsilon) \circ (\beta R) \overset{\varepsilon^C}{=} \epsilon \circ \left(\varepsilon^C LR\right) \circ (\beta R) \overset{(L,\beta)}{=} \epsilon.$$

Therefore we deduce that $\Lambda (\beta) \in \mathfrak{M}$. Let now $\varphi \in \mathfrak{M}$ and let us calculate

$$\Xi\Theta (\varphi) = (\epsilon C) \circ (LR\varphi) \circ (L\eta R) \overset{\epsilon}{=} \varphi \circ (\epsilon LR) \circ (L\eta R) = \varphi.$$

Let now $\alpha \in \mathfrak{R}$ and let us calculate

$$\Theta\Xi (\alpha) = (R\Xi (\alpha)) \circ (\eta R) = (R\epsilon C) \circ (RL\alpha) \circ (\eta R) (R\epsilon C) \circ (\eta RC) \circ \alpha = \alpha.$$

Let now $\varphi \in \mathfrak{M}$ and let us calculate

$$\Lambda\Gamma (\varphi) = (C\epsilon) \circ (\Gamma (\varphi) R) = (C\epsilon) \circ (\varphi LR) \circ (L\eta R) \overset{\varphi}{=} \varphi \circ (LR\epsilon) \circ (L\eta R) = \varphi.$$

Let now $\beta \in \mathfrak{L}$ and let us calculate

$$\Gamma\Lambda (\beta) = (\Lambda (\beta) L) \circ (L\eta) = (C\epsilon L) \circ (\beta RL) \circ (L\eta) \overset{\beta}{=} (C\epsilon L) \circ (CL\eta) \circ \beta = \beta.$$

□

THEOREM 4.43 ([D, Theorem II.1.1] and [GT, Theorem 1.2]). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$ and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. There exists a bijective correspondence between the following collections of data:*

$\mathfrak{K}$ *Functors $K : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ such that ${}^{\mathbb{C}}U \circ K = L$,*

$\mathfrak{M}$ *comonad morphisms $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$*

given by

$$\Psi \ : \ \mathfrak{K} \to \mathfrak{M} \text{ where } \Psi (K) = (C\epsilon) \circ \left(\left[{}^{\mathbb{C}}U \left(\gamma^C K\right)\right] R\right)$$

$$\Upsilon \ : \ \mathfrak{M} \to \mathfrak{K} \text{ where } \Upsilon (\varphi) (Y) = (LY, (\varphi LY) \circ (L\eta Y)) \text{ and } \Upsilon (\varphi) (f) = L (f).$$

Proof. By Corollary 4.25, there exists a bijective correspondence between $\mathfrak{K}$ and the collection $\mathfrak{L}$ of functorial morphisms $\beta : L \to CL$ such that (L, β) is a left comodule functor for the comonad $\mathbb{C}$ given by

$$\Phi \ : \ \mathfrak{K} \to \mathfrak{L} \text{ where } \Phi (K) = {}^{\mathbb{C}}U \left(\gamma^C K\right) : L \to CL$$

$$\Omega \ : \ \mathfrak{L} \to \mathfrak{K} \text{ where } \Omega (\beta) (Y) = (LY, \beta Y) \text{ and } \Omega (\beta) (f) = L (f).$$

By Proposition 4.42, there exists a bijective correspondence between $\mathfrak{L}$ and the collection $\mathfrak{M}$ of comonad morphisms $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ given by

$$\Lambda \ : \ \mathfrak{L} \to \mathfrak{M} \text{ where } \Lambda (\beta) = (C\epsilon) \circ (\beta R)$$

$$\Gamma \ : \ \mathfrak{M} \to \mathfrak{L} \text{ where } \Gamma (\varphi) = (\varphi L) \circ (L\eta).$$

We compute

$$(\Lambda \circ \Phi) (K) = (C\epsilon) \circ \left(\left[{}^{\mathbb{C}}U \left(\gamma^C K\right)\right] R\right) = \Psi (K)$$

and

$$[(\Omega\circ\Gamma)(\varphi)](Y)=(LY,(\varphi LY)\circ(L\eta Y))=\Upsilon(\varphi)(Y)\text{ and }[(\Omega\circ\Gamma)(\varphi)](f)=Lf.$$

□

REMARK 4.44. When $\mathbb{C}=\mathbb{LR}=(LR,L\eta R,\epsilon)$ and $\varphi=\mathrm{Id}_{\mathbb{LR}}$ the functor $K=\Upsilon(\varphi):\mathcal{B}\to{}^{\mathbb{LR}}\mathcal{A}$ such that ${}^{\mathbb{LR}}U\circ K=L$ is called the *Eilenberg-Moore comparison functor*.

COROLLARY 4.45. *Let $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ and $\mathbb{D}=\left(D,\Delta^D,\varepsilon^D\right)$ be comonads on a category $\mathcal{A}$. There exists a bijective correspondence between the following collections of data:*

$\mathcal{K}$ *Functors $K:{}^{\mathbb{C}}\mathcal{A}\to{}^{\mathbb{D}}\mathcal{A}$ such that ${}^{\mathbb{D}}U\circ K={}^{\mathbb{C}}U$,*
$\mathcal{M}$ *comonad morphisms $\varphi:\mathbb{C}\to\mathbb{D}$*

given by

$$\Psi\ :\ \mathcal{K}\to\mathcal{M}\text{ where }\Psi(K)=(C\epsilon)\circ\left(\left[{}^{\mathbb{D}}U\left(\gamma^DK\right)\right]{}^{\mathbb{C}}F\right)$$
$$\Upsilon\ :\ \mathcal{M}\to\mathcal{K}\text{ where }\Upsilon(\varphi)(Y)=\left({}^{\mathbb{C}}UY,\left(\varphi{}^{\mathbb{C}}UY\right)\circ\left({}^{\mathbb{C}}U\gamma^CY\right)\right)\text{ and }\Upsilon(\varphi)(f)={}^{\mathbb{C}}U(f).$$

Proof. Apply Theorem 4.43 to the case $L={}^{\mathbb{C}}U:{}^{\mathbb{C}}\mathcal{A}\to\mathcal{A}$ and $R={}^{\mathbb{C}}F:\mathcal{A}\to{}^{\mathbb{C}}\mathcal{A}$ and note that $(LR,L\eta R,\epsilon)=\left({}^{\mathbb{C}}U{}^{\mathbb{C}}F,{}^{\mathbb{C}}U\gamma^{C\mathbb{C}}F,\varepsilon^C\right)=\left(C,\Delta^C,\varepsilon^C\right)$. □

PROPOSITION 4.46. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$, let $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ be a comonad on the category $\mathcal{A}$ and let $\varphi:\mathbb{LR}=(LR,L\eta R,\epsilon)\to\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ be a comonad morphism. Let $\alpha=\Theta(\varphi)=(R\varphi)\circ(\eta R)$. Then the isomorphism $a_{X,Y}:\mathrm{Hom}_{\mathcal{A}}(LY,X)\to\mathrm{Hom}_{\mathcal{B}}(Y,RX)$ of the adjunction (L,R) induces an isomorphism*

$$\widehat{a}_{X,Y}:\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y,\left(X,{}^C\rho_X\right)\right)\to\mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{B}}(Y,\alpha X),\mathrm{Hom}_{\mathcal{B}}\left(Y,R^{\mathbb{C}}\rho_X\right)\right).$$

Proof. Let

$$a_{X,Y}:\mathrm{Hom}_{\mathcal{A}}(LY,X)\to\mathrm{Hom}_{\mathcal{B}}(Y,RX)$$

be the isomorphism of the adjunction (L,R) for every $Y\in\mathcal{B}$ and for every $X\in\mathcal{A}$. Recall that $a_{X,Y}(\xi)=(R\xi)\circ(\eta Y)$ and $a_{X,Y}^{-1}(\zeta)=(\epsilon X)\circ(L\zeta)$. Let us check that we can apply Lemma 2.15 to the case $Z=\mathrm{Hom}_{\mathcal{A}}(L-,X)$, $Z'=\mathrm{Hom}_{\mathcal{B}}(-,RX)$, $W=\mathrm{Hom}_{\mathcal{A}}(L-,CX)$, $W'=\mathrm{Hom}_{\mathcal{B}}(-,RCX)$, $a={}^C\rho_X\circ-$, $b=C-\circ\Gamma(\varphi)Y$, $a'=\mathrm{Hom}_{\mathcal{B}}(-,\alpha X)$, $b'=\mathrm{Hom}_{\mathcal{B}}\left(-,R^{\mathbb{C}}\rho_X\right)$ and $\varphi=a_{X,-}$, $\psi=a_{CX,-}$, $E=\mathrm{Equ}_{\mathrm{Fun}}\left({}^C\rho_X\circ-,C(-)\circ\Gamma(\varphi)Y\right)$ and
$E'=\mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{B}}(-,\alpha X),\mathrm{Hom}_{\mathcal{B}}\left(-,R^{\mathbb{C}}\rho_X\right)\right)$

$$\begin{tikzcd}
\mathrm{Equ}_{\mathrm{Fun}}\left({}^{\mathbb{C}}\rho_X\circ-,C(-)\circ\Gamma(\varphi)-\right) \arrow[r, dotted, "\hat{a}_{X,-}"] \arrow[d, "i"] & \mathrm{Equ}_{\mathrm{Fun}}\left(\mathrm{Hom}_{\mathcal{B}}(-,\alpha X),\mathrm{Hom}_{\mathcal{B}}\left(-,R^{\mathbb{C}}\rho_X\right)\right) \arrow[d, "i'"] \\
Z=\mathrm{Hom}_{\mathcal{A}}(L-,X) \arrow[r, "a_{X,-}"] \arrow[d, shift right, "a={}^{\mathbb{C}}\rho_X\circ-"'] \arrow[d, shift left, "b=C-\circ\Gamma(\varphi)-"] & Z'=\mathrm{Hom}_{\mathcal{B}}(-,RX) \arrow[d, shift right, "a'=\mathrm{Hom}_{\mathcal{B}}(-,\alpha X)"'] \arrow[d, shift left, "b'=\mathrm{Hom}_{\mathcal{B}}\left(-,R^{\mathbb{C}}\rho_X\right)"] \\
W=\mathrm{Hom}_{\mathcal{A}}(L-,CX) \arrow[r, "a_{CX,-}"] & W'=\mathrm{Hom}_{\mathcal{B}}(-,RCX)
\end{tikzcd}$$

For every $Y \in \mathcal{B}$, $X \in \mathcal{A}$ and for every $\xi \in \mathrm{Hom}_{\mathcal{A}}(LY, X)$, let us compute

$$\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X) \circ a_{X,Y}(\xi) = \alpha X \circ a_{X,Y}(\xi) = (R\varphi X) \circ (\eta RX) \circ a_{X,Y}(\xi)$$
$$\overset{\text{defa}}{=} (R\varphi X) \circ (\eta RX) \circ (R\xi) \circ (\eta Y)$$
$$\overset{\eta}{=} (R\varphi X) \circ (RLR\xi) \circ (RL\eta Y) \circ (\eta Y) \overset{\text{defa}}{=} a_{CX,Y}\left[(\varphi X) \circ (LR\xi) \circ (L\eta Y)\right] \overset{\varphi}{=}$$
$$= a_{CX,Y}\left[(C\xi) \circ (\varphi LY) \circ (L\eta Y)\right]$$

Since $\Gamma(\varphi) = (\varphi L) \circ (L\eta)$ we have obtained that

$$\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X) \circ a_{X,Y} = a_{CX,Y} \circ \left[(C-) \circ (\Gamma(\varphi) Y)\right].$$

Let us calculate

$$\mathrm{Hom}_{\mathcal{B}}\left(Y, R^{C}\rho_X\right) \circ a_{X,Y}(\xi) = \left(R^{C}\rho_X\right) \circ a_{X,Y}(\xi) \overset{\text{defa}}{=} \left(R^{C}\rho_X\right) \circ (R\xi) \circ (\eta Y)$$
$$\overset{\text{defa}}{=} a_{CX,Y}\left({}^{C}\rho_X \circ \xi\right)$$

Therefore we get that

$$\mathrm{Hom}_{\mathcal{B}}\left(Y, R^{C}\rho_X\right) \circ a_{X,Y} = a_{CX,Y} \circ \left({}^{C}\rho_X \circ -\right)$$

Since $K_{\varphi}(Y) = \Upsilon(\varphi)(Y) = (LY, (\varphi LY) \circ (L\eta Y))$, for every $\chi \in \mathrm{Hom}_{\mathcal{A}}(LY, X)$ we have

$$\left[C(-) \circ \Gamma(\varphi) Y\right](\chi) = \Gamma(\varphi) Y = (C\chi) \circ (\varphi LY) \circ (L\eta Y) = (C\chi) \circ {}^{C}\rho_{LY}$$

and

$$\left[{}^{C}\rho_X \circ -\right](\chi) = {}^{C}\rho_X \circ \chi$$

so that

$$\left[C(-) \circ \Gamma(\varphi) Y\right](\chi) = \left[{}^{C}\rho_X \circ -\right](\chi) \text{ if and only if}$$
$$\chi \in \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(((LY), (\varphi LY) \circ (L\eta Y)), \left(X, {}^{C}\rho_X\right)\right).$$

Thus we get

$$\mathrm{Equ}_{\mathrm{Hom}_{\mathcal{A}}(LY,X)}\left({}^{C}\rho_X \circ -, C(-) \circ \Gamma(\varphi) Y\right)$$
$$= \left\{f \in \mathrm{Hom}_{\mathcal{A}}(LY, X) \mid {}^{C}\rho_X \circ f = (Cf) \circ (\Gamma(\varphi) Y)\right\}$$
$$= \left\{f \in \mathrm{Hom}_{\mathcal{A}}(LY, X) \mid {}^{C}\rho_X \circ f = (Cf) \circ (\varphi LY) \circ (L\eta Y)\right\}$$
$$= \left\{f \in \mathrm{Hom}_{\mathcal{A}}\left({}^{\mathbb{C}}U(K_{\varphi}Y), {}^{\mathbb{C}}U\left(X, {}^{C}\rho_X\right)\right) \mid {}^{C}\rho_X \circ f = (Cf) \circ {}^{C}\rho_{{}^{\mathbb{C}}U(K_{\varphi}Y)}\right\}$$
$$= \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_{\varphi}Y, \left(X, {}^{C}\rho_X\right)\right)$$

so that $\mathrm{Equ}_{\mathrm{Fun}}\left({}^{C}\rho_X \circ -, C(-) \circ \Gamma(\varphi) -\right) = \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_{\varphi}-, \left(X, {}^{C}\rho_X\right)\right)$. □

Part of the following Proposition is already in [GT], Proposition 2.3.

Proposition 4.47. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad morphism. Let $\alpha = \Theta(\varphi) = (R\varphi) \circ (\eta R)$. Then the functor $K_{\varphi} = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ has a right adjoint $D_{\varphi} : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$ if and only if,* *for every* $\left(X, {}^{C}\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^{C}\rho_X\right)$*. In this case there exists a functorial morphism* $d_\varphi : D_\varphi \to R^{\mathbb{C}}U$ *such that*

$$(D_\varphi, d_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^{C}\right).$$

and thus

$$\left[D_\varphi\left(\left(X, {}^{C}\rho_X\right)\right), d_\varphi\left(X, {}^{C}\rho_X\right)\right] = \mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R\left({}^{C}\rho_X\right)\right).$$

Proof. Assume first that, for every $\left(X, {}^{C}\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, there exists $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^{C}\rho_X\right)$. By Proposition 4.46, the isomorphism $a_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(LY, X) \to \mathrm{Hom}_{\mathcal{B}}(Y, RX)$ of the adjunction (L, R) induces an isomorphism

$$\widehat{a}_{X,Y} : \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y, \left(X, {}^{C}\rho_X\right)\right) \to \mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X), \mathrm{Hom}_{\mathcal{B}}\left(Y, R^{C}\rho_X\right)\right).$$

Let $\left(D_\varphi\left(\left(X, {}^{C}\rho_X\right)\right), d_\varphi\left(\left(X, {}^{C}\rho_X\right)\right)\right)$ denote the equalizer

$$D_\varphi\left(X, {}^{C}\rho_X\right) \xrightarrow{d_\varphi} RX \underset{\alpha X}{\overset{R^{C}\rho_X}{\rightrightarrows}} RCX$$

where $d_\varphi\left(X, {}^{C}\rho_X\right) : D_\varphi\left(\left(X, {}^{C}\rho_X\right)\right) \to RX$ is the canonical embedding. Then, by Lemma 2.17 we have

$$\begin{aligned}&\left(\mathrm{Hom}_{\mathcal{B}}\left(Y, D_\varphi\left(\left(X, {}^{C}\rho_X\right)\right)\right), \mathrm{Hom}_{\mathcal{B}}\left(Y, d_\varphi\left(\left(X, {}^{C}\rho_X\right)\right)\right)\right)\\ &= \mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X), \mathrm{Hom}_{\mathcal{B}}\left(Y, R^{C}\rho_X\right)\right).\end{aligned}$$

Thus, for every $\left(X, {}^{C}\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$ and for every $Y \in \mathcal{B}$, $a_{X,Y}$ induces an isomorphism $\widehat{a}_{X,Y} : \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y, \left(X, {}^{C}\rho_X\right)\right) \to \mathrm{Hom}_{\mathcal{B}}\left(Y, D_\varphi\left(X, {}^{\mathbb{C}}\rho_X\right)\right)$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y, \left(X, {}^{\mathbb{C}}\rho_X\right)\right) \arrow[r, dotted, "\hat{a}_{X,Y}"] \arrow[d] & \mathrm{Hom}_{\mathcal{B}}\left(Y, D_\varphi\left(X, {}^{\mathbb{C}}\rho_X\right)\right) \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(Y, d_\varphi)"] \\
\mathrm{Hom}_{\mathcal{A}}(LY, X) \arrow[r, "a_{X,Y}"] \arrow[d, shift right, "{}^{\mathbb{C}}\rho_X \circ -"'] \arrow[d, shift left, "(C-)\circ(\beta X)"] & \mathrm{Hom}_{\mathcal{B}}(Y, RX) \arrow[d, shift right, "\mathrm{Hom}_{\mathcal{B}}\left(Y, R^{\mathbb{C}}\rho_X\right)"'] \arrow[d, shift left, "\mathrm{Hom}_{\mathcal{B}}(Y, \alpha X)"] \\
\mathrm{Hom}_{\mathcal{A}}(LY, CX) \arrow[r, "a_{CX,Y}"] & \mathrm{Hom}_{\mathcal{B}}(Y, RCX)
\end{tikzcd} \tag{43}$$

i.e. (K_φ, D_φ) is an adjunction.
Conversely, assume now that the functor $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ has a right adjoint $D_\varphi : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$. Let $\widehat{\epsilon} : K_\varphi D_\varphi \to \mathrm{Id}_{{}^{\mathbb{C}}\mathcal{A}}$ be the counit of the adjunction (K_φ, D_φ) and let

$$d_\varphi = a_{{}^{\mathbb{C}}U, D_\varphi}\left({}^{\mathbb{C}}U\widehat{\epsilon}\right) = \left(R^{\mathbb{C}}U\widehat{\epsilon}\right) \circ (\eta D_\varphi) : D_\varphi \to R^{\mathbb{C}}U.$$

We will prove that

$$(D_\varphi, d_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^{C}\right).$$

First of all let us compute

$$\begin{aligned}\left(\alpha^{\mathbb{C}}U\right) \circ d_\varphi &= \left(\alpha^{\mathbb{C}}U\right) \circ \left(R^{\mathbb{C}}U\widehat{\epsilon}\right) \circ (\eta D_\varphi)\\ &= \left(R\varphi^{\mathbb{C}}U\right) \circ \left(\eta R^{\mathbb{C}}U\right) \circ \left(R^{\mathbb{C}}U\widehat{\epsilon}\right) \circ (\eta D_\varphi)\\ &\overset{\eta}{=} \left(R\varphi^{\mathbb{C}}U\right) \circ \left(RLR^{\mathbb{C}}U\widehat{\epsilon}\right) \circ (RL\eta D_\varphi) \circ (\eta D_\varphi)\end{aligned}$$

$$\overset{\varphi}{=} \left(RC^{\mathbb{C}}U\widehat{\epsilon}\right) \circ \left(R\varphi LD_{\varphi}\right) \circ \left(RL\eta D_{\varphi}\right) \circ \left(\eta D_{\varphi}\right)$$

and also

$$\begin{aligned}\left(R^{\mathbb{C}}U\gamma^{C}\right) \circ d_{\varphi} &= \left(R^{\mathbb{C}}U\gamma^{C}\right) \circ \left(R^{\mathbb{C}}U\widehat{\epsilon}\right) \circ \left(\eta D_{\varphi}\right) \\ &\overset{\widehat{\epsilon}\mathrm{morph}^{\mathbb{C}}\mathcal{A}}{=} \left(RC^{\mathbb{C}}U\widehat{\epsilon}\right) \circ \left(R^{\mathbb{C}}U\gamma^{C}K_{\varphi}D_{\varphi}\right) \circ \left(\eta D_{\varphi}\right) \\ &\overset{\mathrm{def}K_{\varphi}}{=} \left(RC^{\mathbb{C}}U\widehat{\epsilon}\right) \circ \left(R\varphi LD_{\varphi}\right) \circ \left(RL\eta D_{\varphi}\right) \circ \left(\eta D_{\varphi}\right)\end{aligned}$$

so that

$$\left(\alpha^{\mathbb{C}}U\right) \circ d_{\varphi} = \left(R^{\mathbb{C}}U\gamma^{C}\right) \circ d_{\varphi}.$$

Now, we will prove that the following diagram is commutative

$$\begin{tikzcd} \mathrm{Hom}_{^{\mathbb{C}}\mathcal{A}}\left(K_{\varphi}Y, \left(X, {}^{C}\rho_{X}\right)\right) \arrow[r, dotted, "\widehat{a}_{\left(X, {}^{C}\rho_{X}\right), Y}"] \arrow[d, "^{\mathbb{C}}U"'] & \mathrm{Hom}_{\mathcal{B}}\left(Y, D_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right) \arrow[d, "\mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right)"] \\ \mathrm{Hom}_{\mathcal{A}}\left(LY, X\right) \arrow[r, "a_{X,Y}"] & \mathrm{Hom}_{\mathcal{B}}\left(Y, RX\right). \end{tikzcd}$$

In fact, for every $\zeta \in \mathrm{Hom}_{^{\mathbb{C}}\mathcal{A}}\left(K_{\varphi}Y, \left(X, {}^{C}\rho_{X}\right)\right)$, we have

$$\begin{aligned}\left[\mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right) \circ \widehat{a}_{\left(X, {}^{C}\rho_{X}\right), Y}\right](\zeta) &\overset{\mathrm{def}\widehat{a}}{=} \mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right)\left[\left(D_{\varphi}\zeta\right) \circ \left(\widehat{\eta}Y\right)\right] \\ &= \left(d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right) \circ \left(D_{\varphi}\zeta\right) \circ \left(\widehat{\eta}Y\right) \\ &\overset{\mathrm{def}d_{\varphi}}{=} \left(R^{\mathbb{C}}U\widehat{\epsilon}\left(X, {}^{C}\rho_{X}\right)\right) \circ \left(\eta D_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right) \circ \left(D_{\varphi}\zeta\right) \circ \left(\widehat{\eta}Y\right) \\ &\overset{\eta}{=} \left(R^{\mathbb{C}}U\widehat{\epsilon}\left(X, {}^{C}\rho_{X}\right)\right) \circ \left(RLD_{\varphi}\zeta\right) \circ \left(RL\widehat{\eta}Y\right) \circ \left(\eta Y\right) \\ &\overset{\mathrm{def}K_{\varphi}}{=} \left(R^{\mathbb{C}}U\widehat{\epsilon}\left(X, {}^{C}\rho_{X}\right)\right) \circ \left(R^{\mathbb{C}}UK_{\varphi}D_{\varphi}\zeta\right) \circ \left(R^{\mathbb{C}}UK_{\varphi}\widehat{\eta}Y\right) \circ \left(\eta Y\right) \\ &\overset{\widehat{\epsilon}}{=} \left(R^{\mathbb{C}}U\zeta\right) \circ \left(R^{\mathbb{C}}U\widehat{\epsilon}K_{\varphi}Y\right) \circ \left(R^{\mathbb{C}}UK_{\varphi}\widehat{\eta}Y\right) \circ \left(\eta Y\right) \overset{\left(K_{\varphi}, D_{\varphi}\right)}{=} \left(R^{\mathbb{C}}U\zeta\right) \circ \left(\eta Y\right)\end{aligned}$$

and on the other hand

$$\left(a_{X,Y} \circ {}^{\mathbb{C}}U\right)(\zeta) = a_{X,Y}\left({}^{\mathbb{C}}U\zeta\right) \overset{\mathrm{def}a}{=} \left(R^{\mathbb{C}}U\zeta\right) \circ \left(\eta Y\right)$$

so that, for every $\left(X, {}^{\mathbb{C}}\rho_{X}\right) \in {}^{\mathbb{C}}\mathcal{A}$ we have

$$\mathrm{Hom}_{\mathcal{B}}\left(-, d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right) \circ \widehat{a}_{\left(X, {}^{C}\rho_{X}\right), -} = a_{X,-} \circ {}^{\mathbb{C}}U.$$

Since $a_{X,-}$ and $\widehat{a}_{\left(X, {}^{C}\rho_{X}\right), -}$ are isomorphisms, we deduce that $\mathrm{Hom}_{\mathcal{B}}\left(-, d_{\varphi}\left(X, {}^{C}\rho_{X}\right)\right)$ is mono. Applying the commutativity of this diagram in the particular case of $\left(X, {}^{C}\rho_{X}\right) = K_{\varphi}Y$, we get that

$$\begin{aligned}\left(d_{\varphi}K_{\varphi}Y\right) \circ \left(\widehat{\eta}Y\right) &= \mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}K_{\varphi}Y\right)\left(\widehat{\eta}Y\right) \\ &= \mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}K_{\varphi}Y\right)\left(\widehat{a}_{K_{\varphi}Y,Y}\left(\mathrm{Id}_{K_{\varphi}Y}\right)\right) \\ &= \left[\mathrm{Hom}_{\mathcal{B}}\left(Y, d_{\varphi}K_{\varphi}Y\right) \circ \widehat{a}_{K_{\varphi}Y,Y}\right]\left(\mathrm{Id}_{K_{\varphi}Y}\right) \\ &= \left[a_{^{\mathbb{C}}UK_{\varphi}Y,Y} \circ {}^{\mathbb{C}}U\right]\left(\mathrm{Id}_{K_{\varphi}Y}\right) = \left(a_{LY,Y}\right)\left({}^{\mathbb{C}}U\mathrm{Id}_{K_{\varphi}Y}\right) \\ &= \left(a_{LY,Y}\right)\left(\mathrm{Id}_{^{\mathbb{C}}UK_{\varphi}Y}\right) = a_{LY,Y}\left(\mathrm{Id}_{LY}\right) = \eta Y\end{aligned}$$

i.e.

$$(d_\varphi K_\varphi Y)\circ(\widehat{\eta}Y)=\eta Y. \tag{44}$$

Now, we have to prove the universal property of the equalizer. Let $Z\in\mathcal{B}$ and let $\zeta : Z\to RX$ be a morphism such that $(\alpha X)\circ\zeta=\left(R^C\rho_X\right)\circ\zeta$, i.e.

$$(R\varphi X)\circ(\eta RX)\circ\zeta=\left(R^C\rho_X\right)\circ\zeta.$$

This means $\zeta\in\mathrm{Equ}_{\mathrm{Sets}}\left(\mathrm{Hom}_{\mathcal{B}}(Y,\alpha X),\mathrm{Hom}_{\mathcal{B}}\left(Y,R^C\rho_X\right)\right)\simeq\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y,\left(X,{}^C\rho_X\right)\right)$ by Proposition 4.46. Then,

$$\begin{aligned}a_{X,Z}^{-1}(\zeta)=(\epsilon X)\circ(L\zeta)&\in\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left((LZ,(\varphi LZ)\circ(L\eta Z)),\left(X,{}^C\rho_X\right)\right)\\&=\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Z,\left(X,{}^C\rho_X\right)\right).\end{aligned}$$

We want to prove that there exists $\zeta' : Z\to D_\varphi\left(X,{}^{\mathbb{C}}\rho_X\right)$ such that $d_\varphi\left(X,{}^{\mathbb{C}}\rho_X\right)\circ\zeta'=\zeta$. By hypothesis the map

$$\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Y,\left(X,{}^C\rho_X\right)\right)\xrightarrow{\widehat{a}_{(X,{}^C\rho_X),Y}}\mathrm{Hom}_{\mathcal{B}}\left(Y,D_\varphi\left(X,{}^C\rho_X\right)\right)$$

is bijective. Hence, given $(\epsilon X)\circ(L\zeta)\in\mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi Z,\left(X,{}^C\rho_X\right)\right)$,
$\widehat{a}_{(X,{}^C\rho_X),Z}((\epsilon X)\circ(L\zeta))=(D_\varphi\epsilon X)\circ(D_\varphi L\zeta)\circ(\widehat{\eta}Z)\in\mathrm{Hom}_{\mathcal{B}}\left(Z,D_\varphi\left(X,{}^C\rho_X\right)\right)$. We want to prove that

$$\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ(D_\varphi\epsilon X)\circ(D_\varphi L\zeta)\circ(\widehat{\eta}Z)=\zeta.$$

We compute

$$\begin{aligned}\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ(D_\varphi\epsilon)\circ(D_\varphi L\zeta)\circ(\widehat{\eta}Z)&\overset{d_\varphi}{=}(R\epsilon X)\circ(RL\zeta)\circ(d_\varphi K_\varphi Z)\circ(\widehat{\eta}Z)\\&\overset{(44)}{=}(R\epsilon X)\circ(RL\zeta)\circ(\eta Z)\overset{\eta}{=}(R\epsilon X)\circ(\eta RX)\circ\zeta\overset{(L,R)}{=}\zeta.\end{aligned}$$

Let us denote by $\zeta'=(D_\varphi\epsilon X)\circ(D_\varphi L\zeta)\circ(\widehat{\eta}Z)$ the morphism such that $\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ\zeta'=\zeta$. We have to prove that ζ' is unique with respect to this property. Let $\zeta'' : Z\to D_\varphi\left(X,{}^C\rho_X\right)$ be another morphism in $\mathcal{B}$ such that $\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ\zeta''=\zeta$. Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{B}}\left(Z,d_\varphi\left(X,{}^C\rho_X\right)\right)(\zeta'')&=\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ\zeta''=\zeta\\&=\left(d_\varphi\left(X,{}^C\rho_X\right)\right)\circ\zeta'=\mathrm{Hom}_{\mathcal{B}}\left(Z,d_\varphi\left(X,{}^C\rho_X\right)\right)(\zeta')\end{aligned}$$

and since $\mathrm{Hom}_{\mathcal{B}}\left(Z,d_\varphi\left(X,{}^C\rho_X\right)\right)$ is mono we deduce that

$$\zeta''=\zeta'.$$

$\square$

COROLLARY 4.48. *Let (L,R) be an adjunction where $L:\mathcal{B}\to\mathcal{A}$ and $R:\mathcal{A}\to\mathcal{B}$. Let $\alpha=\Theta(\mathrm{Id}_{\mathbb{LR}})=\eta R$. Then the functor $K=\Upsilon(\mathrm{Id}_{\mathbb{LR}}):\mathcal{B}\to{}^{\mathbb{LR}}\mathcal{A}$ has a right adjoint $D:{}^{\mathbb{LR}}\mathcal{A}\to\mathcal{B}$ if and only if, for every $\left(X,{}^{\mathbb{LR}}\rho_X\right)\in{}^{\mathbb{LR}}\mathcal{A}$, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\eta RX,R^{LR}\rho_X\right)$. *In this case there exists a functorial morphism $d:D\to R^{\mathbb{LR}}U$ such that*

$$(D,d)=\mathrm{Equ}_{\mathrm{Fun}}\left(\eta R^{\mathbb{LR}}U,R^{\mathbb{LR}}U\gamma^{LR}\right).$$

and thus

$$\left[D\left(\left(X,{}^{LR}\rho_X\right)\right),d\left(X,{}^{LR}\rho_X\right)\right]=\mathrm{Equ}_{\mathcal{B}}\left(\eta RX,R\left({}^{LR}\rho_X\right)\right).$$

Proof. We can apply Proposition 4.47 with "φ" $=\mathrm{Id}_{\mathbb{LR}}$. $\square$

REMARK 4.49 ([GT]). In the setting of Proposition 4.47, for every $Y\in\mathcal{B}$, we note that the unit of the adjunction (K_φ,D_φ) is given by

$$\widehat{\eta}Y=\widehat{a}_{K_\varphi Y,Y}\left(\mathrm{Id}_{K_\varphi Y}\right):Y\to D_\varphi K_\varphi\left(Y\right).$$

We will consider the diagram (43) in the particular case of $\left(X,{}^{C}\rho_X\right)=K_\varphi Y$. Note that since $K_\varphi Y=(LY,(\varphi LY)\circ(L\eta Y))=(LY,\beta Y)$ we have

$$\left(D_\varphi K_\varphi\left(Y\right),d_\varphi K_\varphi\left(Y\right)\right)=\left(D_\varphi\left(\left(LY,\beta Y\right)\right),d_\varphi K_\varphi\left(Y\right)\right)=\mathrm{Equ}_{\mathcal{B}}\left(\alpha LY,R\beta Y\right)$$

i.e.

$$\left(D_\varphi K_\varphi\left(Y\right),d_\varphi K_\varphi\left(Y\right)\right)=\mathrm{Equ}_{\mathcal{B}}\left(\alpha LY,R\beta Y\right) \tag{45}$$

where $\beta=\Gamma\left(\varphi\right)=\left(\varphi L\right)\circ\left(L\eta\right)$. We compute

$$\begin{aligned}\left(d_\varphi K_\varphi Y\right)\circ\left(\widehat{\eta}Y\right)&=\mathrm{Hom}_{\mathcal{B}}\left(Y,d_\varphi K_\varphi Y\right)\left(\widehat{\eta}Y\right)=\mathrm{Hom}_{\mathcal{B}}\left(Y,d_\varphi K_\varphi Y\right)\left(\widehat{a}_{K_\varphi Y,Y}\left(\mathrm{Id}_{K_\varphi Y}\right)\right)\\&=\left[\mathrm{Hom}_{\mathcal{B}}\left(Y,d_\varphi K_\varphi Y\right)\circ\widehat{a}_{K_\varphi Y,Y}\right]\left(\mathrm{Id}_{K_\varphi Y}\right)\overset{(43)}{=}a_{K_\varphi Y,Y}{}^{\mathbb{C}}U\left(\mathrm{Id}_{K_\varphi Y}\right)\\&=a_{K_\varphi Y,Y}\left(\mathrm{Id}_{{}^{\mathbb{C}}UK_\varphi Y}\right)=a_{K_\varphi Y,Y}\left(\mathrm{Id}_{RY}\right)=\eta Y\end{aligned}$$

so that

$$\left(d_\varphi K_\varphi Y\right)\circ\left(\widehat{\eta}Y\right)=\eta Y. \tag{46}$$

On the other hand, for every $\left(X,{}^{C}\rho_X\right)\in{}^{\mathbb{C}}\mathcal{A}$, the counit of the adjunction (K_φ,D_φ) is given by

$$\widehat{\epsilon}\left(X,{}^{C}\rho_X\right)=\widehat{a}^{-1}_{X,D_\varphi(X,{}^{C}\rho_X)}\left(\mathrm{Id}_{D_\varphi(X,{}^{C}\rho_X)}\right):K_\varphi D_\varphi\left(\left(X,{}^{C}\rho_X\right)\right)\to\left(X,{}^{C}\rho_X\right).$$

Then we have that

$$\widehat{a}_{X,D_\varphi(X,{}^{C}\rho_X)}\left(\widehat{\epsilon}\left(X,{}^{C}\rho_X\right)\right)=\mathrm{Id}_{D_\varphi(X,{}^{C}\rho_X)}.$$

By commutativity of the diagram (43), we deduce that

$$\begin{aligned}d_\varphi\left(X,{}^{C}\rho_X\right)&=d_\varphi\left(X,{}^{C}\rho_X\right)\circ\left(\widehat{a}_{X,D_\varphi(X,{}^{C}\rho_X)}\left(\widehat{\epsilon}\left(X,{}^{C}\rho_X\right)\right)\right)\\&=a_{X,D_\varphi(X,{}^{C}\rho_X)}\left({}^{\mathbb{C}}U\widehat{\epsilon}\left(X,{}^{C}\rho_X\right)\right).\end{aligned}$$

Thus we obtain that

$${}^{\mathbb{C}}U\widehat{\epsilon}\left(X,{}^{C}\rho_X\right)=a^{-1}_{X,D_\varphi(X,{}^{C}\rho_X)}\left(d_\varphi\left(X,{}^{C}\rho_X\right)\right)=\left(\epsilon X\right)\circ\left(Ld_\varphi\left(X,{}^{C}\rho_X\right)\right). \tag{47}$$

Observe that, for every $X\in\mathcal{A}$, we have that ${}^{\mathbb{C}}F\left(X\right)=\left(CX,\Delta^{C}X\right)\in{}^{\mathbb{C}}\mathcal{A}$. Moreover

$$\begin{aligned}\left(D_\varphi{}^{\mathbb{C}}F\left(X\right),d_\varphi{}^{\mathbb{C}}F\left(X\right)\right)&=\left(D_\varphi\left(CX,\Delta^{C}X\right),d_\varphi\left(CX,\Delta^{C}X\right)\right)=\\&=\mathrm{Equ}_{\mathcal{B}}\left(\alpha CX,R\Delta^{C}X\right)\overset{(4.15)}{=}\left(RX,\alpha X\right)\end{aligned}$$

so that we get

$$\left(D_\varphi{}^{\mathbb{C}}F,d_\varphi{}^{\mathbb{C}}F\right)=\left(R,\alpha\right). \tag{48}$$

In particular

$$d_{\varphi}\left(CX,\Delta^{C}X\right)=\alpha X. \tag{49}$$

COROLLARY 4.50. *In the setting of Proposition 4.47, assume that, for every* $\left(X,{}^{C}\rho_{X}\right)\in{}^{\mathbb{C}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X,R^{C}\rho_{X}\right)$*. Then for every* $X\in\mathcal{A}$ *we have*

$$^{\mathbb{C}}U\widehat{\epsilon}\left(CX,\Delta^{C}X\right)=\varphi X$$

and hence

$$^{\mathbb{C}}U\widehat{\epsilon}^{\mathbb{C}}F=\varphi$$

where $\widehat{\epsilon}$ *is the counit of the adjunction* $(K_{\varphi},D_{\varphi})$.

Proof. Let us calculate

$$\begin{aligned}
{}^{\mathbb{C}}U\widehat{\epsilon}\left(CX,\Delta^{C}X\right) &\overset{(47)}{=}(\epsilon CX)\circ\left(Ld_{\varphi}\left(CX,\Delta^{C}X\right)\right)\\
&\overset{(48)}{=}(\epsilon CX)\circ(L\alpha X)=\Xi(\alpha)(X)=\varphi X.
\end{aligned}$$

□

COROLLARY 4.51. *In the setting of Proposition 4.47, assume that, for every* $\left(X,{}^{C}\rho_{X}\right)\in{}^{\mathbb{C}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X,R^{C}\rho_{X}\right)$*. Then, the functor* D_{φ} *is full and faithful if and only if* $\widehat{\epsilon}$ *is a functorial isomorphism.*

Proof. By Proposition 4.47, $(K_{\varphi},D_{\varphi})$ is an adjunction with counit $\widehat{\epsilon}:K_{\varphi}D_{\varphi}\to{}^{\mathbb{C}}\mathcal{A}$. Then we can apply Proposition 2.32. □

LEMMA 4.52 ([GT, Lemma 2.5]). *In the setting of Proposition 4.47, assume that, for every* $\left(X,{}^{C}\rho_{X}\right)\in{}^{\mathbb{C}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X,R^{C}\rho_{X}\right)$*. Then, for every* $\left(X,{}^{C}\rho_{X}\right)\in{}^{\mathbb{C}}\mathcal{A}$ *the following diagram*

$$\begin{tikzcd}
LD_{\varphi}\left(X,{}^{\mathbb{C}}\rho_{X}\right) \arrow[r, "{}^{\mathbb{C}}U\widehat{\epsilon}\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] \arrow[d, "Ld_{\varphi}\left(X,{}^{\mathbb{C}}\rho_{X}\right)"'] & {}^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right) \arrow[d, "{}^{\mathbb{C}}U^{C}\gamma\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] \\
LR^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right) \arrow[r, "\varphi^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] \arrow[d, shift right, "L\alpha^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right)"'] \arrow[d, shift left, "LR^{\mathbb{C}}U^{\mathbb{C}}\gamma\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] & C^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right) \arrow[d, shift right, "\Delta^{C\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right)"'] \arrow[d, shift left, "C^{\mathbb{C}}U^{\mathbb{C}}\gamma\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] \\
LRC^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right) \arrow[r, "\varphi C^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right)"] & CC^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_{X}\right)
\end{tikzcd}$$

serially commutes. Therefore we get

$$\left({}^{\mathbb{C}}U\gamma^{C}\right)\circ\left({}^{\mathbb{C}}U\widehat{\epsilon}\right)=\left(\varphi^{\mathbb{C}}U\right)\circ\left(Ld_{\varphi}\right)\quad\textit{and}\quad\left(\Delta^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)=\left(\varphi C^{\mathbb{C}}U\right)\circ\left(L\alpha^{\mathbb{C}}U\right).$$

Proof. Let us compute

$$\begin{aligned}
{}^{C}\rho_{X}\circ{}^{\mathbb{C}}U\widehat{\epsilon}\left(X,{}^{C}\rho_{X}\right) &\overset{(47)}{=}{}^{C}\rho_{X}\circ(\epsilon X)\circ\left(Ld_{\varphi}\left(X,{}^{C}\rho_{X}\right)\right)\\
&\overset{\varphi\text{morphcomonads}}{=}{}^{C}\rho_{X}\circ\left(\varepsilon^{C}X\right)\circ(\varphi X)\circ\left(Ld_{\varphi}\left(X,{}^{C}\rho_{X}\right)\right)\\
&\overset{\varepsilon^{C}}{=}\left(\varepsilon^{C}CX\right)\circ\left(C^{C}\rho_{X}\right)\circ(\varphi X)\circ\left(Ld_{\varphi}\left(X,{}^{C}\rho_{X}\right)\right)\\
&\overset{\varphi}{=}\left(\varepsilon^{C}CX\right)\circ(\varphi CX)\circ\left(LR^{C}\rho_{X}\right)\circ\left(Ld_{\varphi}\left(X,{}^{C}\rho_{X}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{def}d_\varphi}{=} \left(\varepsilon^C CX\right) \circ (\varphi CX) \circ (L\alpha X) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right) \\
&\overset{\text{def}\alpha}{=} \left(\varepsilon^C CX\right) \circ (\varphi CX) \circ (LR\varphi X) \circ (L\eta RX) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right) \\
&\overset{\varphi}{=} \left(\varepsilon^C CX\right) \circ (\varphi\varphi X) \circ (L\eta RX) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right) \\
&\overset{\varphi\text{morphcomonads}}{=} \left(\varepsilon^C CX\right) \circ \left(\Delta^C X\right) \circ (\varphi X) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right) \\
&\overset{C\text{comonad}}{=} (\varphi X) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right)
\end{aligned}$$

so that we deduce that

$${}^C\rho_X \circ \left({}^{\mathbb{C}}U\widehat{\epsilon}\left(X, {}^C\rho_X\right)\right) = (\varphi X) \circ \left(Ld_\varphi \left(X, {}^C\rho_X\right)\right)$$

and thus

$$\left({}^{\mathbb{C}}U\gamma^C\right) \circ \left({}^{\mathbb{C}}U\widehat{\epsilon}\right) = \left(\varphi{}^{\mathbb{C}}U\right) \circ Ld_\varphi.$$

Let us calculate

$$\begin{aligned}
\left(\Delta^{C\mathbb{C}}U\right) \circ \left(\varphi{}^{\mathbb{C}}U\right) &\overset{\varphi\text{comonadmorph}}{=} \left(\varphi\varphi{}^{\mathbb{C}}U\right) \circ \left(L\eta R^{\mathbb{C}}U\right) \\
= \left(\varphi C^{\mathbb{C}}U\right) \circ \left(LR\varphi{}^{\mathbb{C}}U\right) \circ \left(L\eta R^{\mathbb{C}}U\right) &\overset{\text{def}\alpha}{=} \left(\varphi C^{\mathbb{C}}U\right) \circ \left(L\alpha{}^{\mathbb{C}}U\right).
\end{aligned}$$

□

THEOREM 4.53 ([GT] Theorem 2.6). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad morphism. Let $\alpha = \Theta(\varphi) = (R\varphi) \circ (\eta R)$ and assume that, for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, there exists $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^C\rho_X\right)$. Then we can consider the functor $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ and its right adjoint $D_\varphi : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$. D_φ is full and faithful if and only if*

1) *L preserves the equalizer*

$$(D_\varphi, d_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha{}^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right).$$

2) *$\varphi : \mathbb{LR} \to \mathbb{C}$ is a comonad isomorphism.*

Proof. Recall that, by Corollary 4.50,

$$
{}^{\mathbb{C}}U\widehat{\epsilon}{}^{\mathbb{C}}F = \varphi. \tag{50}
$$

By Corollary 4.51, D_φ is full and faithful if and only if $\widehat{\epsilon}$ is a functorial isomorphism.

Let us assume that $\widehat{\epsilon}$ is a functorial isomorphism, hence φ is an isomorphism too. Recall that, by Lemma4.52, we have

$$
\left({}^{\mathbb{C}}U\gamma^C\right) \circ \left({}^{\mathbb{C}}U\widehat{\epsilon}\right) = \left(\varphi{}^{\mathbb{C}}U\right) \circ (Ld_\varphi) \tag{51}
$$

so that

$$
{}^{\mathbb{C}}U\gamma^C = \left(\varphi{}^{\mathbb{C}}U\right) \circ (Ld_\varphi) \circ \left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right). \tag{52}
$$

Let us consider the diagram

$$LD_\varphi \xrightarrow{d_\varphi} LR^{\mathbb{C}}U \underset{L\alpha^{\mathbb{C}}U}{\overset{LR^{\mathbb{C}}U\gamma^C}{\rightrightarrows}} LRC^{\mathbb{C}}U$$

We have to prove that $(LD_\varphi, Ld_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(L\alpha^{\mathbb{C}}U, LR^{\mathbb{C}}U\gamma^C\right)$. Since L is a functor, we clearly have $\left(L\alpha^{\mathbb{C}}U\right)\circ(Ld_\varphi) = \left(LR^{\mathbb{C}}U\gamma^C\right)\circ(Ld_\varphi)$. Let $Z : \mathcal{Z} \to {}^{\mathbb{C}}\mathcal{A}$ be a functor and let $\xi : Z \to LR^{\mathbb{C}}U$ be a functorial morphism such that

$$\left(L\alpha^{\mathbb{C}}U\right)\circ\xi = \left(LR^{\mathbb{C}}U\gamma^C\right)\circ\xi.$$

Recall that $\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left({}^{\mathbb{C}}U\gamma^C\right) = {}^{\mathbb{C}}U$ and $\left({}^{\mathbb{C}}F\varepsilon^C\right)\circ\left(\gamma^{C\mathbb{C}}F\right) = {}^{\mathbb{C}}F$. We compute

$$\begin{gathered}
\left(\varphi^{\mathbb{C}}U\right)\circ\xi = \mathrm{Id}_{C^{\mathbb{C}}U}\circ\varphi^{\mathbb{C}}U\circ\xi \overset{C\text{comonad}}{=} \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(\Delta^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi = \\
\overset{\varphi\text{comonadmorp}}{=} \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(\varphi\varphi^{\mathbb{C}}U\right)\circ\left(L\eta R^{\mathbb{C}}U\right)\circ\xi = \\
= \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(\varphi C^{\mathbb{C}}U\right)\circ\left(LR\varphi^{\mathbb{C}}U\right)\circ\left(L\eta R^{\mathbb{C}}U\right)\circ\xi = \\
= \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(\varphi C^{\mathbb{C}}U\right)\circ\left(L\alpha^{\mathbb{C}}U\right)\circ\xi = \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(\varphi C^{\mathbb{C}}U\right)\circ\left(LR^{\mathbb{C}}U\gamma^C\right)\circ\xi = \\
\overset{\varphi}{=} \left(\varepsilon^C C^{\mathbb{C}}U\right)\circ\left(C^{\mathbb{C}}U\gamma^C\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi \overset{\varepsilon^C}{=} \left({}^{\mathbb{C}}U\gamma^C\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi = \\
\overset{(52)}{=} \left(\varphi^{\mathbb{C}}U\right)\circ(Ld_\varphi)\circ\left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi.
\end{gathered}$$

Since φ is iso, we get

$$\xi = (Ld_\varphi)\circ\left[\left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi\right].$$

Let now $w : Z \to LD_\varphi$ be a functorial morphism such that

$$\xi = (Ld_\varphi)\circ w.$$

We compute

$$\begin{gathered}
\left({}^{\mathbb{C}}U\gamma^C\right)\circ\left({}^{\mathbb{C}}U\widehat{\epsilon}\right)\circ\left[\left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi\right] = \\
\overset{(51)}{=} \left(\varphi^{\mathbb{C}}U\right)\circ(Ld_\varphi)\circ\left[\left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi\right] = \\
= \left(\varphi^{\mathbb{C}}U\right)\circ\xi = \left(\varphi^{\mathbb{C}}U\right)\circ(Ld_\varphi)\circ w \overset{(51)}{=} \left({}^{\mathbb{C}}U\gamma^C\right)\circ\left({}^{\mathbb{C}}U\widehat{\epsilon}\right)\circ w
\end{gathered}$$

and since ${}^{\mathbb{C}}U\gamma^C$ is a monomorphism (since it is an equalizer) and $\widehat{\epsilon}$ is an isomorphism we obtain that

$$\left({}^{\mathbb{C}}U\widehat{\epsilon}^{-1}\right)\circ\left(\varepsilon^{C\mathbb{C}}U\right)\circ\left(\varphi^{\mathbb{C}}U\right)\circ\xi = w.$$

Conversely, assume that 1) and 2) hold. Then φ is a functorial isomorphism. Consider the diagram

$$\begin{tikzcd}
LD_\varphi\left(X, {}^{\mathbb{C}}\rho_X\right) \arrow[r, "{}^{\mathbb{C}}U\widehat{\epsilon}\left(X,{}^{\mathbb{C}}\rho_X\right)"] \arrow[d, "Ld_\varphi\left(X,{}^{\mathbb{C}}\rho_X\right)"'] & {}^{\mathbb{C}}U\left(X, {}^{\mathbb{C}}\rho_X\right) \arrow[d, "{}^{\mathbb{C}}U^C\gamma\left(X,{}^{\mathbb{C}}\rho_X\right)"] \\
LR^{\mathbb{C}}U\left(X, {}^{\mathbb{C}}\rho_X\right) \arrow[r, "\varphi^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_X\right)"] \arrow[d, shift right, "L\alpha^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_X\right)"'] \arrow[d, shift left, "LR^{\mathbb{C}}U^{\mathbb{C}}\gamma\left(X,{}^{\mathbb{C}}\rho_X\right)"] & C^{\mathbb{C}}U\left(X, {}^{\mathbb{C}}\rho_X\right) \arrow[d, shift right, "\Delta^{C\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_X\right)"'] \arrow[d, shift left, "C^{\mathbb{C}}U^{\mathbb{C}}\gamma\left(X,{}^{\mathbb{C}}\rho_X\right)"] \\
LRC^{\mathbb{C}}U\left(X, {}^{\mathbb{C}}\rho_X\right) \arrow[r, "\varphi C^{\mathbb{C}}U\left(X,{}^{\mathbb{C}}\rho_X\right)"] & CC^{\mathbb{C}}U\left(X, {}^{\mathbb{C}}\rho_X\right)
\end{tikzcd}$$

of Lemma 4.52 where the last row is always an equalizer (see Proposition 4.13) and the first row is also an equalizer by the assumption 1). Then we can apply Lemma

2.15 and hence we get that ${}^{\mathbb{C}}U\widehat{\epsilon}$ is a functorial isomorphism. Since,by Proposition 4.17, ${}^{\mathbb{C}}U$ reflects isomorphism we deduce that $\widehat{\epsilon}$ is a functorial isomorphism. $\square$

COROLLARY 4.54. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Let $\alpha = \Theta(\mathrm{Id}_{\mathbb{LR}}) = \eta R$ and assume that, for every $\left(X, {}^{LR}\rho_X\right) \in {}^{\mathbb{LR}}\mathcal{A}$, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\eta RX, R^{LR}\rho_X\right)$*. Then we can consider the functor $K = \Upsilon(\mathrm{Id}_{\mathbb{LR}}) : \mathcal{B} \to {}^{\mathbb{LR}}\mathcal{A}$ and its right adjoint $D : {}^{\mathbb{LR}}\mathcal{A} \to \mathcal{B}$. D is full and faithful if and only if L preserves the equalizer*

$$(D, d) = \mathrm{Equ}_{\mathrm{Fun}}\left(\eta R^{\mathbb{LR}}U, R^{\mathbb{LR}}U\gamma^{LR}\right).$$

Proof. We can apply Theorem 4.53 with "φ" $= \mathrm{Id}_{\mathbb{LR}}$. $\square$

THEOREM 4.55 ([GT, Theorem 2.7]). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad morphism. Let $\alpha = \Theta(\varphi) = (R\varphi) \circ (\eta R)$ and assume that, for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^C\rho_X\right)$*. Then we can consider the functor $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ and its right adjoint $D_\varphi : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$. The functor K_φ is an equivalence of categories if and only if*

1) *L preserves the equalizer*

$$(D_\varphi, d_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right)$$

2) *L reflects isomorphisms and*
3) *$\varphi : \mathbb{LR} \to \mathbb{C}$ is a comonad isomorphism.*

Proof. If K_φ is an equivalence then, by Lemma 2.33, D_φ is an equivalence of categories so that, by Theorem 4.53, 1) and 3) hold. By Proposition 4.17, the functor ${}^{\mathbb{C}}U$ reflects isomorphisms. Since $L = {}^{\mathbb{C}}UK_\varphi$ we get that 2) holds.

Conversely assume that 1), 2) and 3) hold. By Theorem 4.53 , D_φ is full and faithful and hence by Corollary 4.51 $\widehat{\epsilon}$ is a functorial isomorphism. Let us prove that $\widehat{\eta}$ is an isomorphism as well. Since L reflects isomorphisms, it is enough to prove that $L\widehat{\eta}$ is an isomorphism. As observed in Remark 4.49, by (44), $\widehat{\eta}Y$ is the unique morphism such that

$$(d_\varphi K_\varphi Y) \circ (\widehat{\eta}Y) = \eta Y.$$

Hence we get

$$(Ld_\varphi K_\varphi Y) \circ (L\widehat{\eta}Y) = L\eta Y$$

so that

$$(\epsilon LY) \circ (Ld_\varphi K_\varphi Y) \circ (L\widehat{\eta}Y) = (\epsilon LY) \circ (L\eta Y) = LY.$$

We now want to prove that $(\epsilon LY) \circ (Ld_\varphi K_\varphi Y)$ is also a right inverse for $L\widehat{\eta}Y$. We compute

$$\begin{aligned}(Ld_\varphi K_\varphi Y) \circ (L\widehat{\eta}Y) \circ (\epsilon LY) \circ (Ld_\varphi K_\varphi Y) &\overset{(44)}{=} (L\eta Y) \circ (\epsilon LY) \circ (Ld_\varphi K_\varphi Y) \\ &\overset{(L,R)\mathrm{adj}}{=} (Ld_\varphi K_\varphi Y).\end{aligned}$$

Since L preserves the equalizer

$$(D_\varphi, d_\varphi) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right)$$

we have that $Ld_\varphi K_\varphi Y$ is mono and hence we obtain

$$(L\widehat{\eta}Y)\circ(\epsilon LY)\circ(Ld_\varphi K_\varphi Y)=LD_\varphi K_\varphi Y$$

so that $L\widehat{\eta}$ is a functorial isomorphism. □

Definition 4.56. Let $\left(L,{}^{C}\rho_L\right)$ be a left comodule functor for a comonad $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ such that L has a right adjoint R. Then we can consider a canonical comonad morphism

$$\mathrm{can}:=(C\epsilon)\circ\left({}^{C}\rho_L R\right):\mathbb{LR}\to\mathbb{C}$$

where ϵ denotes the counit of the adjunction (L,R). A *left* $\mathbb{C}$*-Galois functor* is a left $\mathbb{C}$-comodule functor $\left(L,{}^{C}\rho_L\right)$ with a right adjoint R such that can is a comonad isomorphism.

Corollary 4.57. *Let* $\left(L,{}^{C}\rho_L\right)$ *be a left* C*-Galois comodule functor such that* L *preserves equalizers,* L *reflects isomorphisms and let* $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ *be a comonad on* $\mathcal{A}$*. Assume that, for every* $\left(X,{}^{C}\rho_X\right)\in{}^{\mathbb{C}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X,R^{C}\rho_X\right)$ *where* $\alpha=(R\mathrm{can})\circ(\eta R)$ *where* R *is the right adjoint of* L *and* η *is the unit of the adjunction. Then we can consider the functor* $K_{\mathrm{can}}:\mathcal{B}\to{}^{\mathbb{C}}\mathcal{A}$*. Then the functor* K_{can} *is an equivalence of categories.*

Proof. We can apply Theorem 4.55 to the case $\varphi=\mathrm{can}$. □

Theorem 4.58 (Beck's Theorem). *Let* (L,R) *be an adjunction where* $L:\mathcal{B}\to\mathcal{A}$ *and* $R:\mathcal{A}\to\mathcal{B}$*. Let* $\alpha=\Theta\left(\mathrm{Id}_{LR}\right)=\eta R$ *and assume that, for every* $\left(X,{}^{LR}\rho_X\right)\in{}^{\mathbb{LR}}\mathcal{A}$*, there exists* $\mathrm{Equ}_{\mathcal{B}}\left(\eta RX,R^{LR}\rho_X\right)$*. Then we can consider the functor* $K=\Upsilon\left(\mathrm{Id}_{LR}\right):\mathcal{B}\to{}^{\mathbb{LR}}\mathcal{A}$ *and its right adjoint* $D:{}^{\mathbb{LR}}\mathcal{A}\to\mathcal{B}$*. The functor* K *is an equivalence of categories if and only if*

1) *L preserves the equalizer*

$$(D,d)=\mathrm{Equ}_{\mathrm{Fun}}\left(\eta R^{\mathbb{LR}}U,R^{\mathbb{LR}}U\gamma^{LR}\right).$$

2) *L reflects isomorphisms.*

Proof. Apply Theorem 4.55 taking $\varphi=\mathrm{Id}_{\mathbb{LR}}$ and thus $\alpha=\Theta\left(\mathrm{Id}_{\mathbb{LR}}\right)=\eta R$. □

Definition 4.59. Let $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ be a comonad on the category $\mathcal{A}$ and let $L:\mathcal{B}\to\mathcal{A}$. The functor L is called φ*-comonadic* if it has a right adjoint $R:\mathcal{A}\to\mathcal{B}$ for which there exists $\varphi:\mathbb{LR}\to\mathbb{C}$ a comonad morphism such that the functor $K_\varphi=\Upsilon(\varphi):\mathcal{B}\to{}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories with $D_\varphi:{}^{\mathbb{C}}\mathcal{A}\to\mathcal{B}$ which is right adjoint.

Definition 4.60. Let $L:\mathcal{B}\to\mathcal{A}$ be a functor. The functor L is called *comonadic* if it has a right adjoint $R:\mathcal{A}\to\mathcal{B}$ such that the functor $K=\Upsilon\left(\mathrm{Id}_{\mathbb{LR}}\right):\mathcal{B}\to{}^{\mathbb{LR}}\mathcal{A}$ is an equivalence of categories with right adjoint $D:{}^{\mathbb{LR}}\mathcal{A}\to\mathcal{B}$.

Lemma 4.61. *Let* $L:\mathcal{B}\to\mathcal{A}$ *be a* φ*-comonadic functor and let*

$$X'\overset{d_0}{\underset{d_1}{\rightrightarrows}}X\tag{53}$$

be a L-contractible equalizer pair in $\mathcal{B}$. Then (53) *has an equalizer $d : X'' \to X'$ in $\mathcal{B}$ and*

$$\begin{tikzcd} LX'' \arrow[r, "Ld"] & LX' \arrow[r, shift left, "Ld_0"] \arrow[r, shift right, "Ld_1"'] & LX \end{tikzcd}$$

is an equalizer in $\mathcal{A}$.

Proof. Since L is a φ-comonadic functor we know that $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories. Then, instead of considering

$$\begin{tikzcd} X' \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X \end{tikzcd}$$

in the category $\mathcal{B}$, we can consider

$$\begin{tikzcd} K_\varphi X' \arrow[r, shift left, "K_\varphi d_0"] \arrow[r, shift right, "K_\varphi d_1"'] & K_\varphi X \end{tikzcd}$$

in ${}^{\mathbb{C}}\mathcal{A}$ which is a ${}^{\mathbb{C}}U$-contractible equalizer pair. Let us denote by $\left(Z', {}^{C}\rho_{Z'}\right) := K_\varphi X'$ and $\left(Z, {}^{C}\rho_{Z}\right) := K_\varphi X$ so that we can rewrite the ${}^{\mathbb{C}}U$-contractible equalizer pair as follows

$$\begin{tikzcd} \left(Z', {}^{C}\rho_{Z'}\right) \arrow[r, shift left, "K_\varphi d_0"] \arrow[r, shift right, "K_\varphi d_1"'] & \left(Z, {}^{C}\rho_{Z}\right) \end{tikzcd}$$

We want to prove that this pair has an equalizer in ${}^{\mathbb{C}}\mathcal{A}$. Since the pair $(K_\varphi d_0, K_\varphi d_1)$ is a ${}^{\mathbb{C}}U$-contractible equalizer in ${}^{\mathbb{C}}\mathcal{A}$, we have that

$$\begin{tikzcd} Z'' \arrow[r, shift left, "d"] & Z' \arrow[l, shift left, "s"] \arrow[r, shift left=2, "Ld_0"] \arrow[r, shift right=2, "Ld_1"'] & Z \arrow[l, "t"'] \end{tikzcd}$$

is a contractible equalizer and thus, by Proposition 2.19, an equalizer in $\mathcal{A}$. Let us consider the following diagram

$$\begin{tikzcd} Z'' \arrow[rr, "d"] \arrow[d, dotted, "{}^{\mathbb{C}}\rho_{Z''}"'] && Z' \arrow[r, shift left, "Ld_0"] \arrow[r, shift right, "Ld_1"'] \arrow[d, "{}^{\mathbb{C}}\rho_{Z'}"] & Z \arrow[d, "{}^{\mathbb{C}}\rho_{Z}"] \\ CZ'' \arrow[rr, "Cd"] \arrow[d, shift right, "C{}^{\mathbb{C}}\rho_{Z''}"'] \arrow[d, shift left, "\Delta^C Z''"] && CZ' \arrow[r, shift left, "CLd_0"] \arrow[r, shift right, "CLd_1"'] \arrow[d, shift right, "C{}^{\mathbb{C}}\rho_{Z'}"'] \arrow[d, shift left, "\Delta^C Z'"] & CZ \arrow[d, shift right, "C{}^{\mathbb{C}}\rho_{Z}"'] \arrow[d, shift left, "\Delta^C Z"] \\ CCZ'' \arrow[rr, "CCd"] && CCZ' \arrow[r, shift left, "CCLd_0"] \arrow[r, shift right, "CCLd_1"'] & CCZ \end{tikzcd}$$

By Proposition 2.20, all the rows are contractible equalizers. Since $Ld_0 = {}^{\mathbb{C}}UK_\varphi d_0$ and $Ld_1 = {}^{\mathbb{C}}UK_\varphi d_1$ where $K_\varphi d_0$ and $K_\varphi d_1$ are morphisms in ${}^{\mathbb{C}}\mathcal{A}$, we have that the upper right square serially commutes. Moreover, since we also have that Δ^C is a functorial morphism, the lower right square serially commutes. We also have that ${}^{C}\rho_{Z'} \circ d$ is a fork for (CLd_0, CLd_1) and, since $(CZ'', CZ', CZ, Cd, CLd_0, CLd_1, Cs, Ct)$ is a contractible equalizer, in particular $(CZ'', Cd) = \mathrm{Equ}_{\mathcal{A}}(CLd_0, CLd_1)$; by the universal property of the equalizer, there exists a unique morphism ${}^{C}\rho_{Z''} : Z'' \to CZ''$ such that

$$ {}^{C}\rho_{Z'} \circ d = (Cd) \circ {}^{C}\rho_{Z''}. \tag{54}$$

Let us prove that $\left(Z'',{}^{C}\rho_{Z''}\right)\in{}^{\mathbb{C}}\mathcal{A}$ and thus formula (54) will say that d is a morphism in ${}^{\mathbb{C}}\mathcal{A}$. Since Δ^{C} is a functorial morphism and by definition of ${}^{C}\rho_{Z''}$, the lower left square serially commutes. We have

$$\begin{gathered}(CCd)\circ\left(C^{C}\rho_{Z''}\right)\circ{}^{C}\rho_{Z''}\overset{(54)}{=}\left(C^{C}\rho_{Z'}\right)\circ(Cd)\circ{}^{C}\rho_{Z''}\\ \overset{(54)}{=}\left(C^{C}\rho_{Z'}\right)\circ{}^{C}\rho_{Z'}\circ d\overset{{}^{C}\rho_{Z'}\text{coass}}{=}\left(\Delta^{C}Z'\right)\circ{}^{C}\rho_{Z'}\circ d\\ \overset{(54)}{=}\left(\Delta^{C}Z'\right)\circ(Cd)\circ{}^{C}\rho_{Z''}\overset{\Delta^{C}}{=}(CCd)\circ\left(\Delta^{C}Z''\right)\circ{}^{C}\rho_{Z''}\end{gathered}$$

and since CCd is a monomorphism we get

$$\left(C^{C}\rho_{Z''}\right)\circ{}^{C}\rho_{Z''}=\left(\Delta^{C}Z''\right)\circ{}^{C}\rho_{Z''}$$

that is that ${}^{C}\rho_{Z''}$ is coassociative. Moreover we have

$$\begin{gathered}d\circ\left(\varepsilon^{C}Z''\right)\circ{}^{C}\rho_{Z''}\overset{\varepsilon^{C}}{=}\left(\varepsilon^{C}Z'\right)\circ(Cd)\circ{}^{C}\rho_{Z''}\\ \overset{(54)}{=}\left(\varepsilon^{C}Z'\right)\circ{}^{C}\rho_{Z'}\circ d\overset{{}^{C}\rho_{Z'}\text{counit}}{=}d\end{gathered}$$

and since d is mono we get that

$$\left(\varepsilon^{C}Z''\right)\circ{}^{C}\rho_{Z''}=Z''$$

so that ${}^{C}\rho_{Z''}$ is also counital. Therefore $\left(Z'',{}^{C}\rho_{Z''}\right)\in{}^{\mathbb{C}}\mathcal{A}$ and d is a morphism in ${}^{\mathbb{C}}\mathcal{A}$. Now we want to prove that it is an equalizer in ${}^{\mathbb{C}}\mathcal{A}$. Let $\left(E,{}^{C}\rho_{E}\right)\in{}^{\mathbb{C}}\mathcal{A}$ and $f:\left(E,{}^{C}\rho_{E}\right)\to\left(Z',{}^{C}\rho_{Z'}\right)$ be a morphism in ${}^{\mathbb{C}}\mathcal{A}$ such that $(K_{\varphi}d_{0})\circ f=(K_{\varphi}d_{1})\circ f$. Then, by regarding f as a morphism in $\mathcal{A}$ we also have that

$$(Ld_{0})\circ f=(Ld_{1})\circ f.$$

Since $(Z'',d)=\mathrm{Equ}_{\mathcal{A}}\left(Ld_{0},Ld_{1}\right)$, there exists a unique morphism $h:E\to Z''$ such that

$$d\circ h=f.$$

Now we want to prove that h is a morphism in ${}^{\mathbb{C}}\mathcal{A}$. In fact, let us consider the following diagram

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "{}^{C}\rho_{E}"'] & Z'' \arrow[r, "d"] \arrow[d, "{}^{C}\rho_{Z''}"'] & Z' \arrow[d, "{}^{C}\rho_{Z'}"'] \\ CE \arrow[r, "Ch"] & CZ'' \arrow[r, "Cd"] & CZ'. \end{tikzcd}$$

Since $d\in{}^{\mathbb{C}}\mathcal{A}$, the right square commutes. Since $f\in{}^{\mathbb{C}}\mathcal{A}$ we have

$$(Cd)\circ(Ch)\circ{}^{C}\rho_{E}=(Cf)\circ{}^{C}\rho_{E}={}^{C}\rho_{Z'}\circ f={}^{C}\rho_{Z'}\circ d\circ h$$

so that we have

$$(Cd)\circ{}^{C}\rho_{Z''}\circ h\overset{(54)}{=}{}^{C}\rho_{Z'}\circ d\circ h=(Cd)\circ(Ch)\circ{}^{C}\rho_{E}$$

and since Cd is a monomorphism, we deduce that

$${}^{C}\rho_{Z''}\circ h=(Ch)\circ{}^{C}\rho_{E}$$

i.e. $h \in {}^{\mathbb{C}}\mathcal{A}$. Therefore $(Z'', d) = \mathrm{Equ}_{{}^{\mathbb{C}}\mathcal{A}}(K_\varphi d_0, K_\varphi d_1)$. Now, since $K_\varphi : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories, there exist $X'', e \in \mathcal{B}$ such that

$$K_\varphi X'' = \left(Z'', {}^C\rho_{Z''}\right) \text{ and } K_\varphi e = d$$

and thus $(X'', e) = \mathrm{Equ}_{\mathcal{B}}(d_0, d_1)$. Moreover, since

$$\begin{tikzcd} Z'' \arrow[r, shift left, "d"] & Z' \arrow[l, shift left, "s"] \arrow[r, shift left=2, "Ld_0"] \arrow[r, shift right=2, "Ld_1"'] & Z \arrow[l, "t"'] \end{tikzcd}$$

is a contractible coequalizer and $(Z'', d) = \left({}^{\mathbb{C}}UK_\varphi X'', {}^{\mathbb{C}}UK_\varphi e\right)$, we deduce that $\left({}^{\mathbb{C}}UK_\varphi X'', {}^{\mathbb{C}}UK_\varphi e\right)$ is a contractible coequalizer of (Ld_0, Ld_1). Then $(LX'', Le) = \left({}^{\mathbb{C}}UK_\varphi X'', {}^{\mathbb{C}}UK_\varphi e\right)$ is a contractible coequalizer of (Ld_0, Ld_1) so that $(LX'', Le) = \mathrm{Equ}_{\mathcal{A}}(Ld_0, Ld_1)$. □

The following is a slightly improved version of Theorem 3.14 p. 101 [BW] for the dual case.

THEOREM 4.62 (Generalized Beck's Precise Cotripleability Theorem). *Let $L : \mathcal{B} \to \mathcal{A}$ be a functor, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$. Then L is φ-comonadic if and only if*

1) *L has a right adjoint $R : \mathcal{A} \to \mathcal{B}$,*
2) *$\varphi : \mathbb{LR} \to \mathbb{C}$ is a comonads isomorphism where $\mathbb{LR} = (LR, L\eta R, \epsilon)$ with η and ϵ unit and counit of (L, R),*
3) *for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, there exist $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^C\rho_X\right)$, where $\alpha = (R\varphi) \circ (\eta R)$, and L preserves the equalizer*
$$\mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right),$$
4) *L reflects isomorphisms.*

In this case in $\mathcal{B}$ there exist equalizers of reflexive L-contractible equalizer pairs and L preserves them.

Proof. Assume first that L is φ-comonadic. Then by definition L has a right adjoint $R : \mathcal{A} \to \mathcal{B}$ and a comonad morphism $\varphi : \mathbb{LR} \to \mathbb{C}$ such that the functor $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories. Let K'_φ be an inverse of K_φ. Then in particular $K'_\varphi : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$ is a right adjoint of K_φ so that by Proposition 4.47 for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, there exists $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^C\rho_X\right)$ where $\alpha = \Theta(\varphi) = (R\varphi) \circ (\eta R)$ and thus $\left(K'_\varphi, k'_\varphi\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right)$. Then we can apply Theorem 4.55 to get that L preserves the equalizer $\left(K'_\varphi, k'_\varphi\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\alpha^{\mathbb{C}}U, R^{\mathbb{C}}U\gamma^C\right)$, L reflects isomorphisms and $\varphi : \mathbb{LR} \to \mathbb{C}$ is a comonads isomorphism.

Conversely, by assumption 1) L has a right adjoint $R : \mathcal{A} \to \mathcal{B}$ so that (L, R) is an adjunction and by assumption 2) there exist $\mathrm{Equ}_{\mathcal{B}}\left(\alpha X, R^C\rho_X\right)$, for every $\left(X, {}^C\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$ so that we can apply one direction of Proposition 4.47. Thus the functor $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ has a right adjoint $D_\varphi : {}^{\mathbb{C}}\mathcal{A} \to \mathcal{B}$. Now, by applying Theorem 4.55 in the converse direction, we deduce that $K_\varphi = \Upsilon(\varphi) : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is an equivalence of categories, i.e. L is φ-comonadic.

In the case L is φ-comonadic, by Lemma 4.61, in $\mathcal{B}$ there exist equalizers of reflexive L-contractible equalizer pairs and L preserves them. □

COROLLARY 4.63 (Beck's Precise Cotripleability Theorem). *Let $L : \mathcal{B} \to \mathcal{A}$ be a functor. Then L is comonadic if and only if*

1) *L has a right adjoint $R : \mathcal{A} \to \mathcal{B}$,*
2) *for every $\left(X, {}^{LR}\rho_X\right) \in {}^{\mathbb{LR}}\mathcal{A}$, there exist $\mathrm{Equ}_{\mathcal{B}}\left(\eta RX, R^{LR}\rho_X\right)$ and L preserves the equalizer*
$$\mathrm{Equ}_{\mathrm{Fun}}\left((\eta R) \circ \left({}^{\mathbb{LR}}U\right), R^{\mathbb{LR}}U\gamma^{LR}\right)$$
3) *L reflects isomorphisms.*

In this case in $\mathcal{B}$ there exist equalizers of reflexive L-contractible equalizer pairs and L preserves them.

Proof. Apply Theorem 4.62 to the case $\varphi = \mathrm{Id}_{\mathbb{LR}}$. $\square$

LEMMA 4.64. *Let (L, R) be an adjunction, where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, with unit η and counit ϵ. Then for every $Y \in \mathcal{B}$, $(LY, LRLY, LRLRLY, L\eta Y, L\eta RLY, LRL\eta Y, \epsilon LY, \epsilon LRLY)$ is a contractible equalizer and in particular, for every $Y \in \mathcal{B}$*
$$(LY, L\eta Y) = \mathrm{Equ}_{\mathcal{A}}\left(L\eta RLY, LRL\eta Y\right).$$

Proof. Consider the following diagram
$$\begin{tikzcd}[column sep=large] LY \arrow[r, shift left, "L\eta Y"] & LRLY \arrow[l, shift left, "\epsilon LY"] \arrow[r, shift left=2.5, "L\eta RLY"] \arrow[r, shift right=2.5, "LRL\eta Y"'] & LRLRLY \arrow[l, "\epsilon LRLY"description] \end{tikzcd}$$
and let us compute
$$\begin{aligned}
(\epsilon LRLY) \circ (L\eta RLY) &= \mathrm{Id}_{LRLY} \\
(\epsilon LY) \circ (L\eta Y) &= \mathrm{Id}_{LY} \\
(\epsilon LRLY) \circ (LRL\eta Y) &= (L\eta Y) \circ (\epsilon LY) = \mathrm{Id}_{LRLY} \\
(L\eta RLY) \circ (L\eta Y) &= (LRL\eta Y) \circ (L\eta Y).
\end{aligned}$$
Thus $(LY, LRLY, LRLRLY, L\eta Y, L\eta RLY, LRL\eta Y, \epsilon LY, \epsilon LRLY)$ is a contractible equalizer for every $Y \in \mathcal{B}$ and by Proposition 2.19 we get that $(LY, L\eta Y) = \mathrm{Equ}_{\mathcal{A}}\left(L\eta RLY, LRL\eta Y\right)$. $\square$

LEMMA 4.65. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad morphism. Let $K_\varphi = \Upsilon(\varphi) = (L, (\varphi L) \circ (L\eta))$ and ${}^{\mathbb{C}}UK_\varphi(f) = L(f)$ for every morphism f in $\mathcal{B}$. For every $Y \in \mathcal{B}$ we have*

$$(K_\varphi Y, K_\varphi \eta Y) = \mathrm{Equ}_{{}^{\mathbb{C}}\mathcal{A}}\left(K_\varphi \eta RLY, K_\varphi RL\eta Y\right). \tag{55}$$

Proof. By Lemma 4.64 we have that $(LY, L\eta Y) = \mathrm{Equ}_{\mathcal{A}}\left(L\eta RLY, LRL\eta Y\right)$. Let $h : Z \to K_\varphi RLY = (LRLY, (\varphi LRLY) \circ (L\eta RLY))$ be a morphism in ${}^{\mathbb{C}}\mathcal{A}$ such that
$$(K_\varphi RL\eta Y) \circ h = (K_\varphi \eta RLY) \circ h.$$
Then
$$(LRL\eta Y) \circ \left({}^{\mathbb{C}}Uh\right) = (L\eta RLY) \circ \left({}^{\mathbb{C}}Uh\right) \tag{56}$$

and hence there exists a $\zeta : {}^{\mathbb{C}}UZ \to LY = {}^{\mathbb{C}}UK_{\varphi}Y$ such that

$$\left({}^{\mathbb{C}}Uh\right) = (L\eta Y) \circ \zeta = \left({}^{\mathbb{C}}UK_{\varphi}\eta Y\right) \circ \zeta. \tag{57}$$

Let us prove that ζ gives rise to a morphism in ${}^{\mathbb{C}}\mathcal{A}$. Since h is a morphism in ${}^{\mathbb{C}}\mathcal{A}$ we have that

$$(\varphi LRLY) \circ (L\eta RLY) \circ \left({}^{\mathbb{C}}Uh\right) = \left(C{}^{\mathbb{C}}Uh\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}Z\right). \tag{58}$$

Let us compute

$$\begin{aligned}
(CL\eta Y) \circ (C\zeta) \circ \left({}^{\mathbb{C}}U\gamma^{C}Z\right) &\overset{(57)}{=} \left(C{}^{\mathbb{C}}Uh\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}Z\right) \\
&\overset{(58)}{=} (\varphi LRLY) \circ (L\eta RLY) \circ \left({}^{\mathbb{C}}Uh\right) \\
&\overset{(56)}{=} (\varphi LRLY) \circ (LRL\eta Y) \circ \left({}^{\mathbb{C}}Uh\right) \\
&\overset{(57)}{=} (\varphi LRLY) \circ (LRL\eta Y) \circ (L\eta Y) \circ \zeta \\
&\overset{\varphi}{=} (CL\eta Y) \circ (\varphi LY) \circ (L\eta Y) \circ \zeta
\end{aligned}$$

so that

$$(CL\eta Y) \circ (C\zeta) \circ \left({}^{\mathbb{C}}U\gamma^{C}Z\right) = (CL\eta Y) \circ (\varphi LY) \circ (L\eta Y) \circ \zeta.$$

Since $(C\epsilon LY) \circ (CL\eta Y) = CLRLY$, $CL\eta Y$ is mono and hence we get

$$(C\zeta) \circ \left({}^{\mathbb{C}}U\gamma^{C}Z\right) = (\varphi LY) \circ (L\eta Y) \circ \zeta$$

i.e. $\zeta : {}^{\mathbb{C}}UZ \to LY = {}^{\mathbb{C}}UK_{\varphi}Y$ is a morphism of $\mathbb{C}$-comodules. $\square$

PROPOSITION 4.66. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad morphism. Let $K_{\varphi} = \Upsilon(\varphi) = (L, (\varphi L) \circ (L\eta))$ and ${}^{\mathbb{C}}UK_{\varphi}(f) = L(f)$ for every morphism f in $\mathcal{B}$. If φX is a monomorphism for every $X \in \mathcal{A}$, the assignment $\mathcal{K}_{Y,RLY'} : \mathrm{Hom}_{\mathcal{B}}(Y, RLY') \to \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}(K_{\varphi}Y, K_{\varphi}RLY')$ defined by setting*

$$\mathcal{K}_{Y,RLY'}(f) = K_{\varphi}(f)$$

is an isomorphism whose inverse is defined by

$$\mathcal{K}_{Y,RLY'}^{-1}(h) = (R\epsilon LY') \circ \left(R{}^{\mathbb{C}}Uh\right) \circ (\eta Y).$$

Proof. Let $f \in \mathrm{Hom}_{\mathcal{B}}(Y, RLY')$. We compute

$$\begin{aligned}
\widetilde{\mathcal{K}}_{Y,RLY'}^{-1}\left(\widetilde{\mathcal{K}}_{Y,RLY'}(f)\right) &= (R\epsilon LY') \circ \left(R{}^{\mathbb{C}}UK_{\varphi}f\right) \circ (\eta Y) = (R\epsilon LY') \circ (RLf) \circ (\eta Y) \\
&\overset{\eta}{=} (R\epsilon LY') \circ (\eta RLY') \circ f = f.
\end{aligned}$$

Let $h \in \mathrm{Hom}_{{}^{\mathbb{C}}\mathcal{A}}(K_{\varphi}Y, K_{\varphi}RLY')$. This means that

$$\begin{aligned}
(\varphi LRLY') \circ (L\eta RLY') \circ \left({}^{\mathbb{C}}Uh\right) &= \left(C{}^{\mathbb{C}}Uh\right) \circ (\varphi LY) \circ (L\eta Y) \\
&\overset{\varphi}{=} (\varphi LRLY') \circ \left(LR{}^{\mathbb{C}}Uh\right) \circ (L\eta Y).
\end{aligned}$$

Since φX is a monomorphism for every $X \in \mathcal{A}$, we deduce that

$$(L\eta RLY') \circ \left({}^{\mathbb{C}}Uh\right) = \left(LR{}^{\mathbb{C}}Uh\right) \circ (L\eta Y). \tag{59}$$

We compute

$$(LR\epsilon LY') \circ \left(LR^{\mathbb{C}}Uh\right) \circ (L\eta Y) \overset{(59)}{=} (LR\epsilon LY') \circ (L\eta RLY') \circ \left({}^{\mathbb{C}}Uh\right)$$
$$= {}^{\mathbb{C}}Uh$$

and since $L = {}^{\mathbb{C}}UK_\varphi$ and ${}^{\mathbb{C}}U$ reflects, we get

$$(K_\varphi R\epsilon LY') \circ \left(K_\varphi R^{\mathbb{C}}Uh\right) \circ (K_\varphi \eta Y) = h.$$

Then we deduce that

$$\mathcal{K}_{Y,RLY'}\left(\mathcal{K}_{Y,RLY'}^{-1}(h)\right) = \mathcal{K}_{Y,RLY'}\left((R\epsilon LY') \circ \left(R^{\mathbb{C}}Uh\right) \circ (\eta Y)\right)$$
$$= (K_\varphi R\epsilon LY') \circ \left(K_\varphi R^{\mathbb{C}}Uh\right) \circ (K_\varphi \eta Y) = h.$$

□

Proposition 4.67. *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad morphism. Let $K_\varphi = \Upsilon(\varphi) = (L, (\varphi L) \circ (L\eta))$ and ${}^{\mathbb{C}}UK_\varphi(f) = L(f)$ for every morphism f in $\mathcal{B}$. If K_φ is full and faithful then, for every $Y \in \mathcal{B}$, we have*

$$(Y, \eta Y) = \mathrm{Equ}_{\mathcal{B}}(RL\eta Y, \eta RLY).$$

Proof. By Lemma 4.65 we have

$$(K_\varphi Y, K_\varphi \eta Y) = \mathrm{Equ}_{{}^{\mathbb{C}}\mathcal{A}}(K_\varphi RL\eta Y, K_\varphi \eta RLY).$$

Then we can apply Lemma 2.16 and deduce that $(Y, \eta Y) = \mathrm{Equ}_{\mathcal{B}}(RL\eta Y, \eta RLY)$.
□

Theorem 4.68 (Generalized Beck's Theorem for comonads). *Let (L, R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on a category $\mathcal{A}$ and let $\varphi : \mathbb{LR} = (LR, L\eta R, \epsilon) \to \mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonads morphism such that φX is a monomorphism for every $X \in \mathcal{A}$. Let $K_\varphi = \Upsilon(\varphi) = (L, (\varphi L) \circ (L\eta))$ and ${}^{\mathbb{C}}UK_\varphi(f) = L(f)$ for every morphism f in $\mathcal{B}$. Then $K_\varphi : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ is full and faithful if and only if for every $Y \in \mathcal{B}$ we have that $(Y, \eta Y) = \mathrm{Equ}_{\mathcal{B}}(\eta RLY, RL\eta Y)$.*

Proof. If K_φ is full and faithful then we can apply Proposition 4.67 to get that for every $Y \in \mathcal{B}$ we have that $(Y, \eta Y) = \mathrm{Equ}_{\mathcal{B}}(RL\eta Y, \eta RLY)$.
Conversely assume that for every $Y \in \mathcal{B}$ we have that $(Y, \eta Y) = \mathrm{Equ}_{\mathcal{B}}(\eta RLY, RL\eta Y)$.
We want to prove that $\mathcal{K}_{Y,Y'}$ is bijective for every $Y, Y' \in \mathcal{B}$. Let us consider the

following diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\mathrm{Hom}_{\mathcal{B}}\left(Y,Y'\right) \arrow[rr, dotted, "\mathcal{K}_{Y,Y'}"] \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(Y,\eta Y')"'] & & \mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}\left(K_{\varphi}Y,K_{\varphi}Y'\right) \arrow[d, "\mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}(K_{\varphi}Y,K_{\varphi}\eta Y')"] \\
\mathrm{Hom}_{\mathcal{B}}\left(Y,RLY'\right) \arrow[rr, "\mathcal{K}_{Y,RLY'}"] \arrow[d, shift right, "\mathrm{Hom}_{\mathcal{B}}(Y,RL\eta Y')"'] \arrow[d, shift left, "\mathrm{Hom}_{\mathcal{B}}(Y,\eta RLY')"] & & \mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}\left(K_{\varphi}Y,K_{\varphi}RLY'\right) \arrow[d, shift right, "\mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}(K_{\varphi}Y,K_{\varphi}RL\eta Y')"'] \arrow[d, shift left, "\mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}(K_{\varphi}Y,K_{\varphi}\eta RLY')"] \\
\mathrm{Hom}_{\mathcal{B}}\left(Y,RLRLY'\right) \arrow[rr, "\mathcal{K}_{Y,RLRLY'}"] & & \mathrm{Hom}_{\mathbb{C}_{\mathcal{A}}}\left(K_{\varphi}Y,K_{\varphi}RLRLY'\right)
\end{tikzcd}$$

Since $(Y',\eta Y') = \mathrm{Equ}_{\mathcal{B}}\left(\eta RLY',RL\eta Y'\right)$ the left column of the diagram is exact by Lemma 2.17. By Lemma 4.65 we have $\left(K_{\varphi}Y,K_{\varphi}\eta Y\right) = \mathrm{Equ}_{\mathbb{C}_{\mathcal{A}}}\left(K_{\varphi}\eta RLY,K_{\varphi}RL\eta Y\right)$ so that also the right column is also exact by Lemma 2.17. Let $f \in \mathrm{Hom}_{\mathcal{B}}\left(Y,Y'\right)$ and $g \in \mathrm{Hom}_{\mathcal{B}}\left(Y,RLY'\right)$. Since

$$K_{\varphi}\left(\eta Y' \circ f\right) = \left(K_{\varphi}\eta Y'\right) \circ \left(K_{\varphi}f\right),$$

$$K_{\varphi}\left(\eta RLY' \circ g\right) = \left(K_{\varphi}\eta RLY'\right) \circ \left(K_{\varphi}g\right) \text{ and } K_{\varphi}\left(RL\eta Y' \circ g\right) = \left(K_{\varphi}RL\eta Y'\right) \circ \left(K_{\varphi}g\right)$$

the diagram is serially commutative. By Proposition 4.66, $\mathcal{K}_{Y,RLY'}$ and $\mathcal{K}_{Y,RLRLY'}$ are isomorphisms and so is $\mathcal{K}_{Y,Y'}$ by Lemma 2.15. $\square$

Corollary 4.69 (Beck's Theorem for comonads). *Let (L,R) be an adjunction where $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$. Then $K = \Upsilon\left(\mathrm{Id}_{LR}\right) : \mathcal{B} \to {}^{\mathbb{LR}}\mathcal{A}$ is full and faithful if and only if for every $Y \in \mathcal{B}$ we have that $(Y,\eta Y) = \mathrm{Equ}_{\mathcal{B}}\left(\eta RLY,RL\eta Y\right)$.*

5. Liftings and distributive laws

5.1. **Distributive laws.**

Definition 5.1. Let $A = (A,m,u)$ be a monad and $C = (C,\Delta,\varepsilon)$ be a comonad on the same category $\mathcal{A}$. A functorial morphism $\Phi : AC \to CA$ is called a *mixed distributive law* (or in some papers an *entwining*) if

- $\Phi \circ (mC) = (Cm) \circ (\Phi A) \circ (A\Phi)$ and $\Phi \circ (uC) = Cu$
- $(\Delta A) \circ \Phi = (C\Phi) \circ (\Phi C) \circ (A\Delta)$ and $(\varepsilon A) \circ \Phi = A\varepsilon$.

Definition 5.2. Let $A = (A,m,u)$ be a monad and $C = (C,\Delta,\varepsilon)$ be a comonad on the same category $\mathcal{A}$. A functorial morphism $\Psi : CA \to AC$ is called an *opposite mixed distributive law* if

- $\Psi \circ (Cm) = (mC) \circ (A\Psi) \circ (\Psi A)$ and $\Psi \circ (Cu) = uC$
- $(A\Delta) \circ \Psi = (\Psi C) \circ (C\Psi) \circ (\Delta A)$ and $(A\varepsilon) \circ \Psi = \varepsilon A$.

Lemma 5.3. *Let $\mathbb{A} = (A,m_A,u_A)$ be a monad and let $\mathbb{C} = \left(C,\Delta^C,\varepsilon^C\right)$ be a comonad on the category $\mathcal{A}$. Let $Q : \mathcal{B} \to \mathcal{A}$ be a functor such that $\left(Q,{}^A\mu_Q\right)$ is left $\mathbb{A}$-module functor. Assume that $\Phi : AC \to CA$ is a mixed distributive law. Then $\left(CQ,{}^A\mu_{CQ}\right) = \left(CQ,\left(C^A\mu_Q\right) \circ (\Phi Q)\right)$ is a left $\mathbb{A}$-module functor.*

Proof. First of all we prove that ${}^{A}\mu_{CQ} = \left(C^{A}\mu_{Q}\right) \circ (\Phi Q)$ is associative. In fact we have

$$\begin{aligned}
{}^{A}\mu_{CQ} \circ \left(A^{A}\mu_{CQ}\right) &\overset{\mathrm{def}^{A}\mu_{CQ}}{=} \left(C^{A}\mu_{Q}\right) \circ (\Phi Q) \circ \left(AC^{A}\mu_{Q}\right) \circ (A\Phi Q) \\
&\overset{\Phi}{=} \left(C^{A}\mu_{Q}\right) \circ \left(CA^{A}\mu_{Q}\right) \circ (\Phi AQ) \circ (A\Phi Q) \\
&\overset{{}^{A}\mu_{Q}\mathrm{ass}}{=} \left(C^{A}\mu_{Q}\right) \circ (Cm_{A}Q) \circ (\Phi AQ) \circ (A\Phi Q) \\
&\overset{\Phi\mathrm{mdl}}{=} \left(C^{A}\mu_{Q}\right) \circ (\Phi Q) \circ (m_{A}CQ) \overset{\mathrm{def}^{A}\mu_{CQ}}{=} {}^{A}\mu_{CQ} \circ (m_{A}CQ)\,.
\end{aligned}$$

Now we prove the unitality condition. We have

$$\begin{aligned}
{}^{A}\mu_{CQ} \circ (u_{A}CQ) &\overset{\mathrm{def}^{A}\mu_{CQ}}{=} \left(C^{A}\mu_{Q}\right) \circ (\Phi Q) \circ (u_{A}CQ) \\
&\overset{\Phi\mathrm{mdl}}{=} \left(C^{A}\mu_{Q}\right) \circ (Cu_{A}Q) \overset{{}^{A}\mu_{CQ}\mathrm{uni}}{=} CQ.
\end{aligned}$$

□

PROPOSITION 5.4. *Let $\mathbb{A} = (A, m_{A}, u_{A})$ be a monad and let $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad on the category $\mathcal{A}$. Assume that $\Phi : AC \to CA$ is a mixed distributive law between them. Let F, G be left $\mathbb{A}$-module functors and $\alpha : F \to G$ be a functorial morphism between them satisfying*

$$ {}^{A}\mu_{G} \circ (A\alpha) = \alpha \circ \left({}^{A}\mu_{F}\right), $$

i.e. there exists a functorial morphism ${}_{A}\alpha : {}_{A}F \to {}_{A}G$ such that ${}_{\mathbb{A}}U_{A}\alpha = \alpha$. Then also $C\alpha$ is a functorial morphism between left $\mathbb{A}$-module functors satisfying

$$ {}^{A}\mu_{CG} \circ (AC\alpha) = (C\alpha) \circ {}^{A}\mu_{CF} $$

i.e. there exists a functorial morphism ${}_{A}(C\alpha) : {}_{A}(CF) \to {}_{A}(CG)$ such that ${}_{\mathbb{A}}U_{A}(C\alpha) = C\alpha$. Moreover we have

$$ {}_{A}(C\alpha) = \widetilde{C}_{A}\alpha $$

where $\widetilde{\mathbb{C}}$ is the lifted comonad on the category ${}_{\mathbb{A}}\mathcal{A}$, i.e. ${}_{\mathbb{A}}U\widetilde{C} = C_{\mathbb{A}}U$.

Proof. By Lemma 3.29 there exists ${}_{A}\alpha : {}_{A}F \to {}_{A}G$ such that ${}_{\mathbb{A}}U_{A}\alpha = \alpha$. Moreover, by Lemma 5.3, we know that $\left(CF, {}^{A}\mu_{CF}\right) = \left(CF, \left(C^{A}\mu_{F}\right) \circ (\Phi F)\right)$ and $\left(CG, {}^{A}\mu_{CG}\right) = \left(CG, \left(C^{A}\mu_{G}\right) \circ (\Phi G)\right)$ are left $\mathbb{A}$-module functors. Then we have

$$\begin{aligned}
{}^{A}\mu_{CG} \circ (AC\alpha) &\overset{\mathrm{def}^{A}\mu_{CG}}{=} \left(C^{A}\mu_{G}\right) \circ (\Phi G) \circ (AC\alpha) \\
&\overset{\Phi}{=} \left(C^{A}\mu_{G}\right) \circ (CA\alpha) \circ (\Phi G) \\
&\overset{\alpha\mathrm{morp}A\mathrm{mod}}{=} (C\alpha) \circ \left(C^{A}\mu_{F}\right) \circ (\Phi G) \overset{\mathrm{def}^{A}\mu_{CF}}{=} (C\alpha) \circ {}^{A}\mu_{CF}
\end{aligned}$$

i.e. $C\alpha$ is a functorial morphism between left $\mathbb{A}$-module functors. Then there exists a functorial morphism ${}_{A}(C\alpha) : {}_{A}(CF) \to {}_{A}(CG)$ such that ${}_{\mathbb{A}}U_{A}(C\alpha) = C\alpha$. Since we also have

$$ {}_{\mathbb{A}}U\widetilde{C}_{A}\alpha = C_{\mathbb{A}}U_{A}\alpha = C\alpha $$

we deduce that

$$ {}_{\mathbb{A}}U_{A}(C\alpha) = {}_{\mathbb{A}}U\widetilde{C}_{A}\alpha. $$

Since ${}_{\mathbb{A}}U$ is faithful, this implies that ${}_{A}(C\alpha) = \widetilde{C}_{A}\alpha$. $\square$

LEMMA 5.5. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on the category $\mathcal{A}$. Let $Q : \mathcal{B} \to \mathcal{A}$ be a functor such that $\left(Q, {}^{C}\rho_Q\right)$ is a left $\mathbb{C}$-comodule functor. Assume that $\Phi : AC \to CA$ is a mixed distributive law. Then $\left(AQ, {}^{C}\rho_{AQ}\right) = \left(AQ, (\Phi Q) \circ \left(A^C\rho_Q\right)\right)$ is a left $\mathbb{C}$-comodule functor.*

Proof. First of all we prove that ${}^{C}\rho_{AQ} = (\Phi Q) \circ \left(A^C\rho_Q\right)$ is coassociative. In fact we have

$$\begin{aligned}
\left(C^C\rho_{AQ}\right) \circ {}^{C}\rho_{AQ} &\overset{\mathrm{def}{}^{C}\rho_{AQ}}{=} (C\Phi Q) \circ \left(CA^C\rho_Q\right) \circ (\Phi Q) \circ \left(A^C\rho_Q\right) \\
&\overset{\Phi}{=} (C\Phi Q) \circ (\Phi CQ) \circ \left(AC^C\rho_Q\right) \circ \left(A^C\rho_Q\right) \\
&\overset{{}^{C}\rho_Q\mathrm{coass}}{=} (C\Phi Q) \circ (\Phi CQ) \circ \left(A\Delta^C Q\right) \circ \left(A^C\rho_Q\right) \\
&\overset{\Phi\mathrm{mdl}}{=} \left(\Delta^C AQ\right) \circ (\Phi Q) \circ \left(A^C\rho_Q\right) \overset{\mathrm{def}{}^{C}\rho_{AQ}}{=} \left(\Delta^C AQ\right) \circ {}^{C}\rho_{AQ}.
\end{aligned}$$

Now we prove the counitality condition. We have

$$\begin{aligned}
\left(\varepsilon^C AQ\right) \circ {}^{C}\rho_{AQ} &\overset{\mathrm{def}{}^{C}\rho_{AQ}}{=} \left(\varepsilon^C AQ\right) \circ (\Phi Q) \circ \left(A^C\rho_Q\right) \\
&\overset{\Phi\mathrm{mdl}}{=} \left(A\varepsilon^C Q\right) \circ \left(A^C\rho_Q\right) \overset{{}^{C}\rho_Q\mathrm{couni}}{=} AQ.
\end{aligned}$$

$\square$

PROPOSITION 5.6. *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad and let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be a comonad on the category $\mathcal{A}$. Assume that $\Phi : AC \to CA$ is a mixed distributive law between them. Let F, G be left $\mathbb{C}$-comodule functors and $\alpha : F \to G$ be a functorial morphism between them satisfying*

$$ {}^{C}\rho_G \circ \alpha = (C\alpha) \circ \left({}^{C}\rho_F\right), $$

i.e. there exists ${}^{C}\alpha : {}^{C}F \to {}^{C}G$ such that ${}^{\mathbb{C}}U^C\alpha = \alpha$. Then also $A\alpha$ is a morphism between left $\mathbb{C}$-comodule functors satisfying

$$ (CA\alpha) \circ {}^{C}\rho_{AF} = {}^{C}\rho_{AG} \circ (A\alpha) $$

i.e. there exists a functorial morphism ${}^{C}(A\alpha) : {}^{C}(AF) \to {}^{C}(AG)$ such that ${}^{\mathbb{C}}U^C(A\alpha) = A\alpha$. Moreover we have

$$ {}^{C}(A\alpha) = \widetilde{A}^C\alpha $$

where $\widetilde{\mathbb{A}}$ is the lifted monad on the category ${}^{\mathbb{C}}\mathcal{A}$, i.e. ${}^{\mathbb{C}}U\widetilde{A} = A^{\mathbb{C}}U$.

Proof. Since F, G are left $\mathbb{C}$-comodule functors, by Lemma 5.5 we know that $\left(AF, {}^{C}\rho_{AF}\right) = \left(AF, (\Phi F) \circ \left(A^C\rho_F\right)\right)$ and $\left(AG, {}^{C}\rho_{AG}\right) = \left(AG, (\Phi G) \circ \left(A^C\rho_G\right)\right)$ are left $\mathbb{C}$-comodule functors. Then we have

$$\begin{aligned}
(CA\alpha) \circ {}^{C}\rho_{AF} &\overset{\mathrm{def}{}^{C}\rho_{AF}}{=} (CA\alpha) \circ (\Phi F) \circ \left(A^C\rho_F\right) \\
&\overset{\Phi}{=} (\Phi G) \circ (AC\alpha) \circ \left(A^C\rho_F\right) \\
&\overset{\alpha\mathrm{morp}{}^{C}\mathrm{com}}{=} (\Phi G) \circ \left(A^C\rho_G\right) \circ (A\alpha) \overset{\mathrm{def}{}^{C}\rho_{AG}}{=} {}^{C}\rho_{AG} \circ (A\alpha)
\end{aligned}$$

i.e. $A\alpha$ is a functorial morphism between left $\mathbb{C}$-comodule functors. Then there exists a functorial morphism ${}^{C}(A\alpha) : {}^{C}(AF) \to {}^{C}(AG)$ such that ${}^{\mathbb{C}}U^{C}(A\alpha) = A\alpha$. Since we also have

$$ {}^{\mathbb{C}}U\widetilde{A}^{C}\alpha = A^{\mathbb{C}}U^{C}\alpha = A\alpha $$

we deduce that

$$ {}^{\mathbb{C}}U^{C}(A\alpha) = {}^{\mathbb{C}}U\widetilde{A}^{C}\alpha. $$

Since ${}^{\mathbb{C}}U$ is faithful, this implies that ${}^{C}(A\alpha) = \widetilde{A}^{C}\alpha$. □

5.2. **Liftings of monads and comonads.**

THEOREM 5.7 ([Be, Proposition p. 122] and [Mesa, Theorem 2.1]). *Let $\mathbb{A} = (A, m_A, u_A)$ be a monad and let $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ be a comonad on a category $\mathcal{A}$. There is a bijection between the following collections of data:*

$\mathfrak{C}$ *liftings of $\mathbb{C}$ to a comonad $\widetilde{\mathbb{C}}$ on the category ${}_{\mathbb{A}}\mathcal{A}$, that is comonads $\widetilde{\mathbb{C}} = \left(\widetilde{C}, \Delta^{\widetilde{C}}, \varepsilon^{\widetilde{C}}\right)$ on ${}_{\mathbb{A}}\mathcal{A}$ such that*

$$ {}_{\mathbb{A}}U\widetilde{C} = C_{\mathbb{A}}U, \qquad {}_{\mathbb{A}}U\Delta^{\widetilde{C}} = \Delta^{C}{}_{\mathbb{A}}U \qquad \text{and} \qquad {}_{\mathbb{A}}U\varepsilon^{\widetilde{C}} = \varepsilon^{C}{}_{\mathbb{A}}U $$

$\mathfrak{D}$ *mixed distributive laws $\Phi : AC \to CA$*

$\mathfrak{M}$ *liftings of $\mathbb{A}$ to a monad $\widetilde{\mathbb{A}}$ on the category ${}^{\mathbb{C}}\mathcal{A}$, that is monads $\widetilde{\mathbb{A}} = \left(\widetilde{A}, m_{\widetilde{A}}, u_{\widetilde{A}}\right)$ on ${}^{\mathbb{C}}\mathcal{A}$ such that*

$$ {}^{\mathbb{C}}U\widetilde{A} = A^{\mathbb{C}}U, \qquad {}^{\mathbb{C}}Um_{\widetilde{A}} = m_{A}{}^{\mathbb{C}}U \qquad \text{and} \qquad {}^{\mathbb{C}}Uu_{\widetilde{A}} = u_{A}{}^{\mathbb{C}}U $$

given by

$$ \overline{a} : \mathfrak{C} \to \mathfrak{D} \text{ where } \overline{a}\left(\widetilde{C}\right) = \left({}_{\mathbb{B}}U\lambda_{B}\widetilde{C}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{B}}U_{\mathbb{B}}FCu_{A}\right) $$

$$ \overline{b} : \mathfrak{D} \to \mathfrak{C} \text{ where } {}_{\mathbb{A}}U\overline{b}(\Phi) = C_{\mathbb{A}}U \text{ and } {}_{\mathbb{A}}U\lambda_{A}\overline{b}(\Phi) = \left(C_{\mathbb{A}}U\lambda_{A}\right) \circ \Phi \text{ i.e.} $$

$$ \overline{b}(\Phi)\left(\left(X, {}^{A}\mu_{X}\right)\right) = \left(CX, \left(C^{A}\mu_{X}\right) \circ (\Phi X)\right) \text{ and } \overline{b}(\Phi)(f) = C(f) $$

$$ \overline{d} : \mathfrak{M} \to \mathfrak{D} \text{ where } \overline{d}\left(\widetilde{A}\right) = \left({}^{\mathbb{C}}U^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}^{\mathbb{C}}F\right) $$

$$ \overline{m} : \mathfrak{D} \to \mathfrak{M} \text{ where } {}^{\mathbb{C}}U\overline{m}(\Phi) = A^{\mathbb{C}}U \text{ and } {}^{\mathbb{C}}U\gamma^{C}\overline{m}(\Phi) = \Phi \circ \left(A^{\mathbb{C}}U\gamma^{C}\right) \text{ i.e.} $$

$$ \overline{m}(\Phi)\left(\left(X, {}^{C}\rho_{X}\right)\right) = \left(AX, (\Phi X) \circ \left(A^{C}\rho_{X}\right)\right) \text{ and } \overline{m}(\Phi)(f) = A(f). $$

Proof. In order to prove the bijection between $\mathfrak{C}$ and $\mathfrak{D}$, we apply Proposition 3.24, to the case $(A, m_A, u_A) = (B, m_B, u_B)$ monad on $\mathcal{A}$ and $Q = C$. In particular we will prove that the bijection $a : \mathcal{F} \to \mathcal{M}, b : \mathcal{M} \to \mathcal{F}$ of Proposition 3.24 induces a bijection between $\mathfrak{C}$ and $\mathfrak{D}$.

Let $\widetilde{\mathbb{C}} \in \mathfrak{C}$. We have to prove that $\Phi = a\left(\widetilde{\mathbb{C}}\right) = \left({}_{\mathbb{A}}U\lambda_{A}\widetilde{C}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_{A}\right) \in \mathfrak{D}$. We have

$$ (C\Phi) \circ (\Phi C) \circ \left(A\Delta^{C}\right) = $$

$$ \left(C_{\mathbb{A}}U\lambda_{A}\widetilde{C}_{\mathbb{A}}F\right) \circ \left(C_{\mathbb{A}}U_{\mathbb{A}}FCu_{A}\right) \circ \left({}_{\mathbb{A}}U\lambda_{A}\widetilde{C}_{\mathbb{A}}FC\right) \circ \left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_{A}C\right) \circ \left(A\Delta^{C}\right) $$

$$ = \left({}_{\mathbb{A}}U\widetilde{C}\lambda_{A}\widetilde{C}_{\mathbb{A}}F\right) \circ \left({}_{\mathbb{A}}U\widetilde{C}_{\mathbb{A}}FCu_{A}\right) \circ \left({}_{\mathbb{A}}U\lambda_{A}\widetilde{C}_{\mathbb{A}}FC\right) \circ \left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_{A}CX\right) \circ \left(A\Delta^{C}\right) $$

$$= {}_{\mathbb{A}}U\left[\left(\widetilde{C}\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left(\widetilde{C}_{\mathbb{A}}FCu_A\right)\circ\left(\lambda_A\widetilde{C}_{\mathbb{A}}FC\right)\circ\left({}_{\mathbb{A}}FCu_AC\right)\circ\left({}_{\mathbb{A}}F\Delta^C\right)\right]$$

$$\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F_{\mathbb{A}}U\widetilde{C}\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F_{\mathbb{A}}U\widetilde{C}_{\mathbb{A}}FCu_A\right)\circ\left({}_{\mathbb{A}}FCu_AC\right)\circ\left({}_{\mathbb{A}}F\Delta^C\right)\right]$$

$$= {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FC_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FC_{\mathbb{A}}U_{\mathbb{A}}FCu_A\right)\circ\left({}_{\mathbb{A}}FCu_AC\right)\circ\left({}_{\mathbb{A}}F\Delta^C\right)\right]$$

$$\overset{u_A}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FC_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_ACA\right)\circ\left({}_{\mathbb{A}}FCCu_A\right)\circ\left({}_{\mathbb{A}}F\Delta^C\right)\right]$$

$$= {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FC_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_{A\mathbb{A}}U\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCCu_A\right)\circ\left({}_{\mathbb{A}}F\Delta^C\right)\right]$$

$$\overset{(\lambda_A,u_A)\mathrm{adj},\Delta^C}{=} {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F\Delta^CA\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$= {}_{\mathbb{A}}U\left[\left(\lambda_A\widetilde{C}\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F_{\mathbb{A}}U\Delta^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(\Delta^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left(\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$= \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_A\right) = \left(\Delta^CA\right)\circ(\Phi)$$

so that

$$(C\Phi)\circ(\Phi C)\circ\left(A\Delta^C\right) = \left(\Delta^CA\right)\circ(\Phi).$$

Moreover

$$\left(\varepsilon^CA\right)\circ(\Phi) = \left(\varepsilon^CA\right)\circ\left({}_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_A\right)$$

$$= \left({}_{\mathbb{A}}U\varepsilon^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}U\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}U_{\mathbb{A}}FCu_A\right)$$

$$= {}_{\mathbb{A}}U\left[\left(\varepsilon^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left(\lambda_A\widetilde{C}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$\overset{\lambda_A}{=} {}_{\mathbb{A}}U\left[\left(\lambda_{A\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F_{\mathbb{A}}U\varepsilon^{\widetilde{C}}{}_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$= {}_{\mathbb{A}}U\left[\left(\lambda_{A\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}F\varepsilon^C{}_{\mathbb{A}}U_{\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}FCu_A\right)\right]$$

$$\overset{\varepsilon^C}{=} {}_{\mathbb{A}}U\left[\left(\lambda_{A\mathbb{A}}F\right)\circ\left({}_{\mathbb{A}}Fu_A\right)\circ\left({}_{\mathbb{A}}F\varepsilon^C\right)\right]$$

$$\overset{(\lambda_A,u_A)\mathrm{adj}}{=} {}_{\mathbb{A}}U_{\mathbb{A}}F\varepsilon^C = A\varepsilon^C$$

so that

$$\left(\varepsilon^CA\right)\circ(\Phi) = A\varepsilon^C.$$

Therefore Φ is a mixed distributive law.
Conversely let $\Phi\in\mathfrak{D}$. Then we know that $b(\Phi)=\widetilde{C}$ is a functor $\widetilde{C}:{}_{\mathbb{A}}\mathcal{A}\to{}_{\mathbb{A}}\mathcal{A}$ that is a lifting of C (i.e. ${}_{\mathbb{A}}U\widetilde{C}=C_{\mathbb{A}}U$). We have to prove that such a $\widetilde{C}$ gives rise to a comonad on the category ${}_{\mathbb{A}}\mathcal{A}$. Let us prove that Δ^C and ε^C are $\mathbb{A}$-modules morphisms. Indeed, for every $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$, by Lemma 5.3 we have

$${}^A\mu_{CX} = \left(C^A\mu_X\right)\circ(\Phi X)$$

and also

$${}^A\mu_{CCX} = \left(C^A\mu_{CX}\right)\circ(\Phi CX) = \left(CC^A\mu_X\right)\circ(C\Phi X)\circ(\Phi CX).$$

Then we have

$$\begin{aligned}
{}^{A}\mu_{CCX} \circ \left(A\Delta^{C}X\right) &= \left(CC^{A}\mu_{X}\right) \circ (C\Phi X) \circ (\Phi CX) \circ \left(A\Delta^{C}X\right) \\
&\overset{\Phi\text{m.d.l.}}{=} \left(CC^{A}\mu_{X}\right) \circ \left(\Delta^{C}AX\right) \circ (\Phi X) \overset{\Delta^{C}}{=} \left(\Delta^{C}X\right) \circ \left(C^{A}\mu_{X}\right) \circ (\Phi X)
\end{aligned}$$

and

$$\begin{aligned}
\left(\varepsilon^{C}X\right) \circ \left({}^{A}\mu_{CX}\right) &= \left(\varepsilon^{C}X\right) \circ \left(C^{A}\mu_{X}\right) \circ (\Phi X) \\
&\overset{\varepsilon^{C}}{=} {}^{A}\mu_{X} \circ \left(\varepsilon^{C}AX\right) \circ (\Phi X) \overset{\Phi\text{m.d.l.}}{=} {}^{A}\mu_{X} \circ \left(A\varepsilon^{C}X\right).
\end{aligned}$$

Thus Δ^{C} and ε^{C} lift to functorial morphisms $\Delta^{\widetilde{C}}$ and $\varepsilon^{\widetilde{C}}$ uniquely defined by

$${}_{\mathbb{A}}U\Delta^{\widetilde{C}} = \Delta^{C}{}_{\mathbb{A}}U \qquad \text{and} \qquad {}_{\mathbb{A}}U\varepsilon^{\widetilde{C}} = \varepsilon^{C}{}_{\mathbb{A}}U.$$

We compute

$$\begin{aligned}
\left({}_{\mathbb{A}}U\widetilde{C}\Delta^{\widetilde{C}}\right) \circ \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right) &= \left(C{}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) = \left(C\Delta^{C}{}_{\mathbb{A}}U\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) \\
= \left[\left(C\Delta^{C}\right) \circ \Delta^{C}\right]{}_{\mathbb{A}}U &\overset{C\text{comonad}}{=} \left[\left(\Delta^{C}C\right) \circ \Delta^{C}\right]{}_{\mathbb{A}}U = \left(\Delta^{C}C{}_{\mathbb{A}}U\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) \\
&= \left(\Delta^{C}{}_{\mathbb{A}}U\widetilde{C}\right) \circ \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right) = \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\widetilde{C}\right) \circ \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right)
\end{aligned}$$

and since ${}_{\mathbb{A}}U$ is faithful , we deduce

$$\left(\widetilde{C}\Delta^{\widetilde{C}}\right) \circ \Delta^{\widetilde{C}} = \left(\Delta^{\widetilde{C}}\widetilde{C}\right) \circ \Delta^{\widetilde{C}}.$$

We compute

$$\begin{aligned}
\left({}_{\mathbb{A}}U\widetilde{C}\varepsilon^{\widetilde{C}}\right) \circ \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right) &= \left(C{}_{\mathbb{A}}U\varepsilon^{\widetilde{C}}\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) = \left(C\varepsilon^{C}{}_{\mathbb{A}}U\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) \\
&= \left[\left(C\varepsilon^{C}\right) \circ \Delta^{C}\right]{}_{\mathbb{A}}U \overset{C\text{comonad}}{=} C{}_{\mathbb{A}}U = {}_{\mathbb{A}}U\widetilde{C}
\end{aligned}$$

and since ${}_{\mathbb{A}}U$ is faithful, we obtain

$$\left(\widetilde{C}\varepsilon^{\widetilde{C}}\right) \circ \Delta^{\widetilde{C}} = \widetilde{C}.$$

Similarly we compute

$$\begin{aligned}
\left({}_{\mathbb{A}}U\varepsilon^{\widetilde{C}}\widetilde{C}\right) \circ \left({}_{\mathbb{A}}U\Delta^{\widetilde{C}}\right) &= \left(\varepsilon^{C}{}_{\mathbb{A}}U\widetilde{C}\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) = \left(\varepsilon^{C}C{}_{\mathbb{A}}U\right) \circ \left(\Delta^{C}{}_{\mathbb{A}}U\right) \\
&= \left[\left(\varepsilon^{C}C\right) \circ \Delta^{C}\right]{}_{\mathbb{A}}U \overset{C\text{comonad}}{=} C{}_{\mathbb{A}}U = {}_{\mathbb{A}}U\widetilde{C}
\end{aligned}$$

and since ${}_{\mathbb{A}}U$ is faithful, we obtain

$$\left(\varepsilon^{\widetilde{C}}\widetilde{C}\right) \circ \Delta^{\widetilde{C}} = \widetilde{C}.$$

Therefore $\widetilde{\mathbb{C}} = \left(\widetilde{C}, \Delta^{\widetilde{C}}, \varepsilon^{\widetilde{C}}\right)$ is a comonad on ${}_{\mathbb{A}}\mathcal{A}$.

Similarly, in order to prove the bijection between $\mathfrak{D}$ and $\mathfrak{M}$, we apply Proposition 4.23, taking both $\left(C, \Delta^{C}, \varepsilon^{C}\right), \left(D, \Delta^{D}, \varepsilon^{D}\right) = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ comonad on $\mathcal{A}$ and $T = A$. In particular we will prove that the bijection $a : \mathcal{F} \to \mathcal{M}, b : \mathcal{M} \to \mathcal{F}$ of Proposition 4.23 induces a bijection between $\mathfrak{M}$ and $\mathfrak{D}$.

Let $\widetilde{\mathbb{A}} \in \mathfrak{M}$. We have to prove that $\Phi = a\left(\widetilde{A}\right) = \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \in \mathfrak{D}$. We have

$$\begin{aligned}
&(Cm_A) \circ (\Phi A) \circ (A\Phi) = \\
&(Cm_A) \circ \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}A\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}FA\right) \circ \left(A{}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left(A{}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \\
&\overset{\text{Alift}}{=} \left({}^{\mathbb{C}}U{}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}A\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}FA\right) \circ \left({}^{\mathbb{C}}U\widetilde{A}{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\widetilde{A}\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \\
&= {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}FA\varepsilon^{C}A\right) \circ \left(\gamma^{C}\widetilde{A}{}^{\mathbb{C}}FA\right) \circ \left(\widetilde{A}{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left(\widetilde{A}\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\gamma^{C}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}FA\varepsilon^{C}A\right) \circ \left({}^{\mathbb{C}}F{}^{\mathbb{C}}U\widetilde{A}{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}F{}^{\mathbb{C}}U\widetilde{A}\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\text{Alift}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}FA\varepsilon^{C}A\right) \circ \left({}^{\mathbb{C}}FA{}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}FA{}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\varepsilon^{C}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}FAA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}FA\varepsilon^{C}AC\right) \circ \left({}^{\mathbb{C}}FA{}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\text{Alift}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fm_A\right) \circ \left({}^{\mathbb{C}}FAA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}FA\varepsilon^{C\mathbb{C}}U\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left({}^{\mathbb{C}}FA{}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\left(\varepsilon^{C},\gamma^{C}\right)\text{adj},m_A}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}Fm_AC\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\text{Alift}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}F{}^{\mathbb{C}}Um_{\widetilde{A}}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C}\widetilde{A}\widetilde{A}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\gamma^{C}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left(\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(m_{\widetilde{A}}{}^{\mathbb{C}}F\right)\right] \\
&= \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left({}^{\mathbb{C}}Um_{\widetilde{A}}{}^{\mathbb{C}}F\right) \overset{\text{Alift}}{=} \Phi \circ (m_AC)
\end{aligned}$$

so that we get

$$(Cm_A) \circ (\Phi A) \circ (A\Phi) = \Phi \circ (m_AC).$$

Moreover we have

$$\begin{aligned}
\Phi \circ (u_AC) &= \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ (u_AC) \\
&\overset{\text{Alift}}{=} \left({}^{\mathbb{C}}U{}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left({}^{\mathbb{C}}Uu_{\widetilde{A}}{}^{\mathbb{C}}F\right) \\
&= {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left(\gamma^{C}\widetilde{A}{}^{\mathbb{C}}F\right) \circ \left(u_{\widetilde{A}}{}^{\mathbb{C}}F\right)\right] \\
&\overset{\gamma^{C}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}F{}^{\mathbb{C}}Uu_{\widetilde{A}}{}^{\mathbb{C}}F\right) \circ \left(\gamma^{C\mathbb{C}}F\right)\right] \\
&\overset{\text{Alift}}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}FA\varepsilon^{C}\right) \circ \left({}^{\mathbb{C}}Fu_AC\right) \circ \left(\gamma^{C\mathbb{C}}F\right)\right] \\
&\overset{u_A}{=} {}^{\mathbb{C}}U\left[\left({}^{\mathbb{C}}Fu_A\right) \circ \left({}^{\mathbb{C}}F\varepsilon^{C}\right) \circ \left(\gamma^{C\mathbb{C}}F\right)\right] \\
&\overset{\left(\varepsilon^{C},\gamma^{C}\right)\text{adj}}{=} {}^{\mathbb{C}}U{}^{\mathbb{C}}Fu_A = Cu_A
\end{aligned}$$

so that we get

$$\Phi \circ (u_AC) = Cu_A.$$

Therefore Φ is a mixed distributive law.
Conversely let $\Phi \in \mathfrak{D}$. Then we know that $\widetilde{A} = b(\Phi)$ (with notations of Proposition

4.23) is a functor $\widetilde{A} : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{C}}\mathcal{A}$ that is a lifting of A (i.e. ${}^{\mathbb{C}}U\widetilde{A} = A{}^{\mathbb{C}}U$). We have to prove that such a $\widetilde{A}$ gives rise to a monad on the category ${}^{\mathbb{C}}\mathcal{A}$. Let us prove that m_A and u_A are $\mathbb{C}$-comodule morphisms. Indeed, for every $\left(X, {}^{C}\rho_X\right) \in {}^{\mathbb{C}}\mathcal{A}$, by Lemma 5.5 we have

$$ {}^{C}\rho_{AX} = (\Phi X) \circ \left(A{}^{C}\rho_X\right) $$

and also

$$ {}^{C}\rho_{AAX} = (\Phi AX) \circ \left(A{}^{C}\rho_{AX}\right) = (\Phi AX) \circ (A\Phi X) \circ \left(AA{}^{C}\rho_X\right). $$

Then we have

$$ \begin{aligned} (Cm_A X) \circ {}^{C}\rho_{AAX} &= (Cm_A X) \circ (\Phi AX) \circ (A\Phi X) \circ \left(AA{}^{C}\rho_X\right) \\ &\overset{\Phi\text{m.d.l.}}{=} (\Phi X) \circ (m_A CX) \circ \left(AA{}^{C}\rho_X\right) \\ &\overset{m_A}{=} (\Phi X) \circ \left(A{}^{C}\rho_X\right) \circ (m_A X) = {}^{C}\rho_{AX} \circ (m_A X) \end{aligned} $$

and

$$ \begin{aligned} {}^{C}\rho_{AX} \circ (u_A X) &= (\Phi X) \circ \left(A{}^{C}\rho_X\right) \circ (u_A X) \\ &\overset{u_A}{=} (\Phi X) \circ (u_A CX) \circ {}^{C}\rho_X \overset{\Phi\text{m.d.l.}}{=} (Cu_A X) \circ {}^{C}\rho_X \end{aligned} $$

Thus m_A and u_A lift to functorial morphisms $m_{\widetilde{A}}$ and $u_{\widetilde{A}}$ uniquely defined by

$$ {}^{\mathbb{C}}Um_{\widetilde{A}} = m_A{}^{\mathbb{C}}U \qquad \text{and} \qquad {}^{\mathbb{C}}Uu_{\widetilde{A}} = u_A{}^{\mathbb{C}}U. $$

We compute

$$ \begin{aligned} \left({}^{\mathbb{C}}Um_{\widetilde{A}}\right) \circ \left({}^{\mathbb{C}}Um_{\widetilde{A}}\widetilde{A}\right) &\overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(m_A{}^{\mathbb{C}}U\widetilde{A}\right) \overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(m_A A{}^{\mathbb{C}}U\right) \\ &\overset{A\text{monad}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(Am_A{}^{\mathbb{C}}U\right) \overset{A\text{lift}}{=} \left({}^{\mathbb{C}}Um_{\widetilde{A}}\right) \circ \left(A{}^{\mathbb{C}}Um_{\widetilde{A}}\right) \overset{A\text{lift}}{=} \left({}^{\mathbb{C}}Um_{\widetilde{A}}\right) \circ \left({}^{\mathbb{C}}U\widetilde{A}m_{\widetilde{A}}\right) \end{aligned} $$

and since ${}^{\mathbb{C}}U$ is faithful , we deduce

$$ m_{\widetilde{A}} \circ \left(m_{\widetilde{A}}\widetilde{A}\right) = m_{\widetilde{A}} \circ \left(\widetilde{A}m_{\widetilde{A}}\right). $$

We compute

$$ \begin{aligned} \left({}^{\mathbb{C}}Um_{\widetilde{A}}\right) \circ \left({}^{\mathbb{C}}Uu_{\widetilde{A}}\widetilde{A}\right) &\overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(u_A{}^{\mathbb{C}}U\widetilde{A}\right) \\ &\overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(u_A A{}^{\mathbb{C}}U\right) \overset{A\text{monad}}{=} A{}^{\mathbb{C}}U \overset{A\text{lift}}{=} {}^{\mathbb{C}}U\widetilde{A} \end{aligned} $$

and since ${}^{\mathbb{C}}U$ is faithful, we obtain

$$ m_{\widetilde{A}} \circ \left(u_{\widetilde{A}}\widetilde{A}\right) = \widetilde{A}. $$

Similarly we compute

$$ \begin{aligned} \left({}^{\mathbb{C}}Um_{\widetilde{A}}\right) \circ \left({}^{\mathbb{C}}U\widetilde{A}u_{\widetilde{A}}\right) &\overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(A{}^{\mathbb{C}}Uu_{\widetilde{A}}\right) \overset{A\text{lift}}{=} \left(m_A{}^{\mathbb{C}}U\right) \circ \left(Au_A{}^{\mathbb{C}}U\right) \\ &\overset{A\text{monad}}{=} A{}^{\mathbb{C}}U \overset{A\text{lift}}{=} {}^{\mathbb{C}}U\widetilde{A} \end{aligned} $$

and since ${}^{\mathbb{C}}U$ is faithful, we obtain

$$ m_{\widetilde{A}} \circ \left(\widetilde{A}u_{\widetilde{A}}\right) = \widetilde{A}. $$

Therefore $\widetilde{\mathbb{A}}=\left(\widetilde{A},m_{\widetilde{A}},u_{\widetilde{A}}\right)$ is a monad on ${}^{\mathbb{C}}\mathcal{A}$. $\square$

6. (Co)Pretorsors and (co)herds

In this section we collect the material we need in the following or we want to introduce in this thesis about them, starting from pretorsor and copretorsor, through herds and coherds, concluding with the tame and cotame case. From time to time we decide whether or not include the details of the results, proving at least one of the two cases and having in mind that the other could also be obtained by dualizing it. In general we give the proof only for the less-known case even if it is not the first presented.

6.1. **Pretorsors.**

Proposition 6.1 ([BM, Lemma 4.8]). *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $P:\mathcal{A}\to\mathcal{B}$, $Q:\mathcal{B}\to\mathcal{A}$ and $A:\mathcal{A}\to\mathcal{A}$ be functors. Assume that all the functors P,Q and A preserve equalizers. Let $u_A:\mathcal{A}\to A$ be a functorial monomorphism and assume that $(\mathcal{A},u_A)=\mathrm{Equ}_{\mathrm{Fun}}\left(u_AA,Au_A\right)$. Let $\tau:Q\to QPQ$ be a functorial morphism such that*

$$(QP\tau)\circ\tau=(\tau PQ)\circ\tau.$$

and let $\sigma^A:QP\to A$ be a functorial morphism such that

$$\left(\sigma^AQ\right)\circ\tau=u_AQ.$$

Let $\omega^l=\left(QP\sigma^A\right)\circ(\tau P)$ and $\omega^r=QPu_A:QP\to QPA$. Set

$$(C,i)=\mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l,\omega^r\right). \tag{60}$$

There exists a functorial morphism ${}^C\rho_Q:Q\to CQ$ such that

$$(iQ)\circ{}^C\rho_Q=\tau. \tag{61}$$

There exist functorial morphisms $\Delta^C:C\to CC$ and $\varepsilon^C:C\to\mathcal{A}$ such that $\mathbb{C}=\left(C,\Delta^C,\varepsilon^C\right)$ is a comonad over $\mathcal{A}$ and C preserves equalizers. The functorial morphisms Δ^C and ε^C are uniquely determined by

$$\left({}^C\rho_QP\right)\circ i=(Ci)\circ\Delta^C\qquad\text{and}\qquad\sigma^A\circ i=u_A\circ\varepsilon^C \tag{62}$$

or equivalently

$$(\tau P)\circ i=(ii)\circ\Delta^C\qquad\text{and}\qquad\sigma^A\circ i=u_A\circ\varepsilon^C. \tag{63}$$

Moreover $(Q,{}^C\rho_Q)$ is a left $\mathbb{C}$-comodule functor.

Proposition 6.2 ([BM, Lemma 4.8]). *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $P:\mathcal{A}\to\mathcal{B}$, $Q:\mathcal{B}\to\mathcal{A}$, and $B:\mathcal{B}\to\mathcal{B}$ be functors. Assume that all the functors P,Q and B preserve equalizers. Let $u_B:\mathcal{B}\to B$ be a functorial monomorphism and assume that $(\mathcal{B},u_B)=\mathrm{Equ}_{\mathrm{Fun}}\left(u_BB,Bu_B\right)$. Let $\tau:Q\to QPQ$ be a functorial morphism such that*

$$(QP\tau)\circ\tau=(\tau PQ)\circ\tau$$

Let $\sigma^B:PQ\to B$ be a functorial morphism such that

$$\left(Q\sigma^B\right)\circ\tau=Qu_B.$$

Let $\theta^l = \left(\sigma^B PQ\right) \circ (P\tau)$ *and* $\theta^r = u_B PQ : PQ \to BPQ$*. Set*

$$(D, j) = \mathrm{Equ}_{\mathrm{Fun}}\left(\theta^l, \theta^r\right). \tag{64}$$

There exists a functorial morphism $\rho_Q^D : Q \to QD$ *such that*

$$(Qj) \circ \rho_Q^D = \tau. \tag{65}$$

There exist functorial morphisms $\Delta^D : D \to DD$ *and* $\varepsilon^D : D \to \mathcal{B}$ *such that* $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ *is a comonad over* $\mathcal{B}$ *and* D *preserves equalizers. The functorial morphisms* Δ^D *and* ε^D *are uniquely determined by*

$$(jD) \circ \Delta^D = \left(P\rho_Q^D\right) \circ j \qquad \text{and} \qquad \sigma^B \circ j = u_B \circ \varepsilon^D \tag{66}$$

or equivalently

$$(P\tau) \circ j = (jj) \circ \Delta^D \qquad \text{and} \qquad \sigma^B \circ j = u_B \circ \varepsilon^D. \tag{67}$$

Moreover $\left(Q, \rho_Q^D\right)$ *is a right* $\mathbb{D}$*-comodule functor.*

Definition 6.3. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *preformal dual structure* is a eightuple $\Xi = \left(A, B, P, Q, \sigma^A, \sigma^B, u_A, u_B\right)$ where $A : \mathcal{A} \to \mathcal{A}$, $B : \mathcal{B} \to \mathcal{B}$, $P : \mathcal{A} \to \mathcal{B}$ and $Q : \mathcal{B} \to \mathcal{A}$ are functors, $\sigma^A : QP \to A, \sigma^B : PQ \to B, u_A : \mathcal{A} \to A, u_B : \mathcal{B} \to B$ are functorial morphisms. A *pretorsor* τ for Ξ is a functorial morphism $\tau : Q \to QPQ$ satisfying the following conditions.

1) Associativity, in the sense that

$$(QP\tau) \circ \tau = (\tau PQ) \circ \tau \tag{68}$$

2) Unitality, in the sense that

$$(\sigma^A Q) \circ \tau = u_A Q \tag{69}$$

and

$$(Q\sigma^B) \circ \tau = Qu_B. \tag{70}$$

Definition 6.4. A preformal dual structure $\Xi = \left(P, Q, A, B, \sigma^A, \sigma^B, u_A, u_B,\right)$ will be called *regular* whenever $(\mathcal{A}, u_A) = \mathrm{Equ}_{\mathrm{Fun}}(u_A A, A u_A)$ and
$(\mathcal{B}, u_B) = \mathrm{Equ}_{\mathrm{Fun}}(u_B B, B u_B)$. In this case a pretorsor for Ξ will be called a *regular pretorsor*.

Theorem 6.5 ([BM, Lemma 4.8]). *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories with equalizers and let* $\tau : Q \to QPQ$ *be a regular pretorsor for* $\Xi = \left(A, B, P, Q, \sigma^A, \sigma^B, u_A, u_B\right)$*. Assume that the underlying functors* P, Q, A *and* B *preserve equalizers. Let* $\omega^l = \left(QP\sigma^A\right) \circ (\tau P)$ *and* $\omega^r = QPu_A : QP \to QPA$*. Set*

$$(C, i) = \mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l, \omega^r\right). \tag{71}$$

Then there exists a functorial morphism ${}^C\rho_Q : Q \to CQ$ *such that*

$$(iQ) \circ {}^C\rho_Q = \tau. \tag{72}$$

There exist functorial morphisms $\Delta^C : C \to CC$ *and* $\varepsilon^C : C \to \mathcal{A}$ *such that* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ *is a comonad over* $\mathcal{A}$ *and* C *preserves equalizers. The functorial morphisms* Δ^C *and* ε^C *are uniquely determined by*

$$(\tau P) \circ i = (ii) \circ \Delta^C \qquad \text{and} \qquad \sigma^A \circ i = u_A \circ \varepsilon^C. \tag{73}$$

Moreover $(Q, {}^{C}\rho_{Q})$ *is a left* $\mathbb{C}$*-comodule functor.*

Let $\theta^{l} = \left(\sigma^{B}PQ\right) \circ (P\tau)$ *and* $\theta^{r} = u_{B}PQ : PQ \to BPQ$*. Set*

$$(D, j) = \mathrm{Equ}_{\mathrm{Fun}}\left(\theta^{l}, \theta^{r}\right). \tag{74}$$

There exists a functorial morphism $\rho_{Q}^{D} : Q \to QD$ *such that*

$$(Qj) \circ \rho_{Q}^{D} = \tau. \tag{75}$$

There exist functorial morphisms $\Delta^{D} : D \to DD$ *and* $\varepsilon^{D} : D \to \mathcal{B}$ *such that* $\mathbb{D} = \left(D, \Delta^{D}, \varepsilon^{D}\right)$ *is a comonad over* $\mathcal{B}$ *and* D *preserves equalizers. The functorial morphisms* Δ^{D} *and* ε^{D} *are uniquely determined by*

$$(P\tau) \circ j = (jj) \circ \Delta^{D} \qquad \text{and} \qquad \sigma^{B} \circ j = u_{B} \circ \varepsilon^{D}. \tag{76}$$

Moreover $\left(Q, \rho_{Q}^{D}\right)$ *is a right* $\mathbb{D}$*-comodule functor.*
Finally $\left(Q, {}^{C}\rho_{Q}, \rho_{Q}^{D}\right)$ *is a* $\mathbb{C}$*-*$\mathbb{D}$*-bicomodule functor.*

Proof. See the dual Theorem 6.29. $\square$

THEOREM 6.6. *Let* $\Xi = \left(P, Q, A, B, \sigma^{A}, \sigma^{B}, u_{A}, u_{B},\right)$ *be a regular preformal dual structure on categories* $\mathcal{A}$ *and* $\mathcal{B}$ *such that the functors* P, Q, A, B *preserve equalizers and let* $\tau : Q \to QPQ$ *be a pretorsor for* Ξ*. Assume that* A *and* B *are monads,* $\left(P, {}^{B}\mu_{P}\right)$ *is a left* $\mathbb{B}$*-module functor and* $\left(P, \mu_{P}^{A}\right)$ *is a right* $\mathbb{A}$*-module functor. Moreover assume that the functorial morphism* σ^{A} *is right* A*-linear, that is* $\sigma^{A} \circ \left(Q\mu_{P}^{A}\right) = m_{A} \circ \left(\sigma^{A}A\right)$ *and the functorial morphism* σ^{B} *is left* B*-linear that is* $\sigma^{B} \circ \left({}^{B}\mu_{P}Q\right) = m_{B} \circ \left(B\sigma^{B}\right)$ *and that they are compatible in the sense that*

$${}^{B}\mu_{P} \circ \left(\sigma^{B}P\right) = \mu_{P}^{A} \circ \left(P\sigma^{A}\right). \tag{77}$$

Then there exists a comonad $\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ *on the category* $\mathcal{A}$ *together with a functorial morphism* ${}^{C}\rho_{Q} : Q \to CQ$ *such that* $\left(Q, {}^{C}\rho_{Q}\right)$ *is a left* $\mathbb{C}$*-comodule functor and a comonad* $\mathbb{D} = \left(D, \Delta^{D}, \varepsilon^{D}\right)$ *together with a functorial morphism* $\rho_{Q}^{D} : Q \to QD$ *such that* $\left(Q, \rho_{Q}^{D}\right)$ *is a right* $\mathbb{D}$*-comodule functor. The underlying functors are defined as follows*

$$(C, i) = \mathrm{Equ}_{\mathrm{Fun}}\left(\left(QP\sigma^{A}\right) \circ (\tau P), QPu_{A}\right)$$

and

$$(D, j) = \mathrm{Equ}_{\mathrm{Fun}}\left(\left(\sigma^{B}PQ\right) \circ (P\tau), u_{B}PQ\right).$$

satisfying

$$(iQ) \circ {}^{C}\rho_{Q} = \tau \text{ and } (Qj) \circ \rho_{Q}^{D} = \tau.$$

Furthermore

1) *The morphism* $\mathrm{can}_{1} := \left(C\sigma^{A}\right) \circ \left({}^{C}\rho_{Q}P\right) : QP \to CA$ *is an isomorphism.*
2) *The morphism* $\overline{\mathrm{can}_{1}} := \left(\sigma^{B}D\right) \circ \left(P\rho_{Q}^{D}\right) : PQ \to BD$ *is an isomorphisms.*
3)
$$\left(QP\sigma^{A}\right) \circ (\tau P) = (iA) \circ \mathrm{can}_{1} \text{ and } \left(\sigma^{B}PQ\right) \circ (P\tau) = (Bj) \circ \overline{\mathrm{can}_{1}}$$
4)

$$i = \mathrm{can}_{1}^{-1} \circ (Cu_{A}) \tag{78}$$

$$j = \left(\overline{\mathrm{can}_{1}}\right)^{-1} \circ (u_{B}D) \tag{79}$$

5)
$$\sigma^A = \left(\varepsilon^C A\right) \circ \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right)$$

6)
$$\sigma^B = \left(B\varepsilon^D\right) \circ \left(\sigma^B D\right) \circ \left(P\rho_Q^D\right)$$

From the last equalities, we deduce that, σ^A is a regular epimorphism if and only if so is $\varepsilon^C A$ and σ^B is a regular epimorphism if and only if so is $B\varepsilon^D$.

Proof. See the dual Theorem 6.30. ∎

6.2. **Herds.** Following [BV], we recall some definition about herds.

Definition 6.7. A *formal dual structure on two categories $\mathcal{A}$ and $\mathcal{B}$* is a sextuple $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ where $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ are monads on $\mathcal{A}$ and $\mathcal{B}$ respectively and $\left(A, B, P, Q, \sigma^A, \sigma^B, u_A, u_B\right)$ is a preformal dual structure. Moreover $\left(P : \mathcal{A} \to \mathcal{B}, {}^B\mu_P : BP \to P, \mu_P^A : PA \to P\right)$ and $\left(Q : \mathcal{B} \to \mathcal{A}, {}^A\mu_Q : AQ \to Q, \mu_Q^B : QB \to Q\right)$ are bimodule functors; $\sigma^A : QP \to A, \sigma^B : PQ \to B$ are subject to the following conditions: σ^A is A-bilinear and σ^B is B-bilinear

$$\sigma^A \circ \left({}^A\mu_Q P\right) = m_A \circ \left(A\sigma^A\right) \text{ and } \sigma^A \circ \left(Q\mu_P^A\right) = m_A \circ \left(\sigma^A A\right) \tag{80}$$

$$\sigma^B \circ \left({}^B\mu_P Q\right) = m_B \circ \left(B\sigma^B\right) \text{ and } \sigma^B \circ \left(P\mu_Q^B\right) = m_B \circ \left(\sigma^B B\right) \tag{81}$$

and the associative conditions hold

$${}^A\mu_Q \circ \left(\sigma^A Q\right) = \mu_Q^B \circ \left(Q\sigma^B\right) \text{ and } {}^B\mu_P \circ \left(\sigma^B P\right) = \mu_P^A \circ \left(P\sigma^A\right). \tag{82}$$

Definition 6.8. Consider a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ in the sense of the previous definition. A *herd* for $\mathbb{M}$ is a pretorsor $\tau : Q \to QPQ$ i.e.

$$(QP\tau) \circ \tau = (\tau PQ) \circ \tau, \tag{83}$$

$$(\sigma^A Q) \circ \tau = u_A Q \tag{84}$$

and

$$(Q\sigma^B) \circ \tau = Qu_B. \tag{85}$$

Definition 6.9. A formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ will be called *regular* whenever $\left(A, B, P, Q, \sigma^A, \sigma^B, u_A, u_B\right)$ is a regular preformal dual structure. In this case a herd for $\mathbb{M}$ will be called a *regular herd*.

Lemma 6.10. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a formal dual structure and let $\tau : Q \to QPQ$ be a herd for $\mathbb{M}$. Assume that the underlying functors A and B reflect equalizers. Then τ is a regular herd.*

Proof. Since $\mathbb{A}$ and $\mathbb{B}$ are monads, we have $m_A \circ (Au_A) = \mathrm{Id}_A$ and $m_B \circ (Bu_B) = \mathrm{Id}_B$. Thus, Au_A and Bu_B are split monomorphisms and thus monomorphisms. Since A and B reflect equalizers, we deduce that also u_A and u_B are monomorphisms and thus $(\mathcal{A}, u_A) = \mathrm{Equ}_{\mathrm{Fun}}(u_A A, Au_A)$ and $(\mathcal{B}, u_B) = \mathrm{Equ}_{\mathrm{Fun}}(u_B B, Bu_B)$, i.e. τ is a regular herd. ∎

PROPOSITION 6.11. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a formal dual structure such that the lifted functors ${}_AQ_B : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ and ${}_BP_A : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ determine an equivalence of categories. Then $({}_AQ, P_A)$ and $({}_BP, Q_B)$ are adjunctions.*

Proof. Since $({}_{\mathbb{A}}F, {}_{\mathbb{A}}U)$ and $({}_{\mathbb{B}}F, {}_{\mathbb{B}}U)$ are adjunctions, $({}_AQ_{B\mathbb{B}}F, {}_{\mathbb{B}}U_BP_A) = ({}_AQ, P_A)$ and $({}_BP_{A\mathbb{A}}F, {}_{\mathbb{A}}U_AQ_B) = ({}_BP, Q_B)$ are also adjunctions. $\square$

6.3. **Herds and comonads.**

THEOREM 6.12 ([Bo]). *Let $\mathcal{A}$ and $\mathcal{B}$ be categories in both of which the equalizer of any pair of parallel morphisms exists. Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a formal dual structure on two categories $\mathcal{A}$ and $\mathcal{B}$. Then we have*

(1) *If $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ is a comonad on the category $\mathcal{A}$ and $\left(Q, {}^C\rho_Q : Q \to CQ\right)$ is a left C-comodule functor such that*
   (i) *the functorial morphism $\mathrm{can}_1 := \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) : QP \to CA$ is an isomorphism*
   (ii) *the functorial morphism $\mathrm{can}_2 := \left(C\mu_Q^B\right) \circ \left({}^C\rho_Q B\right) : QB \to CQ$ is an isomorphism*
   then $\tau := \left(\mathrm{can}_1^{-1}Q\right) \circ (Cu_AQ) \circ {}^C\rho_Q : Q \to QPQ$ is a pretorsor and thus a herd.
(2) *If $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ is a comonad on the category $\mathcal{B}$ and $\left(Q, \rho_Q^D : Q \to QD\right)$ is a right D-comodule functor such that*
   (i) *the functorial morphism $\overline{\mathrm{can}_1} := \left(\sigma^B D\right) \circ \left(P\rho_Q^D\right) : PQ \to BD$ is an isomorphism*
   (ii) *the functorial morphism $\overline{\mathrm{can}_2} := \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) : AQ \to QD$ is an isomorphism*
   then $\tau := (Q\overline{\mathrm{can}_1}^{-1}) \circ (Qu_BD) \circ \rho_Q^D : Q \to QPQ$ is a pretorsor and thus a herd.

Proof. See the dual Theorem 6.36. $\square$

THEOREM 6.13 ([Bo]). *Let $\mathcal{A}$ and $\mathcal{B}$ be categories in both of which the equalizer of any pair of parallel morphisms exists. Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a regular formal dual structure such that the underlying functors A, B, P and Q preserve equalizers, then the existence of the following structures are equivalent:*

(a) *A herd $\tau : Q \to QPQ$ in $\mathbb{M}$;*
(b) *A comonad $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ on the category $\mathcal{A}$ such that the functor C preserves equalizers and $\left(Q, {}^C\rho_Q : Q \to CQ\right)$ is a left $\mathbb{C}$-comodule functor subject to the following conditions*
   (i) *the functorial morphism $\mathrm{can}_1 := \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) : QP \to CA$ is an isomorphism*
   (ii) *the functorial morphism $\mathrm{can}_2 := \left(C\mu_Q^B\right) \circ \left({}^C\rho_Q B\right) : QB \to CQ$ is an isomorphism;*
(c) *A comonad $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ on the category $\mathcal{B}$ such that the functor D preserves equalizers and $\left(Q, \rho_Q^D : Q \to QD\right)$ is a right $\mathbb{D}$-comodule functor subject to the following conditions*
   (i) *the functorial morphism $\overline{\mathrm{can}_1} := \left(\sigma^B D\right) \circ \left(P\rho_Q^D\right) : PQ \to BD$ is an isomorphism*

(ii) *the functorial morphism* $\overline{\mathrm{can}_2} := \left({}^{A}\mu_Q D\right) \circ \left(A\rho_Q^D\right) : AQ \to QD$ *is an isomorphism.*

Proof. See the dual Theorem 6.37. $\square$

6.4. **Herds and distributive laws.**

PROPOSITION 6.14 ([Bo]). *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $\tau : Q \to QPQ$ be a regular herd for $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ where the underlying functors $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{B} \to \mathcal{B}$ preserve equalizers. Let $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ and $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ be the associated comonads constructed in Proposition 6.1 and in Proposition 6.2. Then*

1) *There exists a mixed distributive law between the comonad $\mathbb{C}$ and the monad $\mathbb{A}$, $\Phi : AC \to CA$ such that*
$$(iA) \circ \Phi = \phi = \left(QP\sigma^A\right) \circ (\tau P) \circ \left({}^{A}\mu_Q P\right) \circ (Ai)\,.$$
2) *There exists an opposite mixed distributive law between the comonad $\mathbb{D}$ and the monad $\mathbb{B}$, $\Psi : DB \to BD$ such that*
$$(Bj) \circ \Psi = \psi = \left(\sigma^B PQ\right) \circ (P\tau) \circ \left(P\mu_Q^B\right) \circ (jB)\,.$$

Proof. See the dual Proposition 6.38. $\square$

6.5. **Herds and Galois functors.**

LEMMA 6.15. *Let $\mathbb{M} = \left(\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B\right)$ be a formal dual structure where $Q : \mathcal{B} \to \mathcal{A}$, $P : \mathcal{A} \to \mathcal{B}$ and $\mathbb{A} = (A, m_A, u_A)$ is a monad on the category $\mathcal{A}$ and $\mathbb{B} = (B, m_B, u_B)$ is a monad on $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and that A, QB preserve them. Then $\sigma^A : QP \to A$ induces a morphism $\sigma_A^A : QP_A \to {}_{\mathbb{A}}U$ in ${}_{\mathbb{A}}\mathcal{A}$ and hence there exists a morphism ${}_A\sigma_A^A$: ${}_AQP_A \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ such that*

$$ {}_{\mathbb{A}}U_A\sigma_A^A = \sigma_A^A. \tag{86}$$

Moreover $\sigma_{A\mathbb{A}}^A F = \sigma^A : QP_{A\mathbb{A}}F = QP \to {}_{\mathbb{A}}U_{\mathbb{A}}F = A$.

Proof. Let us consider the following diagram with notations of Proposition 3.30

$$\begin{tikzcd}
QPA_{\mathbb{A}}U \arrow[r, shift left, "Q\mu_{P\mathbb{A}}^A U"] \arrow[r, shift right, "QP_{\mathbb{A}}U\lambda_A"'] \arrow[d, "\sigma_A A_{\mathbb{A}}U"] & QP_{\mathbb{A}}U \arrow[r, "Qp_P"] \arrow[d, "\sigma_{A\mathbb{A}}U"] & QP_A \arrow[d, dotted, "\sigma_A{}'"] \\
AA_{\mathbb{A}}U \arrow[r, shift left, "m_{A\mathbb{A}}U"] \arrow[r, shift right, "A_{\mathbb{A}}U\lambda_A"'] & A_{\mathbb{A}}U \arrow[r, "{}_{\mathbb{A}}U\lambda_A"] & {}_{\mathbb{A}}U
\end{tikzcd}$$

Since by assumption QB preserves coequalizers, by Lemma 3.19 also Q preserves coequalizers. Since $({}_{\mathbb{A}}U\lambda_A) \circ \left(\sigma^A{}_{\mathbb{A}}U\right)$ coequalizes the pair $\left(Q\mu_{P\mathbb{A}}^A U, QP_{\mathbb{A}}U\lambda_A\right)$ and $(QP_A, Qp_P) = \mathrm{Coequ}_{\mathrm{Fun}}\left(Q\mu_{P\mathbb{A}}^A U, QP_{\mathbb{A}}U\lambda_A\right)$, by the universal property of the coequalizer, there exists a unique morphism $\sigma_A^A : QP_A \to {}_{\mathbb{A}}U$ such that

$$\sigma_A^A \circ (Qp_P) = ({}_{\mathbb{A}}U\lambda_A) \circ \left(\sigma^A{}_{\mathbb{A}}U\right).$$

We now want to prove that $\sigma_A^A : QP_A = {}_{\mathbb{A}}U_AQP_A \to {}_{\mathbb{A}}U$ is a morphism between left $\mathbb{A}$-module functors which satisfies

$$({}_{\mathbb{A}}U\lambda_A) \circ \left(A\sigma_A^A\right) = \sigma_A^A \circ \left({}^{A}\mu_Q P_A\right).$$

We have

$$\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(A\sigma_{A}^{A}\right)\circ\left(AQp_{P}\right)\overset{\mathrm{def}\sigma_{A}^{A}}{=}\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(A_{\mathbb{A}}U\lambda_{A}\right)\circ\left(A\sigma_{A\mathbb{A}}^{A}U\right)$$
$$\overset{{}_{\mathbb{A}}U\lambda_{A}\mathrm{coequ}}{=}\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(m_{A\mathbb{A}}U\right)\circ\left(A\sigma^{A}{}_{\mathbb{A}}U\right)\overset{(80)}{=}\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(\sigma^{A}{}_{\mathbb{A}}U\right)\circ\left({}^{A}\mu_{Q}P_{\mathbb{A}}U\right)$$
$$\overset{\mathrm{def}\sigma_{A}^{A}}{=}\sigma_{A}^{A}\circ\left(Qp_{P}\right)\circ\left({}^{A}\mu_{Q}P_{\mathbb{A}}U\right)\overset{{}^{A}\mu_{Q}}{=}\sigma_{A}^{A}\circ\left({}^{A}\mu_{Q}P_{A}\right)\circ\left(AQp_{P}\right)$$

and since A,Q preserve coequalizers, AQp_{P} is an epimorphism, so that we get

$$\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(A\sigma_{A}^{A}\right)=\sigma_{A}^{A}\circ\left({}^{A}\mu_{Q}P_{A}\right).$$

Hence, by Lemma 3.29, there exists a unique morphism ${}_{A}\sigma_{A}^{A}\colon {}_{A}QP_{A}\rightarrow \mathrm{Id}_{\mathbb{A}\mathcal{A}}$ such that

$${}_{\mathbb{A}}U_{A}\sigma_{A}^{A}=\sigma_{A}^{A}.$$

Now, note that, by definition of σ_{A}^{A}, we have

$$\sigma_{A}^{A}\circ\left(Qp_{P}\right)=\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(\sigma^{A}{}_{\mathbb{A}}U\right)$$

so that by applying it to ${}_{\mathbb{A}}F$ we get

$$\left(\sigma_{A\mathbb{A}}^{A}F\right)\circ\left(Qp_{P\mathbb{A}}F\right)=\left({}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F\right)\circ\left(\sigma^{A}{}_{\mathbb{A}}U_{\mathbb{A}}F\right).$$

Hence, by Proposition 3.34, we obtain that

$$\left(\sigma_{A\mathbb{A}}^{A}F\right)\circ\left(Q\mu_{P}^{A}\right)=m_{A}\circ\left(\sigma^{A}A\right)\overset{(80)}{=}\sigma^{A}\circ\left(Q\mu_{P}^{A}\right).$$

Since $Q\mu_{P}^{A}$ is an epimorphism, we deduce that $\sigma_{A\mathbb{A}}^{A}F=\sigma^{A}$. $\square$

Proposition 6.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $\tau:Q\rightarrow QPQ$ be a regular herd for a formal dual structure $\mathbb{M}=(\mathbb{A},\mathbb{B},P,Q,\sigma^{A},\sigma^{B})$ where the underlying functors $P:\mathcal{A}\rightarrow\mathcal{B}$, $Q:\mathcal{B}\rightarrow\mathcal{A}$ and $A:\mathcal{A}\rightarrow\mathcal{A}$ preserve equalizers. Let*

- *$\mathbb{C}=\left(C,\Delta^{C},\varepsilon^{C}\right)$ be the comonad on the category $\mathcal{A}$ constructed in Proposition 6.1;*
- *$\left(Q,{}^{C}\rho_{Q}\right)$ be the left $\mathbb{C}$-comodule functor constructed in Proposition 6.1;*
- *${}_{A}Q:\mathcal{B}\rightarrow{}_{\mathbb{A}}\mathcal{A}$ be the functor defined in Lemma 3.29;*
- *$\Phi:AC\rightarrow CA$ be the mixed distributive law between the comonad $\mathbb{C}$ and the monad $\mathbb{A}$ constructed in Proposition 6.14;*
- *$\widetilde{\mathbb{C}}$ be the lifting of $\mathbb{C}$ on the category ${}_{\mathbb{A}}\mathcal{A}$ constructed in Theorem 5.7.*

Then there exists a functorial morphism ${}^{\widetilde{C}}\rho_{AQ}:{}_{A}Q\rightarrow\widetilde{C}_{A}Q$ such that

$${}_{\mathbb{A}}U^{\widetilde{C}}\rho_{AQ}={}^{C}\rho_{Q}.$$

Moreover, $\left({}_{A}Q,{}^{\widetilde{C}}\rho_{AQ}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor.

Proof. Since $\tau:Q\rightarrow QPQ$ is a regular herd for $\mathbb{M}=(\mathbb{A},\mathbb{B},P,Q,\sigma^{A},\sigma^{B})$, by Proposition 6.14, the mixed distributive law $\Phi:AC\rightarrow CA$ is uniquely defined by

$$(iA)\circ\Phi=\left(QP\sigma^{A}\right)\circ(\tau P)\circ\left({}^{A}\mu_{Q}P\right)\circ(Ai).$$

Now we prove that ${}^{C}\rho_{Q}$ yields a functorial morphism ${}^{\widetilde{C}}\rho_{AQ}$. In fact we have

$$(iQ)\circ\left(C^{A}\mu_{Q}\right)\circ(\Phi Q)\circ\left(A^{C}\rho_{Q}\right)\overset{i}{=}\left(QP^{A}\mu_{Q}\right)\circ(iAQ)\circ(\Phi Q)\circ\left(A^{C}\rho_{Q}\right)$$

$$
\begin{aligned}
&\overset{\mathrm{def}\Phi}{=} \left(QP^{A}\mu_{Q}\right) \circ \left(QP\sigma^{A}Q\right) \circ (\tau PQ) \circ \left({}^{A}\mu_{Q}PQ\right) \circ (AiQ) \circ \left(A^{C}\rho_{Q}\right) \\
&\overset{(61),(82)}{=} \left(QP\mu_{Q}^{B}\right) \circ \left(QPQ\sigma^{B}\right) \circ (\tau PQ) \circ \left({}^{A}\mu_{Q}PQ\right) \circ (A\tau) \\
&\overset{\tau}{=} \left(QP\mu_{Q}^{B}\right) \circ (\tau B) \circ \left(Q\sigma^{B}\right) \circ \left({}^{A}\mu_{Q}PQ\right) \circ (A\tau) \\
&\overset{{}^{A}\mu_{Q}}{=} \left(QP\mu_{Q}^{B}\right) \circ (\tau B) \circ \left({}^{A}\mu_{Q}B\right) \circ \left(AQ\sigma^{B}\right) \circ (A\tau) \\
&\overset{(70)}{=} \left(QP\mu_{Q}^{B}\right) \circ (\tau B) \circ \left({}^{A}\mu_{Q}B\right) \circ (AQu_{B}) \\
&\overset{{}^{A}\mu_{Q}}{=} \left(QP\mu_{Q}^{B}\right) \circ (\tau B) \circ (Qu_{B}) \circ {}^{A}\mu_{Q} \overset{\tau}{=} \left(QP\mu_{Q}^{B}\right) \circ (QPQu_{B}) \circ \tau \circ {}^{A}\mu_{Q} \\
&\overset{Q\mathrm{modfun}}{=} \tau \circ {}^{A}\mu_{Q} \overset{(61)}{=} (iQ) \circ {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q}
\end{aligned}
$$

and since by construction iQ is a monomorphism we get that

$$\left(C^{A}\mu_{Q}\right) \circ (\Phi Q) \circ \left(A^{C}\rho_{Q}\right) = {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q}.$$

By Lemma 5.3 we know that

$$\left(C^{A}\mu_{Q}\right) \circ (\Phi Q) = {}^{A}\mu_{CQ}$$

so that we get

$${}^{A}\mu_{CQ} \circ \left(A^{C}\rho_{Q}\right) = \left(C^{A}\mu_{Q}\right) \circ (\Phi Q) \circ \left(A^{C}\rho_{Q}\right) = {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q}.$$

Hence there exists a morphism ${}^{\widetilde{C}}\rho_{{}_{A}Q} : {}_{A}Q \to \widetilde{C}_{A}Q$ such that

$${}_{\mathbb{A}}U^{\widetilde{C}}\rho_{{}_{A}Q} = {}^{C}\rho_{Q}.$$

By the coassociativity and counitality properties of ${}^{C}\rho_{Q}$, we deduce that ${}^{\widetilde{C}}\rho_{{}_{A}Q}$ is also coassociative and counital so that $\left({}_{A}Q, {}^{\widetilde{C}}\rho_{{}_{A}Q}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor. $\square$

LEMMA 6.17. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B})$ be a formal dual structure where the underlying functors are $A : \mathcal{A} \to \mathcal{A}$, $B : \mathcal{B} \to \mathcal{B}$, $P : \mathcal{A} \to \mathcal{B}$ and $Q : \mathcal{B} \to \mathcal{A}$. Assume that both categories $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and the functors A, QB preserve them. Assume that*

- *$\mathbb{C} = \left(C, \Delta^{C}, \varepsilon^{C}\right)$ is a comonad on the category $\mathcal{A}$ such that C preserves coequalizers*
- *$\widetilde{\mathbb{C}} = \left(\widetilde{C}, \Delta^{\widetilde{C}}, \varepsilon^{\widetilde{C}}\right)$ is a lifting of the comonad of $\mathbb{C}$ to the category ${}_{\mathbb{A}}\mathcal{A}$*
- *$\left({}_{A}Q, {}^{\widetilde{C}}\rho_{{}_{A}Q}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor where ${}_{\mathbb{A}}U^{\widetilde{C}}\rho_{{}_{A}Q} = {}^{C}\rho_{Q}$.*

Consider the functorial morphisms

$$\mathrm{can}_{1} := \left(C\sigma^{A}\right) \circ \left({}^{C}\rho_{Q}P\right) : QP \to CA$$

and

$${}_{A}\mathrm{can}_{A} := \left(\widetilde{C}_{A}\sigma_{A}^{A}\right) \circ \left({}^{\widetilde{C}}\rho_{{}_{A}Q}P_{A}\right) : {}_{A}QP_{A} \to \widetilde{C}$$

Then can_{1} is an isomorphism if and only if ${}_{A}\mathrm{can}_{A}$ is an isomorphism.

Proof. Note that, since $\left({}_AQ, {}^{\widetilde{C}}\rho_{_AQ}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor, then $\left(Q, {}^C\rho_Q\right)$ is a left $\mathbb{C}$-comodule functor where ${}^C\rho_Q = {}_{\mathbb{A}}U{}^{\widetilde{C}}\rho_{_AQ}$. Let (P_A, p_P) be the coequalizer defined in (6). Now, by Lemma 6.15, σ^A induces a morphism $\sigma_A^A : QP_A \to {}_{\mathbb{A}}U$ such that
$\sigma_A^A \circ (Qp_P) = ({}_{\mathbb{A}}U\lambda_A) \circ \left(\sigma^A{}_{\mathbb{A}}U\right)$. Then, we can consider the morphism

$$\mathrm{can}_A := \left(C\sigma_A^A\right) \circ \left({}^C\rho_Q P_A\right) : QP_A = {}_{\mathbb{A}}U_A QP_A \to C_{\mathbb{A}}U = {}_{\mathbb{A}}U\widetilde{C}. \tag{87}$$

Then, by using the naturality of ${}^C\rho_Q$ and the definition of σ_A^A, we obtain

$$\mathrm{can}_A \circ (Qp_P) = (C_{\mathbb{A}}U\lambda_A) \circ (\mathrm{can}_{1\mathbb{A}}U)\,. \tag{88}$$

Moreover, by Lemma 6.15, there exists a morphism ${}_A\sigma_A^A : {}_AQP_A \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ such that ${}_{\mathbb{A}}U_A\sigma_A^A = \sigma_A^A$. Since $\widetilde{\mathbb{C}}$ is a lifting of the comonad $\mathbb{C}$, we know that $C\sigma_A^A = C_{\mathbb{A}}U_A\sigma_A^A = {}_{\mathbb{A}}U\widetilde{C}_A\sigma_A^A$. Let us set

$${}_A\mathrm{can}_A := \left(\widetilde{C}_A\sigma_A^A\right) \circ \left({}^{\widetilde{C}}\rho_{_AQ}P_A\right) : {}_AQP_A \to \widetilde{C}$$

so that we get

$${}_{\mathbb{A}}U_A\mathrm{can}_A = \left({}_{\mathbb{A}}U\widetilde{C}_A\sigma_A^A\right) \circ \left({}_{\mathbb{A}}U{}^{\widetilde{C}}\rho_{_AQ}P_A\right) = \left(C\sigma_A^A\right) \circ \left({}^C\rho_Q P_A\right) = \mathrm{can}_A. \tag{89}$$

By using the naturality of ${}^C\rho_Q$, we calculate

$$\begin{aligned}
(\mathrm{can}_{1\mathbb{A}}U) \circ \left(Q\mu_{P\mathbb{A}}^A U\right) &= \left(C\sigma^A{}_{\mathbb{A}}U\right) \circ \left({}^C\rho_Q P_{\mathbb{A}}U\right) \circ \left(Q\mu_{P\mathbb{A}}^A U\right) \\
&= \left(C\sigma^A{}_{\mathbb{A}}U\right) \circ \left(CQ\mu_{P\mathbb{A}}^A U\right) \circ \left({}^C\rho_Q PA_{\mathbb{A}}U\right) \\
&\overset{(80)}{=} (Cm_{A\mathbb{A}}U) \circ \left(C\sigma^A A_{\mathbb{A}}U\right) \circ \left({}^C\rho_Q PA_{\mathbb{A}}U\right) = (Cm_{A\mathbb{A}}U) \circ (\mathrm{can}_1 A_{\mathbb{A}}U)
\end{aligned}$$

so that we get

$$(\mathrm{can}_{1\mathbb{A}}U) \circ \left(Q\mu_{P\mathbb{A}}^A U\right) = (Cm_{A\mathbb{A}}U) \circ (\mathrm{can}_1 A_{\mathbb{A}}U)\,. \tag{90}$$

Let us consider the following diagram

$$\begin{tikzcd}
QPA_{\mathbb{A}}U \arrow[r, shift left, "Q\mu_{P\mathbb{A}}^A U"] \arrow[r, shift right, "QP_{\mathbb{A}}U\lambda_A"'] \arrow[d, "\mathrm{can}_1 A_{\mathbb{A}}U"'] & QP_{\mathbb{A}}U \arrow[r, "Qp_P"] \arrow[d, "\mathrm{can}_{1\mathbb{A}}U"'] & QP_A \arrow[r] \arrow[d, "\mathrm{can}_A"'] & 0 \\
CAA_{\mathbb{A}}U \arrow[r, shift left, "Cm_{A\mathbb{A}}U"] \arrow[r, shift right, "CA_{\mathbb{A}}U\lambda_A"'] & CA_{\mathbb{A}}U \arrow[r, "C_{\mathbb{A}}U\lambda_A"] & C_{\mathbb{A}}U = {}_{\mathbb{A}}U\widetilde{C} = CA_A \arrow[r] & 0.
\end{tikzcd}$$

Now, since $\mathrm{can}_1 : QP \to CA$ is a functorial morphism and by formula (90), the left square serially commutes. By formula (88) also the right square commutes. Moreover, by definition, p_P and ${}_{\mathbb{A}}U\lambda_A$ are coequalizers. Since Q and C preserve coequalizers, both the rows are coequalizers .

Assume now that can_1 is a functorial isomorphism. Then both $\mathrm{can}_1 A_{\mathbb{A}}U$ and $\mathrm{can}_{1\mathbb{A}}U$ are isomorphism and we deduce that also can_A is an isomorphism. Since ${}_{\mathbb{A}}U_A\mathrm{can}_A = \mathrm{can}_A$ and ${}_{\mathbb{A}}U$ reflects isomorphisms, we get that also ${}_A\mathrm{can}_A$ is an isomorphism.

Conversely, assume that ${}_A\mathrm{can}_A$ is an isomorphism. Then also $\mathrm{can}_A = {}_{\mathbb{A}}U_A\mathrm{can}_A$ is an isomorphism. Then, by using (89), (87), Lemma 6.15 and (15), we obtain

${}_{\mathbb{A}}U_{A}\mathrm{can}_{A\mathbb{A}}F = \mathrm{can}_{A\mathbb{A}}F = \left(C\sigma^{A}_{A\mathbb{A}}F\right) \circ \left({}^{C}\rho_{Q}P_{A\mathbb{A}}F\right) = \left(C\sigma^{A}\right) \circ \left({}^{C}\rho_{Q}P\right) = \mathrm{can}_{1}$ so that also can_{1} is an isomorphism. □

6.6. **The tame case.**

Definition 6.18. A formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B})$ is called a *Morita context* on the categories $\mathcal{A}$ and $\mathcal{B}$ if it satisfies also the balanced conditions

$$\sigma^{A} \circ \left(\mu^{B}_{Q}P\right) = \sigma^{A} \circ \left(Q^{B}\mu_{P}\right) \text{ and } \sigma^{B} \circ \left(P^{A}\mu_{Q}\right) = \sigma^{B} \circ \left(\mu^{A}_{P}Q\right). \tag{91}$$

Lemma 6.19. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B})$ be a Morita context on the categories $\mathcal{A}$ and $\mathcal{B}$ and assume that A, B, P, Q preserve coequalizers. Hence, there exist functorial morphisms*

- ${}_{AB}\sigma^{A}_{BA} : {}_{A}Q_{BB}P_{A} \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ *such that*

$$ {}_{\mathbb{A}}U_{AB}\sigma^{A}_{BA} = {}_{B}\sigma^{A}_{BA} \tag{92}$$

where ${}_{B}\sigma^{A}_{BA}$ is uniquely determined by ${}_{B}\sigma^{A}_{BA} \circ (Q_{B}p_{{}_{B}P}) = ({}_{\mathbb{A}}U\lambda_{A}) \circ \left({}_{B}\sigma^{A}_{B\mathbb{A}}U\right)$ and

$$ {}_{B}\sigma^{A}_{B} \circ (p_{QB}P) = \sigma^{A} \tag{93}$$

- ${}_{BA}\sigma^{B}_{AB} : {}_{B}P_{AA}Q_{B} \to \mathrm{Id}_{{}_{\mathbb{B}}\mathcal{B}}$ *such that*

$$ {}_{\mathbb{B}}U_{BA}\sigma^{B}_{AB} = {}_{A}\sigma^{B}_{AB} \tag{94}$$

where ${}_{A}\sigma^{B}_{AB}$ is uniquely determined by ${}_{A}\sigma^{B}_{AB} \circ (P_{A}p_{{}_{A}Q}) = ({}_{\mathbb{B}}U\lambda_{B}) \circ \left({}_{A}\sigma^{B}_{A\mathbb{B}}U\right)$ and

$$ {}_{A}\sigma^{B}_{A} \circ (p_{PA}Q) = \sigma^{B}. \tag{95}$$

Moreover we have that

$$ {}_{B}\sigma^{A}_{BA\mathbb{A}}F = {}_{B}\sigma^{A}_{B} \qquad \textit{and} \qquad {}_{A}\sigma^{B}_{AB\mathbb{B}}F = {}_{A}\sigma^{B}_{A}. \tag{96}$$

Proof. By definition, $(Q_{BB}P, p_{QB}P) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^{B}_{Q}P, Q^{B}\mu_{P}\right)$ and by assumption σ^{A} is balanced, so that, by the universal property of the coequalizer, there exists a unique functorial morphism ${}_{B}\sigma^{A}_{B} : Q_{BB}P \to A$ such that ${}_{B}\sigma^{A}_{B} \circ (p_{QB}P) = \sigma^{A}$. Now, let us consider the following diagram

$$\begin{tikzcd}
Q_{BB}PA_{\mathbb{A}}U \arrow[r, shift left, "Q_{B}\mu^{A}_{{}_{B}P\mathbb{A}}U"] \arrow[r, shift right, "Q_{BB}P_{\mathbb{A}}U\lambda_{A}"'] \arrow[d, "{}_{B}\sigma^{A}_{B}A_{\mathbb{A}}U"] & Q_{BB}P_{\mathbb{A}}U \arrow[r, "Q_{B}p_{{}_{B}P}"] \arrow[d, "{}_{B}\sigma^{A}_{B\mathbb{A}}U"] & Q_{BB}P_{A} \arrow[d, dotted, "{}_{B}\sigma^{A}_{B}"] \\
AA_{\mathbb{A}}U \arrow[r, shift left, "m_{A\mathbb{A}}U"] \arrow[r, shift right, "A_{\mathbb{A}}U\lambda_{A}"'] & A_{\mathbb{A}}U \arrow[r, "{}_{\mathbb{A}}U\lambda_{A}"] & {}_{\mathbb{A}}U
\end{tikzcd}$$

Note that, by naturality of p_{Q} and definition of ${}_{B}\sigma^{A}_{B}$ we have

$$\left({}_{B}\sigma^{A}_{B\mathbb{A}}U\right) \circ \left(Q_{B}\mu^{A}_{{}_{B}P\mathbb{A}}U\right) \circ (p_{QB}PA_{\mathbb{A}}U) = \left({}_{B}\sigma^{A}_{B\mathbb{A}}U\right) \circ (p_{QB}P_{\mathbb{A}}U) \circ \left(Q_{\mathbb{B}}U\mu^{A}_{{}_{B}P\mathbb{A}}U\right)$$

$$= \left(\sigma^{A}{}_{\mathbb{A}}U\right) \circ \left(Q\mu^{A}_{P\mathbb{A}}U\right) \overset{(80)}{=} (m_{A\mathbb{A}}U) \circ \left(\sigma^{A}A_{\mathbb{A}}U\right) = (m_{A\mathbb{A}}U) \circ \left({}_{B}\sigma^{A}_{B}A_{\mathbb{A}}U\right) \circ (p_{QB}PA_{\mathbb{A}}U)$$

and since $p_{QB}PA_{\mathbb{A}}U$ is an epimorphism, we get that $\left({}_B\sigma^A_{B\mathbb{A}}U\right) \circ \left(Q_B\mu^A_{{}_BP\mathbb{A}}U\right) = (m_{A\mathbb{A}}U) \circ \left({}_B\sigma^A_B A_{\mathbb{A}}U\right)$. Moreover, by using naturality of p_Q, definition of ${}_B\sigma^A_B$, naturality of σ^A we have

$$\begin{aligned}\left({}_B\sigma^A_{B\mathbb{A}}U\right) \circ (Q_{BB}P_{\mathbb{A}}U\lambda_A) \circ (p_{QB}PA_{\mathbb{A}}U) &= \left({}_B\sigma^A_{B\mathbb{A}}U\right) \circ (p_{QB}P_{\mathbb{A}}U) \circ (Q_{\mathbb{B}}U_BP_{\mathbb{A}}U\lambda_A)\\ &= \left(\sigma^A{}_{\mathbb{A}}U\right) \circ (Q_{\mathbb{B}}U_BP_{\mathbb{A}}U\lambda_A) = (A_{\mathbb{A}}U\lambda_A) \circ \left(\sigma^A{}_{\mathbb{A}}U_AF_{\mathbb{A}}U\right)\\ &= (A_{\mathbb{A}}U\lambda_A) \circ \left({}_B\sigma^A_BA_{\mathbb{A}}U\right) \circ (p_{QB}PA_{\mathbb{A}}U)\end{aligned}$$

and since $p_{QB}PA_{\mathbb{A}}U$ is an epimorphism, we get that $\left({}_B\sigma^A_{B\mathbb{A}}U\right) \circ (Q_{BB}P_{\mathbb{A}}U\lambda_A) = (A_{\mathbb{A}}U\lambda_A) \circ \left({}_B\sigma^A_BA_{\mathbb{A}}U\right)$ so that the left square serially commutes. Since B, P, Q preserve coequalizers, by Corollary 2.12, also $Q_{BB}P = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U_BP, Q_{\mathbb{B}}U\lambda_{BB}P\right)$ preserves them so that both the rows are coequalizers. Hence, there exists a unique functorial morphism ${}_B\sigma^A_{BA} : Q_{BB}P_A \to {}_{\mathbb{A}}U$ such that

$$\text{(97)} \qquad {}_B\sigma^A_{BA} \circ (Q_Bp_{{}_BP}) = ({}_{\mathbb{A}}U\lambda_A) \circ \left({}_B\sigma^A_{B\mathbb{A}}U\right).$$

Now, by using naturality of ${}^A\mu_{Q_B}$, definition of ${}_B\sigma^A_{BA}$, definition of ${}_B\sigma^A_B$, coequalizing property of ${}_{\mathbb{A}}U\lambda_A$, we compute

$$\begin{aligned}&{}_B\sigma^A_{BA} \circ \left({}^A\mu_{Q_BB}P_A\right) \circ (AQ_Bp_{{}_BP}) \circ (Ap_{QB}P_{\mathbb{A}}U)\\ &= {}_B\sigma^A_{BA} \circ (Q_Bp_{{}_BP}) \circ \left({}^A\mu_{Q_BB}P_{\mathbb{A}}U\right) \circ (Ap_{QB}P_{\mathbb{A}}U)\\ &\overset{(7)}{=} ({}_{\mathbb{A}}U\lambda_A) \circ \left({}_B\sigma^A_{B\mathbb{A}}U\right) \circ (p_{QB}P_{\mathbb{A}}U) \circ \left({}^A\mu_{Q\mathbb{B}}U_BP_{\mathbb{A}}U\right)\\ &= ({}_{\mathbb{A}}U\lambda_A) \circ \left(\sigma^A{}_{\mathbb{A}}U\right) \circ \left({}^A\mu_{Q\mathbb{B}}U_BP_{\mathbb{A}}U\right)\\ &\overset{(80)}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (m_{A\mathbb{A}}U) \circ \left(A\sigma^A{}_{\mathbb{A}}U\right) = ({}_{\mathbb{A}}U\lambda_A) \circ (A_{\mathbb{A}}U\lambda_A) \circ \left(A\sigma^A{}_{\mathbb{A}}U\right)\\ &= ({}_{\mathbb{A}}U\lambda_A) \circ (A_{\mathbb{A}}U\lambda_A) \circ \left(A_B\sigma^A_{B\mathbb{A}}U\right) \circ (Ap_{QB}P_{\mathbb{A}}U)\\ &= ({}_{\mathbb{A}}U\lambda_A) \circ \left(A_B\sigma^A_{BA}\right) \circ (AQ_Bp_{{}_BP}) \circ (Ap_{QB}P_{\mathbb{A}}U)\end{aligned}$$

and since $(AQ_Bp_{{}_BP}) \circ (Ap_{QB}P_{\mathbb{A}}U)$ is an epimorphism, we get ${}_B\sigma^A_{BA} \circ \left({}^A\mu_{Q_BB}P_A\right) = ({}_{\mathbb{A}}U\lambda_A) \circ \left(A_B\sigma^A_{BA}\right)$ so that ${}_B\sigma^A_{BA}$ induces a functorial morphism ${}_{AB}\sigma^A_{BA} : {}_AQ_{BB}P_A \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ such that ${}_{\mathbb{A}}U_{AB}\sigma^A_{BA} = {}_B\sigma^A_{BA}$. Similarly, one can prove that there exists a unique functorial morphism ${}_A\sigma^B_A : P_{AA}Q \to B$ such that ${}_A\sigma^B_A \circ (p_{PA}Q) = \sigma^B$ and it induces a unique functorial morphism ${}_{BA}\sigma^B_{AB} : {}_BP_{AA}Q_B \to \mathrm{Id}_{{}_{\mathbb{B}}\mathcal{B}}$ such that ${}_{\mathbb{B}}U_{BA}\sigma^B_{AB} = {}_A\sigma^B_{AB}$ where ${}_A\sigma^B_{AB}$ is uniquely determined by ${}_A\sigma^B_{AB} \circ (P_Ap_{AQ}) = ({}_{\mathbb{B}}U\lambda_B) \circ \left({}_A\sigma^B_{A\mathbb{B}}U\right)$.

Finally we compute

$$\begin{aligned}\left({}_B\sigma^A_{BA\mathbb{A}}F\right) \circ (Q_Bp_{{}_BP\mathbb{A}}F) \circ (p_{QB}PA) &\overset{(97)}{=} ({}_{\mathbb{A}}U\lambda_{A\mathbb{A}}F) \circ \left({}_B\sigma^A_{B\mathbb{A}}U_{\mathbb{A}}F\right) \circ (p_{QB}PA)\\ = m_A \circ \left({}_B\sigma^A_BA\right) \circ (p_{QB}PA) &\overset{(93)}{=} m_A \circ \left(\sigma^AA\right) \overset{(80)}{=} \sigma^A \circ \left(Q\mu^A_P\right)\\ \overset{(93)}{=} {}_B\sigma^A_B \circ (p_{QB}P) \circ \left(Q\mu^A_P\right) &\overset{(11)}{=} {}_B\sigma^A_B \circ (p_{QB}P) \circ \left(Q_{\mathbb{B}}U\mu^A_{{}_BP}\right)\\ \overset{p_Q}{=} {}_B\sigma^A_B \circ \left(Q_B\mu^A_{{}_BP}\right) \circ (p_{QB}PA) &\overset{(13),(14)}{=} {}_B\sigma^A_B \circ (Q_Bp_{{}_BP\mathbb{A}}F) \circ (p_{QB}PA).\end{aligned}$$

Since B and Q preserve coequalizers, by Corollary 2.12, also Q_B preserves them so that $\left(Q_B\mu^A_{{}_BP}\right) \circ (p_{QB}PA)$ is epi and we deduce that

$${}_B\sigma^A_{BA\mathbb{A}}F = {}_B\sigma^A_B.$$

Similarly, one can prove the statement for ${}_{A}\sigma^{B}_{AB\mathbb{B}}F = {}_{A}\sigma^{B}_{A}$. $\square$

DEFINITIONS 6.20. Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B})$ be a Morita context. We will say that $\mathbb{M}$ is *tame* if the lifted functorial morphisms ${}_{AB}\sigma^{A}_{BA} : {}_{A}Q_{BB}P_{A} \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ and ${}_{BA}\sigma^{B}_{AB} : {}_{B}P_{AA}Q_{B} \to \mathrm{Id}_{{}_{\mathbb{B}}\mathcal{B}}$ are isomorphisms so that the lifted functors ${}_{A}Q_{B} : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ and ${}_{B}P_{A} : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ yield a category equivalence. In this case, if $\tau : Q \to QPQ$ is a herd for $\mathbb{M}$, we will say that τ is a *tame herd*.

PROPOSITION 6.21. *Let $\mathbb{M} = \left(\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B}\right)$ be a tame Morita context. Then unit and counit of the adjunction $({}_{A}Q_{B}, {}_{B}P_{A})$ are given by*
$\eta_{({}_{A}Q_{B},{}_{B}P_{A})} = \left({}_{BA}\sigma^{B}_{ABB}P_{AA}Q_{B}\right) \circ \left({}_{B}P_{A}\left({}_{AB}\sigma^{A}_{BA}\right)^{-1}{}_{A}Q_{B}\right) \circ \left({}_{BA}\sigma^{B}_{AB}\right)^{-1}$ *and* $\epsilon_{({}_{A}Q_{B},{}_{B}P_{A})} = {}_{AB}\sigma^{A}_{BA}$ *so that*
$\eta_{({}_{A}Q,P_{A})} = \left({}_{\mathbb{B}}U_{BA}\sigma^{B}_{ABB}P_{AA}Q_{B\mathbb{B}}F\right) \circ \left({}_{\mathbb{B}}U_{B}P_{A}\left({}_{AB}\sigma^{A}_{BA}\right)^{-1}{}_{A}Q_{B\mathbb{B}}F\right) \circ \left({}_{\mathbb{B}}U\left({}_{BA}\sigma^{B}_{AB}\right)^{-1}{}_{\mathbb{B}}F\right) \circ u_{B}$ *and* $\epsilon_{({}_{A}Q,P_{A})} = {}_{AB}\sigma^{A}_{BA} \circ \left({}_{A}Q_{B}\lambda_{BB}P_{A}\right)$.

Proof. It is a well-known fact that, given the two functorial isomorphisms $\sigma : \mathrm{Id} \to RL$ and $\epsilon : LR \to \mathrm{Id}$ associated to an equivalence of categories, the unit of an adjunction is given by $\eta = (\sigma^{-1}RL) \circ (R\epsilon^{-1}L) \circ \sigma$ and the counit is ϵ. Hence, since the isomorphisms are $\epsilon = {}_{AB}\sigma^{A}_{BA} : {}_{A}Q_{BB}P_{A} \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ and $\sigma^{-1} = {}_{BA}\sigma^{B}_{AB} : {}_{B}P_{AA}Q_{B} \to \mathrm{Id}_{{}_{\mathbb{B}}\mathcal{B}}$ the unit is $\eta_{({}_{A}Q_{B},{}_{B}P_{A})} = \left({}_{BA}\sigma^{B}_{ABB}P_{AA}Q_{B}\right) \circ \left({}_{B}P_{A}\left({}_{AB}\sigma^{A}_{BA}\right)^{-1}{}_{A}Q_{B}\right) \circ \left({}_{BA}\sigma^{B}_{AB}\right)^{-1}$ and the counit is $\epsilon_{({}_{A}Q_{B},{}_{B}P_{A})} = {}_{AB}\sigma^{A}_{BA}$. Note that, by Proposition 6.11, $({}_{A}Q, P_{A}) = ({}_{A}Q_{B\mathbb{B}}F, {}_{\mathbb{B}}U_{B}P_{A})$ and $({}_{B}P, Q_{B}) = ({}_{B}P_{A\mathbb{A}}F, {}_{\mathbb{A}}U_{A}Q_{B})$ are adjunctions. Hence, the unit of the adjunction $({}_{A}Q, P_{A})$ is $\eta_{({}_{A}Q,P_{A})} = \left({}_{\mathbb{B}}U\eta_{({}_{A}Q_{B},{}_{B}P_{A})\mathbb{B}}F\right) \circ \eta_{({}_{\mathbb{B}}F,{}_{\mathbb{B}}U)}$ and thus
$\eta_{({}_{A}Q,P_{A})} = \left({}_{\mathbb{B}}U_{BA}\sigma^{B}_{ABB}P_{AA}Q_{B\mathbb{B}}F\right) \circ \left({}_{\mathbb{B}}U_{B}P_{A}\left({}_{AB}\sigma^{A}_{BA}\right)^{-1}{}_{A}Q_{B\mathbb{B}}F\right) \circ \left({}_{\mathbb{B}}U\left({}_{BA}\sigma^{B}_{AB}\right)^{-1}{}_{\mathbb{B}}F\right) \circ u_{B}$. The counit of the adjunction $({}_{A}Q_{B\mathbb{B}}F, {}_{\mathbb{B}}U_{B}P_{A}) = ({}_{A}Q, P_{A})$ is given by $\epsilon_{({}_{A}Q,P_{A})} = \epsilon_{({}_{A}Q_{B},{}_{B}P_{A})} \circ \left({}_{A}Q_{B}\epsilon_{({}_{\mathbb{B}}F,{}_{\mathbb{B}}U)B}P_{A}\right) = {}_{AB}\sigma^{A}_{BA} \circ \left({}_{A}Q_{B}\lambda_{BB}P_{A}\right)$. A similar result holds for the other adjunction. $\square$

COROLLARY 6.22. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^{A}, \sigma^{B})$ be a tame Morita context. Assume that the functors A, B, P, Q preserve coequalizers. Then the counits of the adjunctions $({}_{A}Q, P_{A})$ and $({}_{B}P, Q_{B})$ are given by $\epsilon_{({}_{A}Q,P_{A})} = {}_{A}\sigma^{A}_{A}$ and $\epsilon_{({}_{B}P,Q_{B})} = {}_{B}\sigma^{B}_{B}$.*

Proof. By Proposition 6.11 $({}_{A}Q, P_{A})$ and $({}_{B}P, Q_{B})$ are adjunctions. Let us consider the functorial morphism ${}_{A}\sigma^{A}_{A} : {}_{A}QP_{A} \to \mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ constructed in Lemma 6.15 satisfying ${}_{\mathbb{A}}U_{A}\sigma^{A}_{A\mathbb{A}}F = \sigma^{A}_{A\mathbb{A}}F = \sigma^{A}$. By using naturality of μ^{B}_{Q}, definition of σ^{A}_{A}, the balanced property of σ^{A}, we compute

$$\begin{aligned}\sigma^{A}_{A} \circ \left(\mu^{B}_{Q}P_{A}\right) \circ (QBp_{P}) &= \sigma^{A}_{A} \circ (Qp_{P}) \circ \left(\mu^{B}_{Q}P_{\mathbb{A}}U\right)\\ = ({}_{\mathbb{A}}U\lambda_{A}) \circ \left(\sigma^{A}{}_{\mathbb{A}}U\right) \circ \left(\mu^{B}_{Q}P_{\mathbb{A}}U\right) &= ({}_{\mathbb{A}}U\lambda_{A}) \circ \left(\sigma^{A}{}_{\mathbb{A}}U\right) \circ \left(Q^{B}\mu_{P\mathbb{A}}U\right)\\ = \sigma^{A}_{A} \circ (Qp_{P}) \circ \left(Q^{B}\mu_{P\mathbb{A}}U\right) &\overset{(7)}{=} \sigma^{A}_{A} \circ \left(Q^{B}\mu_{P_{A}}\right) \circ (QBp_{P})\end{aligned}$$

and since QBp_{P} is an epimorphism, we get that

$$\sigma^{A}_{A} \circ \left(\mu^{B}_{Q}P_{A}\right) = \sigma^{A}_{A} \circ \left(Q^{B}\mu_{P_{A}}\right)$$

i.e.

$$\left({}_{\mathbb{A}}U_{A}\sigma^{A}_{A}\right) \circ \left({}_{\mathbb{A}}U\mu^{B}_{{}_{A}Q}P_{A}\right) = \left({}_{\mathbb{A}}U_{A}\sigma^{A}_{A}\right) \circ \left({}_{\mathbb{A}}U_{A}Q^{B}\mu_{P_{A}}\right).$$

Since ${}_{\mathbb{A}}U$ reflects and $\left({}_{A}Q_{BB}P_{A},p_{{}_{A}Q_{B}}P_{A}\right)=\mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^{B}_{Q}P_{A},{}_{A}Q^{B}\mu_{P_{A}}\right)$, there exists a unique functorial morphism ${}_{AB}\overline{\sigma}^{A}_{BA}:{}_{A}Q_{BB}P_{A}\rightarrow\mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$ such that

$$ {}_{AB}\overline{\sigma}^{A}_{BA}\circ\left(p_{{}_{A}Q_{B}}P_{A}\right)={}_{A}\sigma^{A}_{A}. $$

Using definition of σ^{A}_{A}, ${}_{B}\sigma^{A}_{B}$ and ${}_{B}\sigma^{A}_{BA}$, naturality of λ_{B} we compute

$$ \begin{aligned} &\left({}_{\mathbb{A}}U_{AB}\overline{\sigma}^{A}_{BA}\right)\circ\left({}_{\mathbb{A}}Up_{{}_{A}Q_{B}}P_{A}\right)\circ\left(Qp_{P}\right)={}_{\mathbb{A}}U\left[{}_{AB}\overline{\sigma}^{A}_{BA}\circ\left(p_{{}_{A}Q_{B}}P_{A}\right)\right]\circ\left(Qp_{P}\right)\\ &=\left({}_{\mathbb{A}}U_{A}\sigma^{A}_{A}\right)\circ\left(Qp_{P}\right)=\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left(\sigma^{A}{}_{\mathbb{A}}U\right)=\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left({}_{B}\sigma^{A}_{B\mathbb{A}}U\right)\circ\left(p_{Q_{B}}P_{\mathbb{A}}U\right)\\ &\overset{(14)}{=}\left({}_{\mathbb{A}}U\lambda_{A}\right)\circ\left({}_{B}\sigma^{A}_{B\mathbb{A}}U\right)\circ\left(Q_{B}\lambda_{BB}P_{\mathbb{A}}U\right)\\ &={}_{B}\sigma^{A}_{BA}\circ\left(Q_{B}p_{{}_{B}P}\right)\circ\left(Q_{B}\lambda_{BB}P_{\mathbb{A}}U\right)={}_{B}\sigma^{A}_{BA}\circ\left(Q_{B}\lambda_{BB}P_{A}\right)\circ\left(Q_{B\mathbb{B}}F_{\mathbb{B}}Up_{{}_{B}P}\right)\\ &={}_{B}\sigma^{A}_{BA}\circ\left(Q_{B}\lambda_{BB}P_{A}\right)\circ\left(Qp_{P}\right)=\left({}_{\mathbb{A}}U_{AB}\sigma^{A}_{BA}\right)\circ\left({}_{\mathbb{A}}U_{A}Q_{B}\lambda_{BB}P_{A}\right)\circ\left(Qp_{P}\right) \end{aligned} $$

and since Qp_{P} is an epimorphism and ${}_{\mathbb{A}}U$ reflects and by definition of ${}_{AB}\overline{\sigma}^{A}_{BA}$ we get

$$ {}_{AB}\sigma^{A}_{BA}\circ\left({}_{A}Q_{B}\lambda_{BB}P_{A}\right)={}_{AB}\overline{\sigma}^{A}_{BA}\circ\left(p_{{}_{A}Q_{B}}P_{A}\right)={}_{A}\sigma^{A}_{A} $$

so that $\epsilon_{({}_{A}Q,P_{A})}={}_{AB}\sigma^{A}_{BA}\circ\left({}_{A}Q_{B}\lambda_{BB}P_{A}\right)={}_{A}\sigma^{A}_{A}$. □

LEMMA 6.23. *Let $\mathbb{M}=(\mathbb{A},\mathbb{B},P,Q,\sigma_{A},\sigma_{B})$ be a formal dual structure where the underlying functors are $A:\mathcal{A}\rightarrow\mathcal{A}$, $B:\mathcal{B}\rightarrow\mathcal{B}$, $P:\mathcal{A}\rightarrow\mathcal{B}$ and $Q:\mathcal{B}\rightarrow\mathcal{A}$. Assume that both categories $\mathcal{A}$ and $\mathcal{B}$ have coequalizers and the functors A,QB preserve them. Assume that*

- *$\mathbb{C}=\left(C,\Delta^{C},\varepsilon^{C}\right)$ is a comonad on the category $\mathcal{A}$ such that C preserves coequalizers*
- *$\widetilde{\mathbb{C}}=\left(\widetilde{C},\Delta^{\widetilde{C}},\varepsilon^{\widetilde{C}}\right)$ is a lifting of the comonad $\mathbb{C}$ to the category ${}_{\mathbb{A}}\mathcal{A}$*
- *$\left({}_{A}Q,{}^{\widetilde{C}}\rho_{{}_{A}Q}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor*
- *$\mathbb{M}$ is a tame Morita context.*

Then can_1 *is an isomorphism if and only if* ${}_{A}\mathrm{can}_{A}$ *is an isomorphism if and only if ${}_{A}Q$ is a left $\widetilde{\mathbb{C}}$-Galois functor.*

Proof. Assume that $\mathbb{M}$ is a tame Morita context. Then, by Corollary 6.22, $({}_{A}Q,P_{A})$ is an adjunction with counit $\epsilon:={}_{A}\sigma^{A}_{A}:{}_{A}QP_{A}\rightarrow\mathrm{Id}_{{}_{\mathbb{A}}\mathcal{A}}$. Then, ${}_{A}Q$ is a left $\widetilde{\mathbb{C}}$-Galois functor if and only if the morphism $\left(\widetilde{C}_{A}\sigma^{A}_{A}\right)\circ\left({}^{\widetilde{C}}\rho_{{}_{A}Q}P_{A}\right)={}_{A}\mathrm{can}_{A}$ is an isomorphism. By using Lemma 6.17 we deduce that can_1 is an isomorphism if and only if ${}_{A}\mathrm{can}_{A}$ is an isomorphism if and only if ${}_{A}Q$ is a left $\widetilde{\mathbb{C}}$-Galois functor. □

The following Theorem is a formulation, in pure categorical terms, of [BV, Theorem 2.18].

THEOREM 6.24. *Let $\mathbb{M}=(\mathbb{A},\mathbb{B},P,Q,\sigma^{A},\sigma^{B})$ be a regular tame Morita context. Assume that*

- *both categories $\mathcal{A}$ and $\mathcal{B}$ have equalizers and coequalizers,*
- *the functors A and B preserve equalizers,*
- *the functors A,B,P,Q preserve coequalizers.*

Then the existence of the following structures are equivalent:

(a) *A herd $\tau:Q\rightarrow QPQ$ for $\mathbb{M}$*

(b) *A comonad* $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ *on the category* $\mathcal{A}$ *such that the functor* C *preserves equalizers and a mixed distributive law* $\Phi : AC \to CA$ *such that* ${}_AQ$ *is a Galois comodule functor over* $\widetilde{\mathbb{C}}$ *(where* $\widetilde{\mathbb{C}}$ *is the lifting of* $\mathbb{C}$*)*

(c) *A comonad* $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ *on the category* $\mathcal{B}$ *such that the functor* D *preserves equalizers and an opposite mixed distributive law* $\Psi : DB \to BD$ *such that* ${}_BP$ *is a Galois comodule functor over* $\widetilde{\mathbb{D}}$ *(where* $\widetilde{\mathbb{D}}$ *is the lifting of* $\mathbb{D}$*).*

Proof. By Proposition 6.11 the pairs $({}_AQ, P_A)$ and $({}_BP, Q_B)$ are adjunctions and hence P_A and Q_B preserve equalizers. Since $A = {}_{\mathbb{A}}U_{\mathbb{A}}F$ and $B = {}_{\mathbb{B}}U_{\mathbb{B}}F$ preserve equalizers, by Lemma 3.22 also ${}_{\mathbb{A}}F$ and ${}_{\mathbb{B}}F$ preserve them so that, in view of (15), we get that $P = P_{A\mathbb{A}}F$ and $Q = Q_{B\mathbb{B}}F$ preserve equalizers.
$(a) \Rightarrow (b)$ Assume that $\tau : Q \to QPQ$ is a herd for $\mathbb{M} = \left(\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B\right)$. By Proposition 6.14 there exists a mixed distributive law $\Phi : AC \to CA$ such that

$$(iA) \circ \Phi = \left(QP\sigma^A\right) \circ (\tau P) \circ \left({}^A\mu_Q P\right) \circ (Ai)\,.$$

Then, by Theorem 5.7, there exists a lifting comonad $\widetilde{\mathbb{C}} = \left(\widetilde{C}, \Delta^{\widetilde{C}}, \varepsilon^{\widetilde{C}}\right)$ on the category ${}_{\mathbb{A}}\mathcal{A}$. By Proposition 6.16, there exists a functorial morphism ${}^{\widetilde{C}}\rho_{{}_AQ} : {}_AQ \to \widetilde{C}{}_AQ$ such that ${}_{\mathbb{A}}U^{\widetilde{C}}\rho_{{}_AQ} = {}^C\rho_Q$ and $\left({}_AQ, {}^{\widetilde{C}}\rho_{{}_AQ}\right)$ is a left $\widetilde{\mathbb{C}}$-comodule functor. Since by assumption we have a regular formal dual structure, by Theorem 6.6, the functorial morphism $\mathrm{can}_1 := \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) : QP \to CA$ is an isomorphism and so, by Lemma 6.23, ${}_AQ$ is a left $\widetilde{\mathbb{C}}$-Galois functor.

$(b) \Rightarrow (a)$ Follows by [BM, Theorem 4.4 (1)] where $(\mathcal{T}, (N_A, R_A), (N_B, R_B), C, \xi) = ({}_{\mathbb{A}}\mathcal{A}, ({}_{\mathbb{A}}F, {}_{\mathbb{A}}U), ({}_AQ, P_A), \mathbb{C}, {}_{\mathbb{A}}U_A\mathrm{can}_A)$ noting that a pretorsor for a formal dual structure is a herd. $\square$

6.7. **Copretorsors.**

Proposition 6.25. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories with coequalizers and let* $P : \mathcal{A} \to \mathcal{B}, Q : \mathcal{B} \to \mathcal{A}$*, and* $C : \mathcal{A} \to \mathcal{A}$ *be functors. Assume that all the functors* P, Q *and* C *preserve coequalizers. Let* $\varepsilon^C : C \to \mathcal{A}$ *be a functorial morphism and assume that* $\left(\mathcal{A}, \varepsilon^C\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(C\varepsilon^C, \varepsilon^C C\right)$*. Let* $\chi : QPQ \to Q$ *be a functorial morphism such that*

$$\chi \circ (QP\chi) = \chi \circ (\chi PQ) \tag{98}$$

and let $\delta_C : C \to QP$ *be a functorial morphism such that*

$$\chi \circ (\delta_C Q) = \varepsilon^C Q. \tag{99}$$

Let $w^l = (\chi P) \circ (QP\delta_C)$ *and* $w^r = QP\varepsilon^C : QPC \to QP$*. Set*

$$(A, x) = \mathrm{Coequ}_{\mathrm{Fun}}\left(w^l, w^r\right). \tag{100}$$

There exists a functorial morphism ${}^A\mu_Q : AQ \to Q$ *such that*

$${}^A\mu_Q \circ (xQ) = \chi. \tag{101}$$

There exist functorial morphisms $m_A : AA \to A$ *and* $u_A : \mathcal{A} \to A$ *such that* $\mathbb{A} = (A, m_A, u_A)$ *is a monad over* $\mathcal{A}$ *that preserves coequalizers. Moreover* m_A *and* u_A

are uniquely determined by

$$x \circ (\chi P) = m_A \circ (xx) \tag{102}$$

and

$$u_A \circ \varepsilon^C = x \circ \delta_C. \tag{103}$$

Finally $\left(Q, {}^A\mu_Q\right)$ *is a left* $\mathbb{A}$*-module functor.*

Proof. We have

$$\begin{aligned} \chi \circ \left(w^l Q\right) = \chi \circ (\chi P Q) \circ (Q P \delta_C Q) &\overset{(98)}{=} \chi \circ (Q P \chi) \circ (Q P \delta_C Q) \\ &\overset{99}{=} \chi \circ \left(Q P \varepsilon^C Q\right) = \chi \circ (w^r Q)\,. \end{aligned}$$

Hence

$$\chi \circ \left(w^l Q\right) = \chi \circ (w^r Q)\,.$$

By Lemma 2.9, we have that

$$(AQ, xQ) = \mathrm{Coequ}_{\mathrm{Fun}}\left(w^l Q, w^r Q\right)$$

and hence there exists a unique functorial morphism ${}^A\mu_Q : AQ \to Q$ which fulfils (101). We compute

$$\begin{aligned} x \circ ({}^A\mu_Q P) \circ (A w^l) \circ (xQPC) &\overset{x}{=} x \circ ({}^A\mu_Q P) \circ (xQP) \circ \left(QPw^l\right) \\ &\overset{(101)}{=} x \circ (\chi P) \circ \left(QPw^l\right) = x \circ (\chi P) \circ (QP\chi P) \circ (QPQP\delta_C) \\ &\overset{(98)}{=} x \circ (\chi P) \circ (\chi PQP) \circ (QPQP\delta_C) \\ &\overset{\chi}{=} x \circ (\chi P) \circ (QP\delta_C) \circ (\chi PC) = x \circ w^l \circ (\chi PC) \\ &\overset{x\mathrm{coequ}}{=} x \circ w^r \circ (\chi PC) = x \circ \left(QP\varepsilon^C\right) \circ (\chi PC) \\ &\overset{\chi}{=} x \circ (\chi P) \circ \left(QPQP\varepsilon^C\right) \overset{(101)}{=} x \circ ({}^A\mu_Q P) \circ (xQP) \circ \left(QPQP\varepsilon^C\right) \\ &\overset{x}{=} x \circ ({}^A\mu_Q P) \circ \left(AQP\varepsilon^C\right) \circ (xQPC) = x \circ ({}^A\mu_Q P) \circ (Aw^r) \circ (xQPC) \end{aligned}$$

so that we get

$$x \circ ({}^A\mu_Q P) \circ (A w^l) \circ (xQPC) = x \circ ({}^A\mu_Q P) \circ (Aw^r) \circ (xQPC)$$

and since $xQPC$ is an epimorphism we deduce that

$$x \circ ({}^A\mu_Q P) \circ (A w^l) = x \circ ({}^A\mu_Q P) \circ (Aw^r).$$

By Corollary 2.12, A preserves coequalizers so that we get

$$(AA, Ax) = \mathrm{Coequ}_{\mathrm{Fun}}\left(Aw^l, Aw^r\right).$$

Hence there exists a unique functorial morphism $m_A : AA \to A$ such that

$$m_A \circ (Ax) = x \circ ({}^A\mu_Q P) \tag{104}$$

or equivalently

$$m_A \circ (xx) = m_A \circ (Ax) \circ (xQP) = x \circ ({}^A\mu_Q P) \circ (xQP) \overset{(101)}{=} x \circ (\chi P)\,.$$

We calculate

$$\begin{aligned}
x \circ \delta_C \circ \left(C\varepsilon^C\right) &\overset{\delta_C}{=} x \circ \left(QP\varepsilon^C\right) \circ (\delta_C C) = x \circ w^r \circ (\delta_C C) \\
&\overset{x\mathrm{coequ}}{=} x \circ w^l \circ (\delta_C C) = x \circ (\chi P) \circ (QP\delta_C) \circ (\delta_C C) \\
&\overset{\delta_C}{=} x \circ (\chi P) \circ (\delta_C QP) \circ (C\delta_C) \overset{(99)}{=} x \circ \left(\varepsilon^C QP\right) \circ (C\delta_C) \\
&\overset{\varepsilon^C}{=} x \circ \delta_C \circ (\varepsilon^C C)
\end{aligned}$$

so that we get

$$x \circ \delta_C \circ (C\varepsilon^C) = x \circ \delta_C \circ (\varepsilon^C C).$$

Since $\left(\mathcal{A}, \varepsilon^C\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(C\varepsilon^C, \varepsilon^C C\right)$ there exists a unique functorial morphism $u_A : \mathcal{A} \to A$ such that (103) is fulfilled. Now we want to show that $\mathbb{A} = (A, m_A, u_A)$ is a monad over $\mathcal{A}$ that is

$$\begin{aligned}
m_A \circ (m_A A) &= m_A \circ (A m_A) \\
m_A \circ (A u_A) &= A = m_A \circ (u_A A)\,.
\end{aligned}$$

We calculate

$$\begin{aligned}
m_A \circ (m_A A) \circ (xxx) &= m_A \circ (m_A A) \circ (xxA) \circ (QPQPx) \\
&\overset{(102)}{=} m_A \circ (xA) \circ (\chi PA) \circ (QPQPx) \\
&\overset{\chi}{=} m_A \circ (xA) \circ (QPx) \circ (\chi PQP) = m_A \circ (xx) \circ (\chi PQP) \\
&\overset{(102)}{=} x \circ (\chi P) \circ (\chi PQP) \overset{(98)}{=} x \circ (\chi P) \circ (QP\chi P) \\
&\overset{(102)}{=} m_A \circ (xx) \circ (QP\chi P) = m_A \circ (xA) \circ (QPx) \circ (QP\chi P) \\
&\overset{(102)}{=} m_A \circ (xA) \circ (QPm_A) \circ (QPxx) \overset{x}{=} m_A \circ (Am_A) \circ (xAA) \circ (QPxx) \\
&= m_A \circ (Am_A) \circ (xxx)\,.
\end{aligned}$$

Thus we get that

$$m_A \circ (m_A A) \circ (xxx) = m_A \circ (Am_A) \circ (xxx)$$

and since xxx is an epimorphism, we deduce that m_A is associative. We compute

$$\begin{aligned}
m_A \circ (Au_A) \circ \left(A\varepsilon^C\right) \circ (xC) &\overset{(103)}{=} m_A \circ (Ax) \circ (A\delta_C) \circ (xC) \\
&\overset{x}{=} m_A \circ (Ax) \circ (xQP) \circ (QP\delta_C) = m_A \circ (xx) \circ (QP\delta_C) \\
&\overset{(102)}{=} x \circ (\chi P) \circ (QP\delta_C) = x \circ w^l \\
&= x \circ w^r = x \circ (QP\varepsilon^C) \overset{x}{=} \left(A\varepsilon^C\right) \circ (xC)\,.
\end{aligned}$$

Thus we get that

$$m_A \circ (Au_A) \circ \left(A\varepsilon^C\right) \circ (xC) = \left(A\varepsilon^C\right) \circ (xC)\,.$$

and since $\left(A\varepsilon^C\right) \circ (xC)$ is epimorphism we deduce that

$$m_A \circ (Au_A) = A.$$

We compute

$$m_A \circ (u_A A) \circ \left(\varepsilon^C A\right) \circ (Cx) \overset{(103)}{=} m_A \circ (xA) \circ (\delta_C A) \circ (Cx)$$
$$\overset{\delta_C}{=} m_A \circ (xA) \circ (QPx) \circ (\delta_C QP) = m_A \circ (xx) \circ (\delta_C QP)$$
$$\overset{(102)}{=} x \circ (\chi P) \circ (\delta_C QP) \overset{(99)}{=} x \circ (\varepsilon^C QP) \overset{\varepsilon^C}{=} \left(\varepsilon^C A\right) \circ (Cx)$$

so that we get

$$m_A \circ (u_A A) \circ \left(\varepsilon^C A\right) \circ (Cx) = \left(\varepsilon^C A\right) \circ (Cx)$$

and since $\left(\varepsilon^C A\right) \circ (Cx)$ is an epimorphism we deduce that

$$m_A \circ (u_A A) = A.$$

Therefore we obtain that m_A is unital. We compute

$$^A\mu_Q \circ \left(A^A\mu_Q\right) \circ (AxQ) \circ (xQPQ)$$
$$\overset{(101)}{=} {}^A\mu_Q \circ (A\chi) \circ (xQPQ) \overset{x}{=} {}^A\mu_Q \circ (xQ) \circ (QP\chi)$$
$$\overset{(101)}{=} \chi \circ (QP\chi) \overset{(98)}{=} \chi \circ (\chi PQ) \overset{(101)}{=} {}^A\mu_Q \circ (xQ) \circ (\chi PQ)$$
$$\overset{(102)}{=} {}^A\mu_Q \circ (m_A Q) \circ (xxQ) = {}^A\mu_Q \circ (m_A Q) \circ (AxQ) \circ (xQPQ).$$

Since $(AxQ) \circ (xQPQ)$ is an epimorphism we get

$$^A\mu_Q \circ \left(A^A\mu_Q\right) = {}^A\mu_Q \circ (m_A Q).$$

We calculate

$$^A\mu_Q \circ (u_A Q) \circ \left(\varepsilon^C Q\right) \overset{(103)}{=} {}^A\mu_Q \circ (xQ) \circ (\delta_C Q)$$
$$\overset{(101)}{=} \chi \circ (\delta_C Q) \overset{(99)}{=} \left(\varepsilon^C Q\right).$$

Since $\left(\varepsilon^C Q\right)$ is an epimorphism we obtain

$$^A\mu_Q \circ (u_A Q) = Q.$$

$\square$

PROPOSITION 6.26. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with coequalizers and let $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, and $D : \mathcal{B} \to \mathcal{B}$ be functors. Assume that all the functors P, Q and D preserve coequalizers. Let $\varepsilon^D : D \to \mathcal{B}$ be a functorial morphism and assume that $\left(\mathcal{B}, \varepsilon^D\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(D\varepsilon^D, \varepsilon^D D\right)$. Let $\chi : QPQ \to Q$ be a functorial morphism such that*

$$\chi \circ (QP\chi) = \chi \circ (\chi PQ).$$

Let $\delta_D : D \to PQ$ be a functorial morphism such that

$$\chi \circ (Q\delta_D) = Q\varepsilon^D. \tag{105}$$

Let $z^l = (P\chi) \circ (\delta_D PQ)$ and $z^r = \varepsilon^D PQ : DPQ \to PQ$. Set

$$(B, y) = \mathrm{Coequ}_{\mathrm{Fun}}\left(z^l, z^r\right). \tag{106}$$

There exists a functorial morphism $\mu_Q^B : QB \to Q$ such that

$$\mu_Q^B \circ (Qy) = \chi. \tag{107}$$

There exist functorial morphisms $m_B : BB \to B$ and $u_B : \mathcal{B} \to B$ such that $\mathbb{B} = (B, m_B, u_B)$ is a monad over $\mathcal{B}$ that preserves coequalizers. Moreover m_B and u_B are uniquely determined by

$$m_B \circ (yB) = y \circ \left(P\mu_Q^B\right) \tag{108}$$

or equivalently

$$m_B \circ (yy) = y \circ (P\chi) \tag{109}$$

and

$$y \circ \delta_D = u_B \circ \varepsilon^D. \tag{110}$$

Moreover $\left(Q, \mu_Q^B\right)$ is a right $\mathbb{B}$-module functor.

Proof. By left-right symmetric argument of those used in proof of Proposition 6.25, one can prove this Proposition. $\square$

Definition 6.27. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *preformal codual structure* is a eightuple $\Theta = \left(C, D, P, Q, \delta_C, \delta_D, \varepsilon^C, \varepsilon^D\right)$ where $C : \mathcal{A} \to \mathcal{A}$, $D : \mathcal{B} \to \mathcal{B}$, $P : \mathcal{A} \to \mathcal{B}$ and $Q : \mathcal{B} \to \mathcal{A}$ are functors, $\delta_C : C \to QP, \delta_D : D \to PQ, \varepsilon^C : C \to \mathcal{A}, \varepsilon^D : D \to \mathcal{B}$ are functorial morphisms. A *copretorsor* χ for Θ is a functorial morphism $\chi : QPQ \to Q$ satisfying the following conditions:

1) Coassociativity, in the sense that

$$\chi \circ (\chi PQ) = \chi \circ (QP\chi) \tag{111}$$

2) Counitality, in the sense that

$$\chi \circ (\delta_C Q) = \varepsilon^C Q \tag{112}$$

and

$$\chi \circ (Q\delta_D) = Q\varepsilon^D. \tag{113}$$

Definition 6.28. A preformal codual structure $\Theta = \left(C, D, P, Q, \delta_C, \delta_D, \varepsilon^C, \varepsilon^D\right)$ will be called *regular* whenever $\left(\mathcal{A}, \varepsilon^C\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(C\varepsilon^C, \varepsilon^C C\right)$ and $\left(\mathcal{B}, \varepsilon^D\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(D\varepsilon^D, \varepsilon^D D\right)$. In this case a copretorsor for Θ will be called a *regular copretorsor.*

Theorem 6.29. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with coequalizers and let $\chi : QPQ \to Q$ be a regular copretorsor for $\Theta = \left(C, D, P, Q, \delta_C, \delta_D, \varepsilon^C, \varepsilon^D\right)$. Assume that the underlying functors P, Q, C and D preserve coequalizers. Let $w^l = (\chi P) \circ (QP\delta_C)$ and $w^r = QP\varepsilon^C : QPC \to QP$. Set*

$$(A, x) = \mathrm{Coequ}_{\mathrm{Fun}}\left(w^l, w^r\right). \tag{114}$$

There exists a functorial morphism ${}^A\mu_Q : AQ \to Q$ such that

$${}^A\mu_Q \circ (xQ) = \chi. \tag{115}$$

There exist functorial morphisms $m_A : AA \to A$ and $u_A : \mathcal{A} \to A$ such that $\mathbb{A} = (A, m_A, u_A)$ is a monad over $\mathcal{A}$ that preserves coequalizers. Moreover m_A and u_A are uniquely determined by

$$x \circ (\chi P) = m_A \circ (xx) \qquad \textit{and} \qquad x \circ \delta_C = u_A \circ \varepsilon^C.$$

Moreover $\left(Q, {}^{A}\mu_{Q}\right)$ *is a left* $\mathbb{A}$*-module functor.*

Let $z^{l} = (P\chi) \circ (\delta_{D} PQ)$ *and* $z^{r} = \varepsilon^{D} PQ : DPQ \rightarrow PQ$*. Set*

$$(B, y) = \mathrm{Coequ}_{\mathrm{Fun}}\left(z^{l}, z^{r}\right). \tag{116}$$

There exists a functorial morphism $\mu_{Q}^{B} : QB \rightarrow Q$ *such that*

$$\mu_{Q}^{B} \circ (Qy) = \chi. \tag{117}$$

There exist functorial morphisms $m_{B} : BB \rightarrow B$ *and* $u_{B} : \mathcal{B} \rightarrow B$ *such that* $\mathbb{B} = (B, m_{B}, u_{B})$ *is a monad over* $\mathcal{B}$ *that preserves coequalizers. Moreover* m_{B} *and* u_{B} *are uniquely determined by*

$$m_{B} \circ (yy) = y \circ (P\chi) \qquad \text{and} \qquad y \circ \delta_{D} = u_{B} \circ \varepsilon^{D}.$$

Moreover $\left(Q, \mu_{Q}^{B}\right)$ *is a right* $\mathbb{B}$*-module functor.*
Finally $\left(Q, {}^{A}\mu_{Q}, \mu_{Q}^{B}\right)$*is an* $\mathbb{A}$*-*$\mathbb{B}$*-module functor.*

Proof. Within these assumption, we can apply Proposition 6.25 to get the monad $\mathbb{A}$ and the functorial morphism ${}^{A}\mu_{Q} : AQ \rightarrow Q$ satisfying 115 such that $\left(Q, {}^{A}\mu_{Q}\right)$ is a left $\mathbb{A}$-module functor and Proposition 6.26 to get the monad $\mathbb{B}$ and the functorial morphism $\mu_{Q}^{B} : QB \rightarrow Q$ satisfying 117 such that $\left(Q, \mu_{Q}^{B}\right)$ is a right $\mathbb{B}$-module functor. Let us check the compatibility condition. We calculate

$$\begin{aligned} {}^{A}\mu_{Q} \circ \left(A\mu_{Q}^{B}\right) \circ (AQy) \circ (xQPQ) &\overset{x}{=} {}^{A}\mu_{Q} \circ (xQ) \circ \left(QP\mu_{Q}^{B}\right) \circ (QPQy) \\ &\overset{(115),(117)}{=} \chi \circ (QP\chi) \overset{(98)}{=} \chi \circ (\chi PQ) \\ &\overset{(117),(115)}{=} \mu_{Q}^{B} \circ (Qy) \circ \left({}^{A}\mu_{Q} PQ\right) \circ (xQPQ) \\ &\overset{{}^{A}\mu_{Q}}{=} \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q} B\right) \circ (AQy) \circ (xQPQ). \end{aligned}$$

Since $(AQy) \circ (xQPQ)$ is an epimorphism we get that

$${}^{A}\mu_{Q} \circ \left(A\mu_{Q}^{B}\right) = \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q} B\right).$$

Therefore $(Q, {}^{A}\mu_{Q}, \mu_{Q}^{B})$ is an $\mathbb{A}$-$\mathbb{B}$-bimodule functor. □

THEOREM 6.30. *Let* $\chi : QPQ \rightarrow Q$ *be a regular copretorsor for a preformal codual structure* $\Theta = \left(C, D, P, Q, \delta_{C}, \delta_{D}, \varepsilon^{C}, \varepsilon^{D}\right)$ *on categories* $\mathcal{A}$ *and* $\mathcal{B}$ *such that the underlying functors* P, Q, C *and* D *preserve coequalizers. Assume that* C *and* D *are comonads,* $\left(P, {}^{D}\rho_{P}\right)$ *is a left* $\mathbb{D}$*-comodule functor and* $\left(P, \rho_{P}^{C}\right)$ *is a right* $\mathbb{C}$*-comodule functor. Moreover assume that the functorial morphism* δ_{C} *is right* C*-colinear, that is* $\left(Q\rho_{P}^{C}\right) \circ \delta_{C} = (\delta_{C} C) \circ \Delta^{C}$ *and the functorial morphism* δ_{D} *is left* D*-colinear that is* $\left({}^{D}\rho_{P} Q\right) \circ \delta_{D} = (D\delta_{D}) \circ \Delta^{D}$ *and that they are compatible in the sense that*

$$(\delta_{D} P) \circ {}^{D}\rho_{P} = (P\delta_{C}) \circ \rho_{P}^{C}. \tag{118}$$

Then there exists a monad $\mathbb{A} = (A, m_{A}, u_{A})$ *on the category* $\mathcal{A}$ *together with a functorial morphism* ${}^{A}\mu_{Q} : AQ \rightarrow Q$ *such that* $\left(Q, {}^{A}\mu_{Q}\right)$ *is a left* $\mathbb{A}$*-module functor and a monad* $\mathbb{B} = (B, m_{B}, u_{B})$ *together with a functorial morphism* $\mu_{Q}^{B} : QB \rightarrow Q$ *such that* $\left(Q, \mu_{Q}^{B}\right)$ *is a right* $\mathbb{B}$*-module functor. The underlying functors are defined as follows*

$$(A, x) = \mathrm{Coequ}_{\mathrm{Fun}}\left((\chi P) \circ (QP\delta_{C}), QP\varepsilon^{C}\right).$$

and

$$(B, y) = \mathrm{Coequ}_{\mathrm{Fun}}\left((P\chi) \circ (\delta_D PQ), \varepsilon^D PQ\right).$$

satisfying

$${}^A\mu_Q \circ (xQ) = \chi \text{ and } \mu_Q^B \circ (Qy) = \chi.$$

Furthermore

1) *The morphisms* $cocan_1 := \left({}^A\mu_Q P\right) \circ (A\delta_C) : AC \to QP$ *is an isomorphism.*
2) *The morphism* $\overline{cocan_1} := \left(P\mu_Q^B\right) \circ (\delta_D B) : DB \to PQ$ *is an isomorphisms.*
3)

$$(\chi P) \circ (QP\delta_C) = cocan_1 \circ (xC) \text{ and } (P\chi) \circ (\delta_D PQ) = \overline{cocan_1} \circ (Dy)$$

4)

$$\begin{aligned} x &= \left(A\varepsilon^C\right) \circ (cocan_1)^{-1} \\ y &= \left(\varepsilon^D B\right) \circ \left(\overline{cocan_1}\right)^{-1} \end{aligned}$$

5)

$$\delta_C = \left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (u_A C)$$

6)

$$\delta_D = \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B).$$

From the last equalities, we deduce that if $\varepsilon^C A$ *is a regular epimorphism, so is* σ^A *and if* $B\varepsilon^D$ *is a regular epimorphism, so is* σ^B.

Proof. Note that we are in the setting of Theorem 6.29.
1) Let us check that $cocan_1$ is an isomorphism.

The inverse of the functorial morphism $cocan_1$ is given by $cocan_1^{-1} = (xC) \circ \left(Q\rho_P^C\right) : QP \to AC$. Indeed we compute

$$\begin{aligned} (xC) \circ \left(Q\rho_P^C\right) \circ cocan_1 \circ (xC) &= (xC) \circ \left(Q\rho_P^C\right) \circ \left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (xC) \\ &= (xC) \circ \left(Q\rho_P^C\right) \circ \left({}^A\mu_Q P\right) \circ (xQP) \circ (QP\delta_C) \\ &\overset{(115)}{=} (xC) \circ \left(Q\rho_P^C\right) \circ (\chi P) \circ (QP\delta_C) = (xC) \circ (\chi PC) \circ \left(QPQ\rho_P^C\right) \circ (QP\delta_C) \\ &\overset{P\mathrm{right}C\mathrm{col}}{=} (xC) \circ (\chi PC) \circ (QP\delta_C C) \circ \left(QP\Delta^C\right) = (xC) \circ \left(w^l C\right) \circ \left(QP\Delta^C\right) \\ &\overset{x \text{ coequ}}{=} (xC) \circ (w^r C) \circ \left(QP\Delta^C\right) = (xC) \circ \left(QP\varepsilon^C C\right) \circ \left(QP\Delta^C\right) \overset{C \text{ comonad}}{=} (xC). \end{aligned}$$

Since xC is an epimorphism, we obtain that

$$(xC) \circ \left(Q\rho_P^C\right) \circ cocan_1 = AC.$$

On the other hand, we have

$$\begin{aligned} cocan_1 \circ (xC) \circ \left(Q\rho_P^C\right) &= \left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (xC) \circ \left(Q\rho_P^C\right) \\ &= \left({}^A\mu_Q P\right) \circ (xQP) \circ (QP\delta_C) \circ \left(Q\rho_P^C\right) \overset{(115)}{=} (\chi P) \circ (QP\delta_C) \circ \left(Q\rho_P^C\right) \\ &\overset{(118)}{=} (\chi P) \circ (Q\delta_D P) \circ \left(Q\,{}^D\rho_P\right) \\ &\overset{(113)}{=} \left(Q\varepsilon^D P\right) \circ \left(Q\,{}^D\rho_P\right) \overset{{}^D\rho_P \text{ counital}}{=} QP \end{aligned}$$

so we obtain that

$$cocan_1^{-1} = (xC) \circ \left(Q\rho_P^C\right).$$

2) Similarly we prove that we prove that $\overline{cocan_1} := \left(P\mu_Q^B\right) \circ (\delta_D B) : DB \to PQ$ is an isomorphism with $(Dy) \circ \left({}^D\rho_P Q\right)$ its inverse. In fact we have

$$\begin{aligned}
(Dy) \circ \left({}^D\rho_P Q\right) \circ \overline{cocan_1} \circ (Dy) &= (Dy) \circ \left({}^D\rho_P Q\right) \circ \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Dy) \\
&\overset{{}^D\rho_P}{=} (Dy) \circ \left(DP\mu_Q^B\right) \circ \left({}^D\rho_P QB\right) \circ (\delta_D B) \circ (Dy) \\
&\overset{\delta_D \mathrm{left} D\mathrm{col}}{=} (Dy) \circ \left(DP\mu_Q^B\right) \circ (D\delta_D B) \circ \left(\Delta^D B\right) \circ (Dy) \\
&\overset{\Delta^D}{=} (Dy) \circ \left(DP\mu_Q^B\right) \circ (D\delta_D B) \circ (DDy) \circ \left(\Delta^D PQ\right) \\
&\overset{\delta_D}{=} (Dy) \circ \left(DP\mu_Q^B\right) \circ (DPQy) \circ (D\delta_D PQ) \circ \left(\Delta^D PQ\right) \\
&\overset{(117)}{=} (Dy) \circ (DP\chi) \circ (D\delta_D PQ) \circ \left(\Delta^D PQ\right) \\
&\overset{y\mathrm{coequ}}{=} (Dy) \circ \left(D\varepsilon^D PQ\right) \circ \left(\Delta^D PQ\right) \overset{D\mathrm{comonad}}{=} Dy
\end{aligned}$$

and since D preserves coequalizers, Dy is an epimorphism, so that we get

$$(Dy) \circ \left({}^D\rho_P Q\right) \circ \overline{cocan_1} = DB.$$

On the other hand we have

$$\begin{aligned}
\overline{cocan_1} \circ (Dy) \circ \left({}^D\rho_P Q\right) &= \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Dy) \circ \left({}^D\rho_P Q\right) \\
&\overset{\delta_D}{=} \left(P\mu_Q^B\right) \circ (PQy) \circ (\delta_D PQ) \circ \left({}^D\rho_P Q\right) \overset{(117)}{=} (P\chi) \circ (\delta_D PQ) \circ \left({}^D\rho_P Q\right) \\
&\overset{(118)}{=} (P\chi) \circ (P\delta_C Q) \circ \left(\rho_P^C Q\right) \overset{(112)}{=} \left(P\varepsilon^C Q\right) \circ \left(\rho_P^C Q\right) \\
&\overset{P\mathrm{com}}{=} PQ
\end{aligned}$$

so that we get

$$\overline{cocan_1} \circ (Dy) \circ \left({}^D\rho_P Q\right) = PQ.$$

3) We have

$$\begin{aligned}
(\chi P) \circ (QP\delta_C) &\overset{(115)}{=} \left({}^A\mu_Q P\right) \circ (xQP) \circ (QP\delta_C) \\
&\overset{x}{=} \left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (xC) \overset{\mathrm{def} cocan_1}{=} cocan_1 \circ (xC)
\end{aligned}$$

so that

(119) $$(\chi P) \circ (QP\delta_C) = cocan_1 \circ (xC).$$

Similarly we have

$$\begin{aligned}
(P\chi) \circ (\delta_D PQ) &\overset{(117)}{=} \left(P\mu_Q^B\right) \circ (PQy) \circ (\delta_D PQ) \\
&\overset{\delta_D}{=} \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Dy) \overset{\mathrm{def}\overline{cocan_1}}{=} \overline{cocan_1} \circ (Dy)
\end{aligned}$$

so that

(120) $$(P\chi) \circ (\delta_D PQ) = \overline{cocan_1} \circ (Dy).$$

4) With notations of Theorem 6.29, we have

$$x\circ cocan_1\circ (xC)\overset{(119)}{=}x\circ (\chi P)\circ (QP\delta_C)$$
$$=x\circ w^l\overset{x\text{coequ}}{=}x\circ w^r=x\circ \left(QP\varepsilon^C\right)\overset{x}{=}\left(A\varepsilon^C\right)\circ (xC).$$

Since xC is an epimorphism, we deduce that

$$x\circ cocan_1=A\varepsilon^C$$

and hence

$$x=\left(A\varepsilon^C\right)\circ (cocan_1)^{-1}.$$

Similarly, we have

$$y\circ \overline{cocan_1}\circ (Dy)\overset{(120)}{=}y\circ (P\chi)\circ (\delta_D PQ)$$
$$=y\circ z^l\overset{y\text{coequ}}{=}y\circ z^r=y\circ \left(\varepsilon^D PQ\right)\overset{\varepsilon^D}{=}\left(\varepsilon^D B\right)\circ (Dy).$$

Since Dy is an epimorphism, we deduce that

$$y\circ \overline{cocan_1}=\varepsilon^D B$$

and hence

$$y=\left(\varepsilon^D B\right)\circ (\overline{cocan_1})^{-1}.$$

5) We have that

$$\delta_C=\left({}^A\mu_Q P\right)\circ (u_A QP)\circ \delta_C\overset{u_A}{=}\left({}^A\mu_Q P\right)\circ (A\delta_C)\circ (u_A C)$$

so that

$$\delta_C=\left({}^A\mu_Q P\right)\circ (A\delta_C)\circ (u_A C). \tag{121}$$

Since $\left({}^A\mu_Q P\right)\circ (A\delta_C)=cocan_1$ is an isomorphism, we will prove that if u_AC is a regular monomorphism, so is δ_C. In fact, let $(C,u_AC)=\mathrm{Equ}_{\mathrm{Fun}}(\upsilon,\varpi)$ where $\upsilon,\varpi:AC\to T$. We know that $\left({}^A\mu_Q P\right)\circ (A\delta_C)=cocan_1$ is an isomorphism with inverse $(xC)\circ \left(Q\rho_P^C\right)$ so that we have

$$\upsilon\circ (cocan_1)^{-1}\circ \delta_C\overset{(121)}{=}\upsilon\circ (cocan_1)^{-1}\circ \left({}^A\mu_Q P\right)\circ (A\delta_C)\circ (u_A C)$$
$$\overset{cocan_1\text{iso}}{=}\upsilon\circ (u_A C)\overset{u_A C\text{equ}}{=}\varpi\circ (u_A C)$$
$$\overset{cocan_1\text{iso}}{=}\varpi\circ (cocan_1)^{-1}\circ \left({}^A\mu_Q P\right)\circ (A\delta_C)\circ (u_A C)$$
$$\overset{(121)}{=}\varpi\circ (cocan_1)^{-1}\circ \delta_C$$

so that

$$\upsilon\circ (cocan_1)^{-1}\circ \delta_C=\varpi\circ (cocan_1)^{-1}\circ \delta_C$$

i.e.

$$\upsilon\circ (xC)\circ \left(Q\rho_P^C\right)\circ \delta_C=\varpi\circ (xC)\circ \left(Q\rho_P^C\right)\circ \delta_C.$$

Moreover, for every $\xi:X\to QP$ such that

$$\upsilon\circ (xC)\circ \left(Q\rho_P^C\right)\circ \xi=\varpi\circ (xC)\circ \left(Q\rho_P^C\right)\circ \xi,$$

since $(C, u_A C) = \mathrm{Equ}_{\mathrm{Fun}}(\upsilon, \varpi)$, there exists a unique functorial morphism $\overline{\xi} : X \to C$ such that

$$(u_A C) \circ \overline{\xi} = (xC) \circ \left(Q\rho_P^C\right) \circ \xi.$$

Then, by composing to the left with the isomorphism $cocan_1 := \left({}^A\mu_Q P\right) \circ (A\delta_C)$ we get

$$\left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (u_A C) \circ \overline{\xi} = \left({}^A\mu_Q P\right) \circ (A\delta_C) \circ (xC) \circ \left(Q\rho_P^C\right) \circ \xi$$

i.e. by (121) we have

$$\delta_C \circ \overline{\xi} = \xi. \tag{122}$$

Then $\overline{\xi} : X \to C$ is the unique morphism satisfying condition (122) so that

$$(C, \delta_C) = \mathrm{Equ}_{\mathrm{Fun}}\left(\upsilon \circ (xC) \circ \left(Q\rho_P^C\right), \varpi \circ (xC) \circ \left(Q\rho_P^C\right)\right)$$

i.e. also δ_C is a regular monomorphism.

6) We have that

$$\delta_D = \left(P\mu_Q^B\right) \circ (PQu_B) \circ \delta_D \overset{u_B}{=} \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B)$$

so that

$$\delta_D = \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B). \tag{123}$$

Since $\left(P\mu_Q^B\right) \circ (\delta_D B) = \overline{cocan_1}$ is an isomorphism, we will prove that if Du_B is a regular monomorphism, so is δ_D. In fact, let $(D, Du_B) = \mathrm{Equ}_{\mathrm{Fun}}(\zeta, \theta)$ where $\zeta, \theta : DB \to L$. We know that $\left(P\mu_Q^B\right) \circ (\delta_D B) = \overline{cocan_1}$ is an isomorphism with inverse $(Dy) \circ \left({}^D\rho_P Q\right)$ so that we have

$$\begin{aligned}
\zeta \circ \left(\overline{cocan_1}\right)^{-1} \circ \delta_D &\overset{(123)}{=} \zeta \circ \left(\overline{cocan_1}\right)^{-1} \circ \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B) \\
&\overset{\overline{cocan_1}\text{iso}}{=} \zeta \circ (Du_B) \overset{Du_B\text{equ}}{=} \theta \circ (Du_B) \\
&\overset{\overline{cocan_1}\text{iso}}{=} \theta \circ \left(\overline{cocan_1}\right)^{-1} \circ \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B) \\
&\overset{(123)}{=} \theta \circ \left(\overline{cocan_1}\right)^{-1} \circ \delta_D
\end{aligned}$$

so that

$$\zeta \circ \left(\overline{cocan_1}\right)^{-1} \circ \delta_D = \theta \circ \left(\overline{cocan_1}\right)^{-1} \circ \delta_D$$

i.e.

$$\zeta \circ (Dy) \circ \left({}^D\rho_P Q\right) \circ \delta_D = \theta \circ (Dy) \circ \left({}^D\rho_P Q\right) \circ \delta_D.$$

Moreover, for every $\nu : Y \to PQ$ such that

$$\zeta \circ (Dy) \circ \left({}^D\rho_P Q\right) \circ \nu = \theta \circ (Dy) \circ \left({}^D\rho_P Q\right) \circ \nu,$$

since $(D, Du_B) = \mathrm{Equ}_{\mathrm{Fun}}(\zeta, \theta)$, there exists a unique functorial morphism $\overline{\nu} : Y \to D$ such that

$$(Du_B) \circ \overline{\nu} = (Dy) \circ \left({}^D\rho_P Q\right) \circ \nu.$$

Then, by composing to the left with the isomorphism $\overline{cocan_1} :=$ we get

$$\left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Du_B) \circ \overline{\nu} = \left(P\mu_Q^B\right) \circ (\delta_D B) \circ (Dy) \circ \left({}^D\rho_P Q\right) \circ \nu$$

i.e. by (123) we have

$$\delta_D \circ \overline{\nu} = \nu. \tag{124}$$

Then $\overline{\nu}: Y \to D$ is the unique morphism satisfying condition (124) so that

$$(D, \delta_D) = \mathrm{Equ}_{\mathrm{Fun}}\left(\zeta \circ (Dy) \circ \left({}^D\rho_P Q\right), \theta \circ (Dy) \circ \left({}^D\rho_P Q\right)\right)$$

i.e. also δ_D is a regular monomorphism. □

6.8. **Coherds.** Following [BV], by formally dualizing definitions of formal dual structure and herd, the notions of formal codual structure and of coherd are introduced.

Definition 6.31. A *formal codual structure on two categories $\mathcal{A}$ and $\mathcal{B}$* is a sextuple $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ where $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ and $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ are comonads on on $\mathcal{A}$ and $\mathcal{B}$ respectively and $\left(C, D, P, Q, \delta_C, \delta_D, \varepsilon^C, \varepsilon^D\right)$ is a preformal codual structure. Moreover $\left(P : \mathcal{A} \to \mathcal{B}, {}^D\rho_P : P \to DP, \rho_P^C : P \to PC\right)$ and $\left(Q : \mathcal{B} \to \mathcal{A}, {}^C\rho_Q : Q \to CQ, \rho_Q^D : Q \to QD\right)$ are bicomodule functors; $\delta_C : C \to QP, \delta_D : D \to PQ$ are subject to the following conditions: δ_C is C-bicolinear and δ_D is D-bicolinear

$$\left({}^C\rho_Q P\right) \circ \delta_C = (C\delta_C) \circ \Delta^C \text{ and } \left(Q\rho_P^C\right) \circ \delta_C = (\delta_C C) \circ \Delta^C \tag{125}$$

$$\left(P\rho_Q^D\right) \circ \delta_D = (\delta_D D) \circ \Delta^D \text{ and } \left({}^D\rho_P Q\right) \circ \delta_D = (D\delta_D) \circ \Delta^D \tag{126}$$

and the coassociative conditions hold

$$(\delta_C Q) \circ {}^C\rho_Q = (Q\delta_D) \circ \rho_Q^D \text{ and } (\delta_D P) \circ {}^D\rho_P = (P\delta_C) \circ \rho_P^C. \tag{127}$$

Definition 6.32. Consider a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ in the sense of the previous definition. A *coherd* for $\mathbb{X}$ is a copretorsor $\chi : QPQ \to Q$ i.e.

$$\chi \circ (\chi PQ) = \chi \circ (QP\chi) \tag{128}$$

$$\chi \circ (\delta_C Q) = \varepsilon^C Q \tag{129}$$

and

$$\chi \circ (Q\delta_D) = Q\varepsilon^D. \tag{130}$$

Definition 6.33. A formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ will be called *regular* whenever $\left(C, D, P, Q, \delta_C, \delta_D, \varepsilon^C, \varepsilon^D\right)$ is a regular preformal codual structure. In this case a coherd for $\mathbb{X}$ will be called a *regular coherd*.

Lemma 6.34. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a formal codual structure and let $\chi : QPQ \to QQ$ be a coherd for $\mathbb{X}$. Assume that the underlying functors C and D reflect coequalizers. Then χ is a regular coherd.*

Proof. Since $\mathbb{C}$ and $\mathbb{D}$ are comonads, we have $\left(C\varepsilon^C\right) \circ \Delta^C = \mathrm{Id}_C$ and $\left(D\varepsilon^D\right) \circ \Delta^D = \mathrm{Id}_D$. Thus, $C\varepsilon^C$ and $D\varepsilon^D$ are split epimorphisms and thus epimorphisms. Since C and D reflect coequalizers, we deduce that also ε^C and ε^D are epimorphisms and thus $\left(\mathcal{A}, \varepsilon^C\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\varepsilon^C C, C\varepsilon^C\right)$ and $\left(\mathcal{B}, \varepsilon^D\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\varepsilon^D D, D\varepsilon^D\right)$, i.e. χ is a regular coherd. □

Proposition 6.35. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a formal codual structure such that the lifted functors ${}^CQ^D : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ and ${}^DP^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ determine an equivalence of categories. Then $\left(Q^D, {}^DP\right)$ and $\left(P^C, {}^CQ\right)$ are adjunctions.*

Proof. Since $\left({}^{\mathbb{C}}U,\right)$ and $\left({}^{\mathbb{D}}U,{}^{\mathbb{D}}F\right)$ are adjunctions, $\left({}^{\mathbb{C}}U^{C}Q^{D},{}^{D}P^{C\mathbb{C}}F\right)=\left(Q^{D},{}^{D}P\right)$ and $\left({}^{\mathbb{D}}U^{D}P^{C},{}^{C}Q^{D\mathbb{D}}F\right)=\left(P^{C},{}^{C}Q\right)$ are also adjunctions. □

6.9. **Coherds and Monads.** In this subsection we prove that in the case when there exist coequalizers in the base categories and all functors occurring in a formal codual structure preserve them, we establish an equivalence between coherds on one hand, and monads on the other hand, together with two natural isomorphism generating a Galois map.

Theorem 6.36. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories in both of which the coequalizer of any pair of parallel morphisms exists. Let $\mathbb{X}=(\mathbb{C},\mathbb{D},P,Q,\delta_C,\delta_D)$ be a formal codual structure on $\mathcal{A}$ and $\mathcal{B}$. Then we have*

(1) *If $\mathbb{A}=(A,m_A,u_A)$ is a monad on the category $\mathcal{A}$ and $\left(Q,{}^{A}\mu_Q:AQ\to Q\right)$ is a left A-module functor such that*
   (i) *the functorial morphism $cocan_1:=\left({}^{A}\mu_QP\right)\circ(A\delta_C):AC\to QP$ is an isomorphism*
   (ii) *the functorial morphism $cocan_2:=\left({}^{A}\mu_QD\right)\circ\left(A\rho_Q^D\right):AQ\to QD$ is an isomorphism*

   then $\chi:={}^{A}\mu_Q\circ\left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right):QPQ\to Q$ is a copretorsor and thus a coherd.

(2) *If $\mathbb{B}=(B,m_B,u_B)$ is a monad on the category $\mathcal{B}$ and $\left(Q,\mu_Q^B\colon QB\to Q\right)$ is a right $\mathbb{B}$-module functor such that*
   (i) *the functorial morphism $\overline{cocan_1}:=\left(P\mu_Q^B\right)\circ(\delta_DB):DB\to PQ$ is an isomorphism*
   (ii) *the functorial morphism $\overline{cocan_2}:=\left(C\mu_Q^B\right)\circ\left({}^{C}\rho_QB\right):QB\to CQ$ is an isomorphism*

   then $\chi:=\mu_Q^B\circ\left(Q\varepsilon^{D}B\right)\circ\left(Q\overline{cocan_1}^{-1}\right):QPQ\to Q$ is a copretorsor and thus a coherd.

Proof. Let us prove 1), the other is similar. We have to prove that

$$\chi:={}^{A}\mu_Q\circ\left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)$$

satisfies $(111),(112),(113)$. We compute

$$\begin{aligned}cocan_1\circ(m_AC)&=\left({}^{A}\mu_QP\right)\circ(A\delta_C)\circ(m_AC)\\&=\left({}^{A}\mu_QP\right)\circ(m_AQP)\circ(AA\delta_C)\\&\overset{\left({}^{A}\mu_Q\text{ ass}\right)}{=}\left({}^{A}\mu_QP\right)\circ\left(A^{A}\mu_QP\right)\circ(AA\delta_C)=\left({}^{A}\mu_QP\right)\circ(Acocan_1)\end{aligned}$$

so we obtain

$$cocan_1\circ(m_AC)=\left({}^{A}\mu_QP\right)\circ(Acocan_1)$$

i.e.

$$(m_AC)\circ Acocan_1^{-1}=\left(cocan_1^{-1}\right)\circ\left({}^{A}\mu_QP\right). \tag{131}$$

We compute

$$\begin{aligned}&\chi\circ(QP\chi)\\&={}^{A}\mu_Q\circ\left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left(QP^{A}\mu_Q\right)\circ\left(QPA\varepsilon^{C}Q\right)\circ\left(QPcocan_1^{-1}Q\right)\end{aligned}$$

$$
\begin{aligned}
&= {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(AC{}^{A}\mu_{Q}\right) \circ \left(ACA\varepsilon^{C}Q\right) \circ \left(ACcocan_{1}^{-1}Q\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&= {}^{A}\mu_{Q} \circ \left(A{}^{A}\mu_{Q}\right) \circ \left(A\varepsilon^{C}AQ\right) \circ \left(ACA\varepsilon^{C}Q\right) \circ \left(ACcocan_{1}^{-1}Q\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&= {}^{A}\mu_{Q} \circ \left(m_{A}Q\right) \circ \left(A\varepsilon^{C}AQ\right) \circ \left(ACA\varepsilon^{C}Q\right) \circ \left(ACcocan_{1}^{-1}Q\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&= {}^{A}\mu_{Q} \circ \left(m_{A}Q\right) \circ \left(AA\varepsilon^{C}Q\right) \circ \left(A\varepsilon^{C}CAQ\right) \circ \left(ACcocan_{1}^{-1}Q\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&= {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(m_{A}CQ\right) \circ \left(Acocan_{1}^{-1}Q\right) \circ \left(A\varepsilon^{C}QPQ\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&\overset{(131)}{=} {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ \left({}^{A}\mu_{Q}PQ\right) \circ \left(A\varepsilon^{C}QPQ\right) \circ \left(cocan_{1}^{-1}QPQ\right) \\
&= \chi \circ (\chi PQ)\,.
\end{aligned}
$$

Note that we have

$$
\begin{aligned}
(cocan_{1}) \circ (u_{A}C) &= \left({}^{A}\mu_{Q}P\right) \circ (A\delta_{C}) \circ (u_{A}C) \\
&= \left({}^{A}\mu_{Q}P\right) \circ (u_{A}QP) \circ \delta_{C} \overset{{}^{A}\mu_{Q}\text{ unital}}{=} \delta_{C}
\end{aligned}
$$

so we have

$$
(cocan_{1}) \circ (u_{A}C) = \delta_{C}. \tag{132}
$$

Now we compute

$$
\begin{aligned}
\chi \circ (\delta_{C}Q) &= {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (\delta_{C}Q) \\
&\overset{(132)}{=} {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (cocan_{1}Q) \circ (u_{A}CQ) \\
&= {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ (u_{A}CQ) = {}^{A}\mu_{Q} \circ (u_{A}Q) \circ \left(\varepsilon^{C}Q\right) \overset{\left({}^{A}\mu_{Q}\text{ unital}\right)}{=} \left(\varepsilon^{C}Q\right)
\end{aligned}
$$

and so we get

$$
\chi \circ (\delta_{C}Q) = \left(\varepsilon^{C}Q\right).
$$

We have

$$
\begin{aligned}
&\left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D}) \circ cocan_{2} \\
&= \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D}) \circ \left({}^{A}\mu_{Q}D\right) \circ \left(A\rho_{Q}^{D}\right) \\
&= \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ \left({}^{A}\mu_{Q}PQ\right) \circ (AQ\delta_{D}) \circ \left(A\rho_{Q}^{D}\right) \\
&\overset{(127)}{=} \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ \left({}^{A}\mu_{Q}PQ\right) \circ (A\delta_{C}Q) \circ \left(A{}^{C}\rho_{Q}\right) \\
&= \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (cocan_{1}Q) \circ \left(A{}^{C}\rho_{Q}\right) \\
&= \left(A\varepsilon^{C}Q\right) \circ \left(A{}^{C}\rho_{Q}\right) \overset{{}^{C}\rho_{Q}\text{ counital}}{=} AQ
\end{aligned}
$$

Since $cocan_{2}$ is an isomorphism, we have that

$$
cocan_{2} \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D}) = QD. \tag{133}
$$

Finally we compute

$$
\begin{aligned}
\chi \circ (Q\delta_{D}) &= {}^{A}\mu_{Q} \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D}) \\
&\overset{\rho_{Q}^{D}\text{ counital}}{=} {}^{A}\mu_{Q} \circ \left(AQ\varepsilon^{D}\right) \circ \left(A\rho_{Q}^{D}\right) \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D}) \\
&= \left(Q\varepsilon^{D}\right) \circ \left({}^{A}\mu_{Q}D\right) \circ \left(A\rho_{Q}^{D}\right) \circ \left(A\varepsilon^{C}Q\right) \circ \left(cocan_{1}^{-1}Q\right) \circ (Q\delta_{D})
\end{aligned}
$$

$$= \left(Q\varepsilon^D\right) \circ cocan_2 \circ \left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ \left(Q\delta_D\right) \overset{(133)}{=} \left(Q\varepsilon^D\right)$$

so that we obtain

$$\chi \circ \left(Q\delta_D\right) = \left(Q\varepsilon^D\right).$$

□

THEOREM 6.37. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories in both of which the coequalizer of any pair of parallel morphisms exist. Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a regular formal co-dual structure on $\mathcal{A}$ and $\mathcal{B}$ such that the underlying functors C, D, P and Q preserve coequalizers, then the existence of the following structures are equivalent:*

(a) *A coherd $\chi : QPQ \to Q$ in $\mathbb{X}$.*

(b) *A monad $\mathbb{A} = (A : \mathcal{A} \to \mathcal{A}, m_A : AA \to A, u_A : \mathcal{A} \to A)$, such that the functor A preserves coequalizers, together with a left action ${}^A\mu_Q : AQ \to Q$, subject to the following conditions:*
  - (i) *The natural transformation $cocan_1 := \left({}^A\mu_Q P\right) \circ (A\delta_C) : AC \to QP$ is an isomorphism.*
  - (ii) *The natural transformation $cocan_2 := \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) : AQ \to QD$ is an isomorphism.*

(c) *A monad $\mathbb{B} = (B : \mathcal{B} \to \mathcal{B}, m_B : BB \to B, u_B : \mathcal{B} \to B)$, such that the functor B preserves coequalizers, together with a right action $\mu_Q^B : QB \to Q$, subject to the following conditions:*
  - (i) *The natural transformation $\overline{cocan_1} := \left(P\mu_Q^B\right) \circ (\delta_D B) : DB \to PQ$ is an isomorphism.*
  - (ii) *The natural transformation $\overline{cocan_2} := \left(C\mu_Q^B\right) \circ \left({}^C\rho_Q B\right) : QB \to CQ$ is an isomorphism.*

Proof. (a) $\Rightarrow$ (b) Under weaker assumptions, the monad $\mathbb{A} = (A, m_A, u_A)$ and the action ${}^A\mu_Q$ have been constructed in Proposition 6.25. Moreover, by Theorem 6.30 1) we already proved that $cocan_1$ is an isomorphism with $(xC) \circ \left(Q\rho_P^C\right)$ its inverse. Now we prove that $\left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ (Q\delta_D)$ is the inverse of $cocan_2$. We compute

$$\begin{aligned}
& cocan_2 \circ \left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ (Q\delta_D) \\
&= \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) \circ \left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ (Q\delta_D) \\
&= \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) \circ \left(A\varepsilon^C Q\right) \circ (xCQ) \circ \left(Q\rho_P^C Q\right) \circ (Q\delta_D) \\
&= \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) \circ (xQ) \circ \left(QP\varepsilon^C Q\right) \circ \left(Q\rho_P^C Q\right) \circ (Q\delta_D) \\
&\overset{\rho_P^C \text{ counital}}{=} \left({}^A\mu_Q D\right) \circ \left(A\rho_Q^D\right) \circ (xQ) \circ (Q\delta_D) = \left({}^A\mu_Q D\right) \circ (xQD) \circ \left(QP\rho_Q^D\right) \circ (Q\delta_D) \\
&\overset{(101)}{=} (\chi D) \circ \left(QP\rho_Q^D\right) \circ (Q\delta_D) \overset{(126)}{=} (\chi D) \circ (Q\delta_D D) \circ \left(Q\Delta^D\right) \\
&\overset{(113)}{=} \left(Q\varepsilon^D D\right) \circ \left(Q\Delta^D\right) \overset{(D \text{ comonad})}{=} QD
\end{aligned}$$

so we obtain

$$cocan_2 \circ \left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ (Q\delta_D) = QD.$$

On the other hand, we have

$$\left(A\varepsilon^C Q\right) \circ \left(cocan_1^{-1} Q\right) \circ (Q\delta_D) \circ cocan_2$$

$$
\begin{aligned}
&= \left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left(Q\delta_D\right)\circ\left({}^{A}\mu_Q D\right)\circ\left(A\rho_Q^{D}\right)\\
&= \left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left({}^{A}\mu_Q PQ\right)\circ\left(AQ\delta_D\right)\circ\left(A\rho_Q^{D}\right)\\
&\overset{(127)}{=} \left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left({}^{A}\mu_Q PQ\right)\circ\left(A\delta_C Q\right)\circ\left(A\,{}^{C}\rho_Q\right)\\
&= \left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left(cocan_1 Q\right)\circ\left(A\,{}^{C}\rho_Q\right)\\
&= \left(A\varepsilon^{C}Q\right)\circ\left(A\,{}^{C}\rho_Q\right) \overset{{}^{C}\rho_Q\ \text{counital}}{=} AQ
\end{aligned}
$$

so we obtain

$$cocan_2^{-1} = \left(A\varepsilon^{C}Q\right)\circ\left(cocan_1^{-1}Q\right)\circ\left(Q\delta_D\right).$$

$(b) \Rightarrow (a)$ Follows from Theorem 6.36. $(a) \Leftrightarrow (c)$ follows by similar computations. □

6.10. **Coherds and distributive laws.** The following result is a reformulation of Theorem 2.16 in [BV] in our categorical setting.

PROPOSITION 6.38. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $\chi : QPQ \to Q$ be a regular coherd for $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ where the underlying functors $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $C : \mathcal{A} \to \mathcal{A}$ and $D : \mathcal{B} \to \mathcal{B}$ preserve coequalizers. Let $\mathbb{A} = (A, m_A, u_A)$ and $\mathbb{B} = (B, m_B, u_B)$ be the associated monads constructed in Proposition 6.25 and in Proposition 6.26. Then*

1) *There exists a mixed distributive law between the monad $\mathbb{A}$ and the comonad $\mathbb{C}$, $\Lambda : AC \to CA$ such that*
$$\Lambda\circ(xC) = \lambda = (Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\delta_C).$$
2) *There exists an opposite mixed distributive law between the monad $\mathbb{B}$ and the comonad $\mathbb{D}$, $\Gamma : DB \to BD$ such that*
$$\Gamma\circ(Dy) = \gamma = (yD)\circ\left(P\rho_Q^{D}\right)\circ(P\chi)\circ(\delta_D PQ).$$

Proof. 1) Consider the functorial morphism given by

$$QPC \xrightarrow{QP\delta_C} QPQP \xrightarrow{\chi P} QP \xrightarrow{{}^{C}\rho_Q P} CQP \xrightarrow{Cx} CA$$
$$\lambda = (Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\delta_C)$$

Recall that $w^l = (\chi P)\circ(QP\delta_C)$ and $w^r = QP\varepsilon^C : QPC \to QP$ and let us prove that

$$\lambda\circ\left(w^l C\right) \overset{?}{=} \lambda\circ(w^r C)$$

that is

$$
\begin{aligned}
&(Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\delta_C)\circ(\chi PC)\circ(QP\delta_C C)\\
&\overset{?}{=} (Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\delta_C)\circ\left(QP\varepsilon^{C}C\right).
\end{aligned}
$$

Let us compute

$$
\begin{aligned}
&(Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\delta_C)\circ(\chi PC)\circ(QP\delta_C C)\\
&\overset{\chi}{=} (Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(\chi PQP)\circ(QPQP\delta_C)\circ(QP\delta_C C)\\
&\overset{\delta_C,(111)}{=} (Cx)\circ\left({}^{C}\rho_Q P\right)\circ(\chi P)\circ(QP\chi P)\circ(QP\delta_C QP)\circ(QPC\delta_C)
\end{aligned}
$$

$$\overset{(112)}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ \left(QP\varepsilon^{C} QP \right) \circ (QPC\delta_{C})$$

$$\overset{\varepsilon^{C}}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ (QP\delta_{C}) \circ \left(QP\varepsilon^{C} C \right)$$

Since $(AC, xC) = \mathrm{Coequ}_{\mathrm{Fun}} \left(w^{l} C, w^{r} C \right)$,by the universal property of coequalizers, there exists a unique functorial morphism $\Lambda : AC \to CA$ such that

$$\Lambda \circ (xC) = \lambda = (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ (QP\delta_{C}) .$$

We want to prove that Λ is a mixed distributive law. We compute

$$(Cm_{A}) \circ (\Lambda A) \circ (A\Lambda) \circ (xxC) \overset{x}{=} (Cm_{A}) \circ (\Lambda A) \circ (A\Lambda) \circ (xAC) \circ (QPxC)$$

$$\overset{x}{=} (Cm_{A}) \circ (\Lambda A) \circ (xCA) \circ (QP\Lambda) \circ (QPxC)$$

$$\overset{\mathrm{def}\lambda}{=} (Cm_{A}) \circ (CxA) \circ \left({}^{C}\rho_{Q} PA \right) \circ (\chi PA) \circ (QP\delta_{C} A) \circ (QPCx)$$
$$\circ \left(QP {}^{C}\rho_{Q} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{\delta_{C}}{=} (Cm_{A}) \circ (CxA) \circ \left({}^{C}\rho_{Q} PA \right) \circ (\chi PA) \circ (QPQPx) \circ (QP\delta_{C} QP)$$
$$\circ \left(QP {}^{C}\rho_{Q} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{\chi}{=} (Cm_{A}) \circ (CxA) \circ \left({}^{C}\rho_{Q} PA \right) \circ (QPx) \circ (\chi PQP) \circ (QP\delta_{C} QP)$$
$$\circ \left(QP {}^{C}\rho_{Q} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{{}^{C}\rho_{Q}}{=} (Cm_{A}) \circ (CxA) \circ (CQPx) \circ \left({}^{C}\rho_{Q} PQP \right) \circ (\chi PQP) \circ (QP\delta_{C} QP)$$
$$\circ \left(QP {}^{C}\rho_{Q} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{x,(127)}{=} (Cm_{A}) \circ (Cxx) \circ \left({}^{C}\rho_{Q} PQP \right) \circ (\chi PQP) \circ (QPQ\delta_{D} P)$$
$$\circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{(102)}{=} (Cx) \circ (C\chi P) \circ \left({}^{C}\rho_{Q} PQP \right) \circ (\chi PQP) \circ (QPQ\delta_{D} P)$$
$$\circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{\chi}{=} (Cx) \circ (C\chi P) \circ \left({}^{C}\rho_{Q} PQP \right) \circ (Q\delta_{D} P) \circ (\chi DP) \circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P)$$
$$\circ (QPQP\delta_{C})$$

$$\overset{{}^{C}\rho_{Q}}{=} (Cx) \circ (C\chi P) \circ (CQ\delta_{D} P) \circ \left({}^{C}\rho_{Q} DP \right) \circ (\chi DP) \circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P)$$
$$\circ (QPQP\delta_{C})$$

$$\overset{(113)}{=} (Cx) \circ \left(CQ\varepsilon^{D} P \right) \circ \left({}^{C}\rho_{Q} DP \right) \circ (\chi DP) \circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P)$$
$$\circ (QPQP\delta_{C})$$

$$\overset{{}^{C}\rho_{Q}}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ \left(Q\varepsilon^{D} P \right) \circ (\chi DP) \circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{\chi}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ \left(QPQ\varepsilon^{D} P \right) \circ \left(QP\rho_{Q}^{D} P \right) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{Q\mathrm{comfun}}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ (QP\chi P) \circ (QPQP\delta_{C})$$

$$\overset{(111)}{=} (Cx) \circ \left({}^{C}\rho_{Q} P \right) \circ (\chi P) \circ (\chi PQP) \circ (QPQP\delta_{C})$$

$$\overset{\chi}{=} (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C}) \circ (\chi PC)$$

$$\overset{\text{def}\lambda}{=} \Lambda \circ (xC) \circ (\chi PC) \overset{(102)}{=} \Lambda \circ (m_{A}C) \circ (xxC)$$

so that we get

$$(Cm_{A}) \circ (\Lambda A) \circ (A\Lambda) \circ (xxC) = \Lambda \circ (m_{A}C) \circ (xxC)$$

and since xxC is an epimorphism we obtain

$$(Cm_{A}) \circ (\Lambda A) \circ (A\Lambda) = \Lambda \circ (m_{A}C) .$$

Let now compute

$$(C\Lambda) \circ (\Lambda C) \circ \left(A\Delta^{C}\right) \circ (xC) \overset{x}{=} (C\Lambda) \circ (\Lambda C) \circ (xCC) \circ \left(QP\Delta^{C}\right)$$

$$\overset{\text{def}\lambda}{=} (C\Lambda) \circ (CxC) \circ \left({}^{C}\rho_{Q}PC\right) \circ (\chi PC) \circ (QP\delta_{C}C) \circ \left(QP\Delta^{C}\right)$$

$$\overset{\text{def}\lambda}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ (C\chi P) \circ (CQP\delta_{C}) \circ \left({}^{C}\rho_{Q}PC\right) \circ (\chi PC) \circ (QP\delta_{C}C) \circ \left(QP\Delta^{C}\right)$$

$$\overset{{}^{C}\rho_{Q},(125)}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ (C\chi P) \circ \left({}^{C}\rho_{Q}PQP\right) \circ (QP\delta_{C}) \circ (\chi PC) \circ \left(QPQ\rho_{P}^{C}\right) \circ (QP\delta_{C})$$

$$\overset{\chi}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ (C\chi P) \circ \left({}^{C}\rho_{Q}PQP\right) \circ (QP\delta_{C}) \circ \left(Q\rho_{P}^{C}\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{{}^{C}\rho_{Q}}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ (C\chi P) \circ (CQP\delta_{C}) \circ \left(CQ\rho_{P}^{C}\right) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{(127)}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ (C\chi P) \circ (CQ\delta_{D}P) \circ \left(CQ^{D}\rho_{P}\right) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{(113)}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ \left(CQ\varepsilon^{D}P\right) \circ \left(CQ^{D}\rho_{P}\right) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{Q\text{comfun}}{=} (CCx) \circ \left(C^{C}\rho_{Q}P\right) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{Q\text{comfun}}{=} (CCx) \circ \left(\Delta^{C}QP\right) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C})$$

$$\overset{\Delta^{C}}{=} \left(\Delta^{C}A\right) \circ (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C}) \overset{\text{def}\lambda}{=} \left(\Delta^{C}A\right) \circ \Lambda \circ (xC)$$

so that we get

$$(C\Lambda) \circ (\Lambda C) \circ \left(A\Delta^{C}\right) \circ (xC) = \left(\Delta^{C}A\right) \circ \Lambda \circ (xC)$$

and since xC is an epimorphism we deduce that

$$(C\Lambda) \circ (\Lambda C) \circ \left(A\Delta^{C}\right) = \left(\Delta^{C}A\right) \circ \Lambda.$$

Now we compute

$$\Lambda \circ (u_{A}C) \circ \left(\varepsilon^{C}C\right) \overset{(103)}{=} \Lambda \circ (xC) \circ (\delta_{C}C)$$

$$= (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_{C}) \circ (\delta_{C}C)$$

$$\overset{\delta_{C}}{=} (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (\delta_{C}QP) \circ (C\delta_{C})$$

$$\overset{(112)}{=} (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ \left(\varepsilon^{C}QP\right) \circ (C\delta_{C}) = (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ \delta_{C} \circ \left(\varepsilon^{C}C\right)$$

$$\overset{(112)}{=} (Cx) \circ (C\delta_C) \circ \Delta^C \circ \left(\varepsilon^C C\right) \overset{(103)}{=} (Cu_A) \circ \left(C\varepsilon^C\right) \circ \Delta^C \circ \left(\varepsilon^C C\right)$$
$$\overset{C\text{comonad}}{=} (Cu_A) \circ \left(\varepsilon^C C\right)$$

and since $\varepsilon^C C$ is an epimorphism we get that

$$\Lambda \circ (u_A C) = (Cu_A) .$$

Finally we compute

$$\left(\varepsilon^C A\right) \circ \Lambda \circ (xC) \overset{\text{def}\lambda}{=} \left(\varepsilon^C A\right) \circ (Cx) \circ \left({}^C\rho_Q P\right) \circ (\chi P) \circ (QP\delta_C)$$
$$\overset{\varepsilon^C}{=} x \circ \left(\varepsilon^C QP\right) \circ \left({}^C\rho_Q P\right) \circ (\chi P) \circ (QP\delta_C)$$
$$\overset{Q\text{comfun}}{=} x \circ (\chi P) \circ (QP\delta_C) \overset{(102)}{=} m_A \circ (xx) \circ (QP\delta_C)$$
$$\overset{x}{=} m_A \circ (xA) \circ (QPx) \circ (QP\delta_C) \overset{(103)}{=} m_A \circ (xA) \circ (QPu_A) \circ \left(QP\varepsilon^C\right)$$
$$\overset{x}{=} m_A \circ (Au_A) \circ x \circ \left(QP\varepsilon^C\right) \overset{A\text{comonad},x}{=} \left(A\varepsilon^C\right) \circ (xC)$$

and since xC is an epimorphism we get that

$$\left(\varepsilon^C A\right) \circ \Lambda = A\varepsilon^C .$$

2) Consider the functorial morphism given by

$$DPQ \xrightarrow{\delta_D PQ} PQPQ \xrightarrow{P\chi} PQ \xrightarrow{P\rho_Q^D} PQD \xrightarrow{yD} BD$$
$$\gamma = (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) .$$

Recall that $z^l = (P\chi) \circ (\delta_D PQ)$ and $z^r = \varepsilon^D PQ : DPQ \to PQ$ and let us compute

$$\gamma \circ \left(Dz^l\right) \overset{?}{=} \gamma \circ (Dz^r)$$

that is

$$(yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \circ (DP\chi) \circ (D\delta_D PQ)$$
$$\overset{?}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \circ \left(D\varepsilon^D PQ\right) .$$

Let us compute

$$(yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \circ (DP\chi) \circ (D\delta_D PQ)$$
$$\overset{\delta_D}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (PQP\chi) \circ (\delta_D PQPQ) \circ (D\delta_D PQ)$$
$$\overset{\delta_D,(111)}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (P\chi PQ) \circ (PQ\delta_D PQ) \circ (\delta_D DPQ)$$
$$\overset{(113)}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ \left(PQ\varepsilon^D PQ\right) \circ (\delta_D DPQ)$$
$$\overset{\delta_D}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \circ \left(D\varepsilon^D PQ\right) .$$

Since $(DB, Dy) = \mathrm{Coequ}_{\mathrm{Fun}}\left(Dz^l, Dz^r\right)$, there exists a functorial morphism $\Gamma : DB \to BD$ such that

$$\Gamma \circ (Dy) = \gamma = (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) .$$

We want to prove that Γ is an opposite mixed distributive law. We compute

$$
\begin{aligned}
(m_B D)\circ(B\Gamma)\circ(\Gamma B)\circ(Dyy) &\overset{y}{=} (m_B D)\circ(B\Gamma)\circ(\Gamma B)\circ(DyB)\circ(DPQy)\\
&\overset{\mathrm{def}\gamma}{=} (m_B D)\circ(B\Gamma)\circ(yDB)\circ\left(P\rho_Q^D B\right)\circ(P\chi B)\circ(\delta_D PQB)\circ(DPQy)\\
&\overset{\delta_D}{=} (m_B D)\circ(B\Gamma)\circ(yDB)\circ\left(P\rho_Q^D B\right)\circ(P\chi B)\circ(PQPQy)\circ(\delta_D PQPQ)\\
&\overset{\chi}{=} (m_B D)\circ(B\Gamma)\circ(yDB)\circ\left(P\rho_Q^D B\right)\circ(PQy)\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{\rho_Q^D}{=} (m_B D)\circ(B\Gamma)\circ(yDB)\circ(PQDy)\circ\left(P\rho_Q^D PQ\right)\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{y}{=} (m_B D)\circ(B\Gamma)\circ(BDy)\circ(yDPQ)\circ\left(P\rho_Q^D PQ\right)\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{\mathrm{def}\gamma}{=} (m_B D)\circ(ByD)\circ\left(BP\rho_Q^D\right)\circ(BP\chi)\circ(B\delta_D PQ)\circ(yDPQ)\circ\left(P\rho_Q^D PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{y}{=} (m_B D)\circ(yDPQ)\circ(PQyD)\circ\left(PQP\rho_Q^D\right)\circ(PQP\chi)\circ(PQ\delta_D PQ)\circ\left(P\rho_Q^D PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{y}{=} (m_B D)\circ(yyD)\circ\left(PQP\rho_Q^D\right)\circ(PQP\chi)\circ(PQ\delta_D PQ)\circ\left(P\rho_Q^D PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{(109)}{=} (yD)\circ(P\chi D)\circ\left(PQP\rho_Q^D\right)\circ(PQP\chi)\circ(PQ\delta_D PQ)\circ\left(P\rho_Q^D PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{(127)}{=} (yD)\circ(P\chi D)\circ\left(PQP\rho_Q^D\right)\circ(PQP\chi)\circ(P\delta_C QPQ)\circ\left(P^C\rho_Q PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{\delta_C}{=} (yD)\circ(P\chi D)\circ(P\delta_C QD)\circ\left(PC\rho_Q^D\right)\circ(PC\chi)\circ\left(P^C\rho_Q PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{(112)}{=} (yD)\circ\left(P\varepsilon^C QD\right)\circ\left(PC\rho_Q^D\right)\circ(PC\chi)\circ\left(P^C\rho_Q PQ\right)\\
&\qquad\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{\varepsilon^C}{=} (yD)\circ\left(P\rho_Q^D\right)\circ(P\chi)\circ\left(P\varepsilon^C QPQ\right)\circ\left(P^C\rho_Q PQ\right)\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{Q\mathrm{comfun}}{=} (yD)\circ\left(P\rho_Q^D\right)\circ(P\chi)\circ(P\chi PQ)\circ(\delta_D PQPQ)\\
&\overset{(111)}{=} (yD)\circ\left(P\rho_Q^D\right)\circ(P\chi)\circ(PQP\chi)\circ(\delta_D PQPQ)\\
&\overset{\delta_D}{=} (yD)\circ\left(P\rho_Q^D\right)\circ(P\chi)\circ(\delta_D PQ)\circ(DP\chi)\\
&\overset{\mathrm{def}\gamma}{=} \Gamma\circ(Dy)\circ(DP\chi)\overset{(109)}{=}\Gamma\circ(Dm_B)\circ(Dyy)
\end{aligned}
$$

and since Dyy is an epimorphism we deduce that

$$(m_B D)\circ(B\Gamma)\circ(\Gamma B)=\Gamma\circ(Dm_B).$$

Let us compute

$$(\Gamma D)\circ(D\Gamma)\circ\left(\Delta^D B\right)\circ(Dy)\overset{\Delta^D}{=}(\Gamma D)\circ(D\Gamma)\circ(DDy)\circ\left(\Delta^D PQ\right)$$

$$
\begin{aligned}
&\overset{\text{def}\gamma}{=} (\Gamma D) \circ (DyD) \circ \left(DP\rho_Q^D\right) \circ (DP\chi) \circ (D\delta_D PQ) \circ \left(\Delta^D PQ\right) \\
&\overset{\text{def}\gamma}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ (P\chi D) \circ (\delta_D PQD) \circ \left(DP\rho_Q^D\right) \circ (DP\chi) \\
&\qquad \circ (D\delta_D PQ) \circ \left(\Delta^D PQ\right) \\
&\overset{\delta_D}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ (P\chi D) \circ \left(PQP\rho_Q^D\right) \circ (PQP\chi) \circ (PQ\delta_D PQ) \\
&\qquad \circ (\delta_D DPQ) \circ \left(\Delta^D PQ\right) \\
&\overset{(126)}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ (P\chi D) \circ \left(PQP\rho_Q^D\right) \circ (PQP\chi) \circ (PQ\delta_D PQ) \\
&\qquad \circ \left(P\rho_Q^D PQ\right) \circ (\delta_D PQ) \\
&\overset{(127)}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ (P\chi D) \circ \left(PQP\rho_Q^D\right) \circ (PQP\chi) \circ (P\delta_C QPQ) \\
&\qquad \circ \left(P^C\rho_Q PQ\right) \circ (\delta_D PQ) \\
&\overset{\delta_C}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ (P\chi D) \circ (P\delta_C QD) \circ \left(PC\rho_Q^D\right) \circ (PC\chi) \circ \left(P^C\rho_Q PQ\right) \\
&\qquad \circ (\delta_D PQ) \\
&\overset{(112)}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ \left(P\varepsilon^C QD\right) \circ \left(PC\rho_Q^D\right) \circ (PC\chi) \circ \left(P^C\rho_Q PQ\right) \\
&\qquad \circ (\delta_D PQ) \\
&\overset{\varepsilon^C}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ \left(P\varepsilon^C QPQ\right) \circ \left(P^C\rho_Q PQ\right) \circ (\delta_D PQ) \\
&\overset{Q\text{comfun}}{=} (yDD) \circ \left(P\rho_Q^D D\right) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \\
&\overset{Q\text{comfun}}{=} (yDD) \circ \left(PQ\Delta^D\right) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \\
&\overset{y}{=} \left(B\Delta^D\right) \circ (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \overset{\text{def}\gamma}{=} \left(B\Delta^D\right) \circ \Gamma \circ (Dy)
\end{aligned}
$$

and since Dy is an epimorphism we get that

$$(\Gamma D) \circ (D\Gamma) \circ \left(\Delta^D B\right) = \left(B\Delta^D\right) \circ \Gamma.$$

Now we compute

$$
\begin{aligned}
&\Gamma \circ (Du_B) \circ \left(D\varepsilon^D\right) \overset{(110)}{=} \Gamma \circ (Dy) \circ (D\delta_D) \\
&\overset{\text{def}\gamma}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \circ (D\delta_D) \\
&\overset{\delta_D}{=} (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (PQ\delta_D) \circ (\delta_D D) \\
&\overset{(113)}{=} (yD) \circ \left(P\rho_Q^D\right) \circ \left(PQ\varepsilon^D\right) \circ (\delta_D D) \\
&\overset{\rho_Q^D}{=} (yD) \circ \left(PQD\varepsilon^D\right) \circ \left(P\rho_Q^D D\right) \circ (\delta_D D) \\
&\overset{(126)}{=} (yD) \circ \left(PQD\varepsilon^D\right) \circ (\delta_D DD) \circ \left(\Delta^D D\right) \\
&\overset{\delta_D}{=} (yD) \circ (\delta_D D) \circ \left(DD\varepsilon^D\right) \circ \left(\Delta^D D\right) \overset{\Delta^D}{=} (yD) \circ (\delta_D D) \circ \Delta^D \circ \left(D\varepsilon^D\right) \\
&\overset{(110)}{=} (u_B D) \circ \left(\varepsilon^D D\right) \circ \Delta^D \circ \left(D\varepsilon^D\right) \overset{D\text{comonad}}{=} (u_B D) \circ \left(D\varepsilon^D\right)
\end{aligned}
$$

and since $D\varepsilon^D$ is an epimorphism we deduce that

$$\Gamma \circ (Du_B) = u_B D.$$

Finally we compute

$$\begin{aligned}
\left(B\varepsilon^D\right) \circ \Gamma \circ (Dy) &\overset{\text{def}\gamma}{=} \left(B\varepsilon^D\right) \circ (yD) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \\
&\overset{y}{=} y \circ \left(PQ\varepsilon^D\right) \circ \left(P\rho_Q^D\right) \circ (P\chi) \circ (\delta_D PQ) \overset{Q\text{comfun}}{=} y \circ (P\chi) \circ (\delta_D PQ) \\
&\overset{(106)}{=} y \circ \left(\varepsilon^D PQ\right) \overset{\varepsilon^D}{=} \left(\varepsilon^D B\right) \circ (Dy)
\end{aligned}$$

and since Dy is an epimorphism we get that

$$\left(B\varepsilon^D\right) \circ \Gamma = \varepsilon^D B.$$

$\square$

6.11. **Coherds and coGalois functors.** We keep the details of this subsection because the coGalois case is not as common as the Galois notion in the literature.

LEMMA 6.39. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a formal codual structure where $Q : \mathcal{B} \to \mathcal{A}$, $P : \mathcal{A} \to \mathcal{B}$ and $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ is a comonad on the category $\mathcal{A}$, $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ is a comonad on $\mathcal{B}$. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have equalizers and that C, QD preserve them. Then, $\delta_C : C \to QP$ induces a morphism $\delta_C : {}^{\mathbb{C}}U \to QP^C$ in ${}^{\mathbb{C}}\mathcal{A}$ so that there exists a morphism ${}^C\delta_C^C : {}^{\mathbb{C}}\mathcal{A} \to {}^CQP^C$ such that*

$$(134) \qquad {}^{\mathbb{C}}U^C \delta_C^C = \delta_C^C.$$

Moreover $\delta_C^{C\mathbb{C}}F = \delta_C : {}^{\mathbb{C}}U^{\mathbb{C}}F = C \to QP^{C\mathbb{C}}F = QP.$

Proof. Let us consider the following diagram with notations of Proposition 4.29

$$\begin{tikzcd}
{}^{\mathbb{C}}U \arrow[r, "{}^{\mathbb{C}}U\gamma^C"] \arrow[d, dotted, "\delta_C^C"] & C{}^{\mathbb{C}}U \arrow[r, shift left, "\Delta^{C\mathbb{C}}U"] \arrow[r, shift right, "C{}^{\mathbb{C}}U\gamma^C"'] \arrow[d, "\delta_C{}^{\mathbb{C}}U"] & CC{}^{\mathbb{C}}U \arrow[d, "\delta_C C{}^{\mathbb{C}}U"] \\
QP^C \arrow[r, "Q\iota^P"] & QP^{\mathbb{C}}U \arrow[r, shift left, "Q\rho_P^{C\mathbb{C}}U"] \arrow[r, shift right, "QP^{\mathbb{C}}U\gamma^C"'] & QPC^{\mathbb{C}}U
\end{tikzcd}$$

Since QD preserves equalizers, by Lemma 4.18, also the functor Q preserves equalizers. Since $\left(\delta_C^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)$ equalizes the pair $\left(Q\rho_P^{C\mathbb{C}}U, QP^{\mathbb{C}}U\gamma^C\right)$ and $\left(QP^C, Q\iota^P\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(Q\rho_P^{C\mathbb{C}}U, QP^{\mathbb{C}}U\gamma^C\right)$, by the universal property of the equalizer, there exists a unique morphism $\delta_C^C : {}^{\mathbb{C}}U \to QP^C$ such that

$$(135) \qquad \left(Q\iota^P\right) \circ \delta_C^C = \left(\delta_C{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right).$$

We now want to prove that $\delta_C^C : {}^{\mathbb{C}}U \to QP^C = {}^{\mathbb{C}}U^C QP^C$ is a morphism between left $\mathbb{C}$-comodule functors which satisfies

$$\left(C\delta_C^C\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) = \left({}^C\rho_Q P^C\right) \circ \delta_C^C.$$

We have

$$\begin{aligned}
\left(CQ\iota^P\right) \circ \left(C\delta_C^C\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) &\overset{(135)}{=} \left(C\delta_C{}^{\mathbb{C}}U\right) \circ \left(C{}^{\mathbb{C}}U\gamma^C\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) \\
\overset{{}^{\mathbb{C}}U\gamma^C\text{equ}}{=} \left(C\delta_C{}^{\mathbb{C}}U\right) \circ \left(\Delta^{C\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right) &\overset{(125)}{=} \left({}^C\rho_Q P^{\mathbb{C}}U\right) \circ \left(\delta_C{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)
\end{aligned}$$

$$\overset{(135)}{=} \left({}^{C}\rho_{Q}P^{\mathbb{C}}U\right) \circ \left(Q\iota^{P}\right) \circ \delta_{C}^{C} \overset{{}^{C}\rho_{Q}}{=} \left(CQ\iota^{P}\right) \circ \left({}^{C}\rho_{Q}P^{C}\right) \circ \delta_{C}^{C}$$

and since C, Q preserve equalizers, $CQ\iota^{P}$ is a monomorphism, so that we get

$$\left(C\delta_{C}^{C}\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\right) = \left({}^{C}\rho_{Q}P^{C}\right) \circ \delta_{C}^{C}.$$

Hence, by Lemma 4.28, there exists a unique morphism there exists a unique morphism ${}^{C}\delta_{C}^{C} : {}^{\mathbb{C}}\mathcal{A} \to {}^{C}QP^{C}$ such that

$${}^{\mathbb{C}}U^{C}\delta_{C}^{C} = \delta_{C}^{C}.$$

Moreover, note that by definition of δ_{C}^{C} we have

$$\left(Q\iota^{P}\right) \circ \delta_{C}^{C} = \left(\delta_{C}{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\right)$$

so that by applying it to ${}^{\mathbb{C}}F$ we get

$$\left(Q\iota^{P\mathbb{C}}F\right) \circ \left(\delta_{C}^{C\mathbb{C}}F\right) = \left(\delta_{C}{}^{\mathbb{C}}U^{\mathbb{C}}F\right) \circ \left({}^{\mathbb{C}}U\gamma^{C\mathbb{C}}F\right).$$

Hence, by Proposition 4.32, we obtain that

$$\left(Q\rho_{Q}^{D}\right) \circ \left(\delta_{C}^{C\mathbb{C}}F\right) = \left(\delta_{C}C\right) \circ \Delta^{C} \overset{(125)}{=} \left(Q\rho_{P}^{C}\right) \circ \delta_{C}.$$

Since $Q\rho_{Q}^{D}$ is a monomorphism, we deduce that $\delta_{C}^{C\mathbb{C}}F = \delta_{C}$. $\square$

Proposition 6.40. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with coequalizers and let $\chi : QPQ \to Q$ be a regular coherd for a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ where the underlying functors $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$ and $C : \mathcal{A} \to \mathcal{A}$ preserve coequalizers. Let*

- *$\mathbb{A} = (A, m_A, u_A)$ be the monad on the category $\mathcal{A}$ constructed in Proposition 6.25;*
- *$\left(Q, {}^{A}\mu_{Q}\right)$ be the left $\mathbb{A}$-module functor constructed in Proposition 6.25;*
- *${}^{C}Q : \mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ be the functor defined in Lemma 4.28;*
- *$\Lambda : AC \to CA$ be the mixed distributive law between the comonad $\mathbb{C}$ and the monad $\mathbb{A}$ constructed in Proposition 6.38;*
- *$\widetilde{\mathbb{A}}$ be the lifting of $\mathbb{A}$ on the category ${}^{\mathbb{C}}\mathcal{A}$ constructed in Theorem 5.7.*

Then there exists a functorial morphism ${}^{\widetilde{A}}\mu_{{}^{C}Q} : \widetilde{A}^{C}Q \to {}^{C}Q$ such that

$${}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^{C}Q} = {}^{A}\mu_{Q}.$$

Moreover, $\left({}^{C}Q, {}^{\widetilde{A}}\mu_{{}^{C}Q}\right)$ is a left $\widetilde{\mathbb{A}}$-module functor.

Proof. Since $\chi : QPQ \to Q$ is a regular coherd for $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$, by Proposition 6.38 the mixed distributive law $\Lambda : AC \to CA$ is uniquely defined by

$$\Lambda \circ (xC) = (Cx) \circ \left({}^{C}\rho_{Q}P\right) \circ (\chi P) \circ (QP\delta_C).$$

Now we prove that ${}^{A}\mu_{Q}$ yields a functorial morphism ${}^{\widetilde{A}}\mu_{{}^{C}Q}$. In fact we have

$$\left(C^{A}\mu_{Q}\right) \circ (\Lambda Q) \circ \left(A^{C}\rho_{Q}\right) \circ (xQ) \overset{x}{=} \left(C^{A}\mu_{Q}\right) \circ (\Lambda Q) \circ (xCQ) \circ \left(QP^{C}\rho_{Q}\right)$$

$$\overset{\mathrm{def}\Lambda}{=} \left(C^{A}\mu_{Q}\right) \circ (CxQ) \circ \left({}^{C}\rho_{Q}PQ\right) \circ (\chi PQ) \circ (QP\delta_C Q) \circ \left(QP^{C}\rho_{Q}\right)$$

$$\overset{(101),(127)}{=} (C\chi) \circ \left({}^{C}\rho_{Q}PQ\right) \circ (\chi PQ) \circ (QPQ\delta_D) \circ \left(QP\rho_{Q}^{D}\right)$$

$$\overset{\chi}{=} (C\chi) \circ \left({}^{C}\rho_{Q}PQ\right) \circ (Q\delta_{D}) \circ (\chi D) \circ \left(QP\rho_{Q}^{D}\right)$$

$$\overset{{}^{C}\rho_{Q}}{=} (C\chi) \circ (CQ\delta_{D}) \circ \left({}^{C}\rho_{Q}D\right) \circ (\chi D) \circ \left(QP\rho_{Q}^{D}\right)$$

$$\overset{(130)}{=} \left(CQ\varepsilon^{D}\right) \circ \left({}^{C}\rho_{Q}D\right) \circ (\chi D) \circ \left(QP\rho_{Q}^{D}\right)$$

$$\overset{{}^{C}\rho_{Q}}{=} {}^{C}\rho_{Q} \circ \left(Q\varepsilon^{D}\right) \circ (\chi D) \circ \left(QP\rho_{Q}^{D}\right) \overset{\chi}{=} {}^{C}\rho_{Q} \circ \chi \circ \left(QPQ\varepsilon^{D}\right) \circ \left(QP\rho_{Q}^{D}\right)$$

$$\overset{Q\text{comfun}}{=} {}^{C}\rho_{Q} \circ \chi \overset{(101)}{=} {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q} \circ (xQ)$$

and since by construction xQ is an epimorphism we get that

$$\left(C^{A}\mu_{Q}\right) \circ (\Lambda Q) \circ \left(A^{C}\rho_{Q}\right) = {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q}.$$

By Lemma 5.5 we know that

$$(\Lambda Q) \circ \left(A^{C}\rho_{Q}\right) = {}^{C}\rho_{AQ}$$

so that

$$\left(C^{A}\mu_{Q}\right) \circ {}^{C}\rho_{AQ} = \left(C^{A}\mu_{Q}\right) \circ (\Lambda Q) \circ \left(A^{C}\rho_{Q}\right) = {}^{C}\rho_{Q} \circ {}^{A}\mu_{Q}.$$

Hence there exists a morphism ${}^{\widetilde{A}}\mu_{{}^{C}Q} : \widetilde{A}^{C}Q \to {}^{C}Q$ such that

$${}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^{C}Q} = {}^{A}\mu_{Q}.$$

By the associativity and unitality properties of ${}^{A}\mu_{Q}$, we deduce that ${}^{\widetilde{A}}\mu_{{}^{C}Q}$ is also associative and unital so that $\left({}^{C}Q, {}^{\widetilde{A}}\mu_{{}^{C}Q}\right)$ is a left $\widetilde{\mathbb{A}}$-module functor. $\square$

Lemma 6.41. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_{C}, \delta_{D})$ be a formal codual structure with underlying functors $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $C : \mathcal{A} \to \mathcal{A}$ and $D : \mathcal{B} \to \mathcal{B}$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are categories with equalizers and C, QD preserves them. Assume that*

- *$\mathbb{A} = (A, m_{A}, u_{A})$ is a monad on the category $\mathcal{A}$ such that A preserves equalizers*
- *$\left(Q, {}^{A}\mu_{Q}\right)$ is a left $\mathbb{A}$-module functor*
- *$\widetilde{\mathbb{A}} = \left(\widetilde{A}, m_{\widetilde{A}}, u_{\widetilde{A}}\right)$ is a lifting of the monad of $\mathbb{A}$ to the category ${}^{\mathbb{C}}\mathcal{A}$*
- *$\left({}^{C}Q, {}^{\widetilde{A}}\mu_{{}^{C}Q}\right)$ is a left $\widetilde{\mathbb{A}}$-module functor where ${}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^{C}Q} = {}^{A}\mu_{Q}$.*

Consider the functorial morphisms

$$cocan_{1} := \left({}^{A}\mu_{Q}P\right) \circ (A\delta_{C}) : AC \to QP$$

and

$${}^{C}cocan^{C} := \left({}^{\widetilde{A}}\mu_{{}^{C}Q}P^{C}\right) \circ \left(\widetilde{A}^{C}\delta_{C}^{C}\right) : \widetilde{A} \to {}^{C}QP^{C}.$$

Then $cocan_{1}$ is an isomorphism if and only if ${}^{C}cocan^{C}$ is an isomorphism.

Proof. Let us consider $cocan_{1} := \left({}^{A}\mu_{Q}P\right) \circ (A\delta_{C}) : AC \to QP$. Let $\left(Q^{D}, \iota^{Q}\right)$ be the equalizer described in Proposition 4.29. Since ${}^{A}\mu_{Q}$ is a functorial morphism, we have that

$$\left(Q\iota^{P}\right) \circ \left({}^{A}\mu_{Q}P^{C}\right) = \left({}^{A}\mu_{Q}P^{\mathbb{C}}U\right) \circ \left(AQ\iota^{P}\right).$$

Now, by Lemma 6.39, δ_{C} induces a morphism $\delta_{C}^{C} : {}^{\mathbb{C}}U \to QP^{C}$ such that

$$\left(Q\iota^{P}\right) \circ \delta_{C}^{C} = \left(\delta_{C}{}^{\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^{C}\right).$$

Then, we can consider the morphism

$$cocan^C := \left({}^A\mu_Q P^C\right)\circ\left(A\delta_C^C\right) : A^{\mathbb{C}}U = {}^{\mathbb{C}}U\widetilde{A} \to QP^C = {}^{\mathbb{C}}U^C QP^C. \tag{136}$$

Then we have

$$\left(Q\iota^P\right)\circ cocan^C = \left(Q\iota^P\right)\circ\left({}^A\mu_Q P^C\right)\circ\left(A\delta_C^C\right) \overset{{}^A\mu_Q}{=} \left({}^A\mu_Q P^{\mathbb{C}}U\right)\circ\left(AQ\iota^P\right)\circ\left(A\delta_C^C\right)$$
$$\overset{(135)}{=} \left({}^A\mu_Q P^{\mathbb{C}}U\right)\circ\left(A\delta_C{}^{\mathbb{C}}U\right)\circ\left(A^{\mathbb{C}}U\gamma^C\right) = \left(cocan_1{}^{\mathbb{C}}U\right)\circ\left(A^{\mathbb{C}}U\gamma^C\right)$$

i.e.

$$\left(Q\iota^P\right)\circ cocan^C = \left(cocan_1{}^{\mathbb{C}}U\right)\circ\left(A^{\mathbb{C}}U\gamma^C\right). \tag{137}$$

Now, by assumption we have

$${}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^C Q} = {}^A\mu_Q$$

so that

$${}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^C Q}P^C = {}^A\mu_Q P^C.$$

Moreover, by Lemma 6.39, there exists a morphism ${}^C\delta_C \colon {}^{\mathbb{C}}\mathcal{A} \to {}^C QP^C$ such that

$${}^{\mathbb{C}}U^C\delta_C^C = \delta_C^C.$$

Since $\widetilde{\mathbb{A}}$ is a lifting of the monad $\mathbb{A}$, by Theorem 5.7 we have a mixed distributive law $\Phi : AC \to CA$ so that we can apply Proposition 5.6 and we get that

$$A\delta_C^C = A^{\mathbb{C}}U^C\delta_C^C = {}^{\mathbb{C}}U\widetilde{A}^C\delta_C^C$$

where $\widetilde{A}^C\delta_C \colon \widetilde{A} \to \widetilde{A}^C QP^C$ is a functorial morphism. Then we can consider the morphism

$${}^C cocan^C := \left({}^{\widetilde{A}}\mu_{{}^C Q}P^C\right)\circ\left(\widetilde{A}^C\delta_C^C\right) : \widetilde{A} \to {}^C QP^C$$

and we get that

$${}^{\mathbb{C}}U^C cocan^C = \left({}^{\mathbb{C}}U^{\widetilde{A}}\mu_{{}^C Q}P^C\right)\circ\left({}^{\mathbb{C}}U\widetilde{A}^C\delta_C^C\right)$$
$$= \left({}^A\mu_Q P^C\right)\circ\left(A^{\mathbb{C}}U^C\delta_C^C\right) = \left({}^A\mu_Q P^C\right)\circ\left(A\delta_C^C\right) = cocan^C.$$

We compute

$$\left(cocan_1 C^{\mathbb{C}}U\right)\circ\left(A\Delta^{C\mathbb{C}}U\right) = \left({}^A\mu_Q PC^{\mathbb{C}}U\right)\circ\left(A\delta_C C^{\mathbb{C}}U\right)\circ\left(A\Delta^{C\mathbb{C}}U\right)$$
$$\overset{(125)}{=} \left({}^A\mu_Q PC^{\mathbb{C}}U\right)\circ\left(AQ\rho_P^{C\mathbb{C}}U\right)\circ\left(A\delta_C{}^{\mathbb{C}}U\right)$$
$$\overset{{}^A\mu_Q}{=} \left(Q\rho_P^{C\mathbb{C}}U\right)\circ\left({}^A\mu_Q P^{\mathbb{C}}U\right)\circ\left(A\delta_C{}^{\mathbb{C}}U\right) = \left(Q\rho_P^{C\mathbb{C}}U\right)\circ\left(cocan_1{}^{\mathbb{C}}U\right)$$

so that we get

$$\left(cocan_1 C^{\mathbb{C}}U\right)\circ\left(A\Delta^{C\mathbb{C}}U\right) = \left(Q\rho_P^{C\mathbb{C}}U\right)\circ\left(cocan_1{}^{\mathbb{C}}U\right). \tag{138}$$

Let us consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{\mathbb{C}}U \arrow[r, "A^{\mathbb{C}}U\gamma^C"] \arrow[d, "cocan^C"'] & AC^{\mathbb{C}}U \arrow[r, shift left, "A\Delta^{C\mathbb{C}}U"] \arrow[r, shift right, "AC^{\mathbb{C}}U\gamma^C"'] \arrow[d, "cocan_1{}^{\mathbb{C}}U"'] & ACC^{\mathbb{C}}U \arrow[d, "cocan_1 C^{\mathbb{C}}U"] \\
0 \arrow[r] & QP^C \arrow[r, "Q\iota^P"] & QP^{\mathbb{C}}U \arrow[r, shift left, "Q\rho_P^{C\mathbb{C}}U"] \arrow[r, shift right, "QP^{\mathbb{C}}U\gamma^C"'] & QPC^{\mathbb{C}}U
\end{tikzcd}$$

Now, since $cocan_1 : AC \to QP$ is a functorial morphism and by formula (138) , the right square serially commutes. By formula (137) also the left square commutes. Moreover, by definition, ι^P and ${}^{\mathbb{C}}U\gamma^C$ are monomorphisms. Since QD preserves equalizers, by Lemma 4.18 also Q preserves equalizers. Since C, Q preserve equalizers, $Q\iota^P$ and $A^{\mathbb{C}}U\gamma^C$ are also monomorphisms. Then, if $cocan_{1\mathbb{A}}U$ is an isomorphism, also $cocan^C$ is an isomorphism. Since ${}^{\mathbb{C}}U^C cocan^C = cocan^C$, by 4.17, also ${}^C cocan^C$ is an isomorphism.

Conversely, assume that ${}^C cocan^C$ is an isomorphism. Then also $cocan^C = {}^{\mathbb{C}}U^C cocan^C$ is an isomorphism. Then we have

$$\begin{aligned} {}^{\mathbb{C}}U^C cocan^{C\mathbb{C}}F &= cocan^{C\mathbb{C}}F = \left({}^A\mu_Q P^{C\mathbb{C}}F\right) \circ \left(A\delta_C^{C\mathbb{C}}F\right) \\ &\overset{\text{Pro4.32,Lem6.39}}{=} \left({}^A\mu_Q P\right) \circ (A\delta_C) = cocan_1 \end{aligned}$$

so that also $cocan_1$ is an isomorphism. $\square$

6.12. **The cotame case.** The following subsection is presented without proofs, which can be obtained as the dual versions of results of the tame case (see Subsection 6.6).

Definition 6.42. A formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ is called a *coMorita context* on the categories $\mathcal{A}$ and $\mathcal{B}$ if it satisfies also the balanced conditions

$$(139) \qquad \left(\rho_Q^D P\right) \circ \delta_C = \left(Q^D \rho_P\right) \circ \delta_C \text{ and } \left(\rho_P^C Q\right) \circ \delta_D = \left(P^C \rho_Q\right) \circ \delta_D.$$

Lemma 6.43. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a coMorita context on the categories $\mathcal{A}$ and $\mathcal{B}$ and assume that C, D, P, Q preserve equalizers. Hence, there exist functorial morphisms*

- ${}^{CD}\delta_C^{DC} : \mathrm{Id}_{{}^{\mathbb{C}}\mathcal{A}} \to {}^C Q^{DD} P^C$ *such that*

$$(140) \qquad {}^{\mathbb{C}}U^{CD}\delta_C^{DC} = {}^D\delta_C^{DC}$$

where ${}^D\delta_C^{DC}$ *is uniquely determined by* $\left(Q^D \iota^{{}^D P}\right) \circ {}^D\delta_C^{DC} = \left({}^D\delta_C^{DC\mathbb{C}}U\right) \circ \left({}^{\mathbb{C}}U\gamma^C\right)$ *and*

$$(141) \qquad \left(\iota^{QD}P\right) \circ {}^D\delta_C^D = \delta_C$$

- ${}^{DC}\delta_D^{CD} : \mathrm{Id}_{{}^{\mathbb{D}}\mathcal{B}} \to {}^D P^{CC} Q^D$ *such that*

$$(142) \qquad {}^{\mathbb{D}}U^{DC}\delta_D^{CD} = {}^C\delta_D^{CD}$$

where ${}^C\delta_D^{CD}$ *is uniquely determined by* $\left(P^C \iota^{{}^C Q}\right) \circ {}^C\delta_D^{CD} = \left({}^C\delta_D^{C\mathbb{D}}U\right) \circ \left({}^{\mathbb{D}}U\gamma^D\right)$ *and*

$$(143) \qquad \left(\iota^{PC}Q\right) \circ {}^C\delta_D^C = \delta_D$$

Definitions 6.44. Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a coMorita context. We will say that $\mathbb{X}$ is *cotame* if the lifted functorial morphisms ${}^{CD}\delta_C^{DC} : \mathrm{Id}_{{}^{\mathbb{C}}\mathcal{A}} \to {}^C Q^{DD} P^C$ and ${}^{DC}\delta_D^{CD} : \mathrm{Id}_{{}^{\mathbb{D}}\mathcal{B}} \to {}^D P^{CC} Q^D$ are isomorphisms so that the lifted functors ${}^C Q^D : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ and ${}^D P^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ yield a category equivalence. In this case, if $\chi : QPQ \to Q$ is a coherd for $\mathbb{X}$, we will say that χ is a *cotame coherd*.

Proposition 6.45. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a cotame coMorita context. Then unit and counit of the adjunction $\left({}^D P^C, {}^C Q^D\right)$ are given by*
$\eta_{({}^D P^C, {}^C Q^D)} = {}^{CD}\delta_C^{DC}$ *and* $\epsilon_{({}^D P^C, {}^C Q^D)} = \left({}^{DC}\delta_D^{CD}\right)^{-1} \circ \left({}^D P^C \left({}^{CD}\delta_C^{DC}\right)^{-1} {}^C Q^D\right) \circ \left({}^{DC}\delta_D^{CD D} P^{CC} Q^D\right)$ *so that*
$\eta_{(P^C, {}^C Q)} = \left({}^C Q^D \gamma^{D D} P^C\right) \circ {}^{CD}\delta_C^{DC}$ *and* $\epsilon_{(P^C, {}^C Q)} = \varepsilon^D \circ \left({}^{\mathbb{D}} U \left({}^{DC}\delta_D^{CD}\right)^{-1} {}^{\mathbb{D}} F\right) \circ \left({}^{\mathbb{D}} U^D P^C \left({}^{CD}\delta_C^{DC}\right)^{-1} {}^C Q^{D\mathbb{D}} F\right) \circ \left({}^{\mathbb{D}} U^{DC}\delta_D^{CD D} P^{CC} Q^{D\mathbb{D}} F\right)$.

Corollary 6.46. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a cotame coMorita context. Assume that the functors A, B, P, Q preserve equalizers. Then the units of the adjunctions $\left(P^C, {}^C Q\right)$ and $\left(Q^D, {}^D P\right)$ are given by $\epsilon_{(P^C, {}^C Q)} = {}^C\delta_C^C$ and $\epsilon_{(Q^D, {}^D P)} = {}^D\delta_D^D$.*

Lemma 6.47. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a formal codual structure where the underlying functors are $C : \mathcal{A} \to \mathcal{A}$, $D : \mathcal{B} \to \mathcal{B}$, $P : \mathcal{A} \to \mathcal{B}$ and $Q : \mathcal{B} \to \mathcal{A}$. Assume that both categories $\mathcal{A}$ and $\mathcal{B}$ have equalizers and the functors C, QD preserve them. Assume that*

- *$\mathbb{A} = (A, m_A, u_A)$ is a monad on the category $\mathcal{A}$ such that A preserves equalizers*
- *$\widetilde{\mathbb{A}} = \left(\widetilde{A}, m_{\widetilde{A}}, u_{\widetilde{A}}\right)$ is a lifting of the monad $\mathbb{A}$ to the category ${}^{\mathbb{C}}\mathcal{A}$*
- *$\left({}^C Q, {}^{\widetilde{A}}\mu_{{}^C Q}\right)$ is a left $\widetilde{\mathbb{A}}$-module functor*
- *$\mathbb{X}$ is a cotame coMorita context.*

Then cocan_1 is an isomorphism if and only if ${}^C\mathrm{cocan}^C$ is an isomorphism if and only if ${}^C Q$ is a left $\widetilde{\mathbb{A}}$-coGalois functor.

The following Theorem is a formulation, in pure categorical terms, for the coherd version of [BV, Theorem 2.18].

Theorem 6.48. *Let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a regular cotame coMorita context. Assume that*

- *both categories $\mathcal{A}$ and $\mathcal{B}$ have equalizers and coequalizers,*
- *the functors C and D preserve coequalizers,*
- *the functors C, D, P, Q preserve equalizers.*

Then the existence of the following structures are equivalent:

(a) *A coherd $\chi : QPQ \to Q$ for $\mathbb{X}$*
(b) *A monad $\mathbb{A} = (A, m_A, u_A)$ on the category $\mathcal{A}$ such that the functor A preserves coequalizers and a mixed distributive law $\Lambda : AC \to CA$ such that ${}^C Q$ is a coGalois module functor over $\widetilde{\mathbb{A}}$ (where $\widetilde{\mathbb{A}}$ is the lifting of $\mathbb{A}$)*
(c) *A monad $\mathbb{B} = (B, m_B, u_B)$ on the category $\mathcal{B}$ such that the functor B preserves coequalizers and an opposite mixed distributive law $\Gamma : DB \to BD$ such that ${}^D P$ is a coGalois module functor over $\widetilde{\mathbb{B}}$ (where $\widetilde{\mathbb{B}}$ is the lifting of $\mathbb{B}$).*

7.1. Constructing the functor $\overline{Q}$. Our next task is to construct a $\mathbb{D}$-$\mathbb{C}$-bicomodule functor $\overline{Q}$. Such a functor appears in [BM, Section 5], but we give here new notations. For the details of the proofs, see the dual results in the following.

Proposition 7.1. *In the setting of Theorem 6.5, we define functors $\overline{Q} : \mathcal{A} \to \mathcal{B}$ via the equalizer*

$$\overline{Q} \xrightarrow{q} PC \underset{(\theta^r P)\circ(Pi)}{\overset{(\theta^l P)\circ(Pi)}{\rightrightarrows}} BPQP$$

Then there exists a unique functorial morphism $\kappa_0' : \overline{Q} \to DP$ such that

$$(Pi) \circ q = (jP) \circ \kappa_0' \tag{144}$$

Moreover

$$\left(\overline{Q},\kappa_0'\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\left(P\omega^l\right) \circ (jP), (P\omega^r) \circ (jP)\right).$$

The functor $\overline{Q}$ can be equipped with the structure of a $\mathbb{D}$-$\mathbb{C}$-bicomodule functor $\left(\overline{Q}, {}^D\rho_{\overline{Q}}, \rho^C_{\overline{Q}}\right)$ where $\rho^C_{\overline{Q}}$ and ${}^D\rho_{\overline{Q}}$ are uniquely determined by

$$(qC) \circ \rho^C_{\overline{Q}} = \left(P\Delta^C\right) \circ q \tag{145}$$

and

$$(D\kappa_0') \circ {}^D\rho_{\overline{Q}} = \left(\Delta^D P\right) \circ \kappa_0'. \tag{146}$$

Proposition 7.2. *In the setting of Theorem 6.5 and Proposition 7.1, there exist two functorial morphisms $\delta_C : C \to Q\overline{Q}$ and $\delta_D : D \to \overline{Q}Q$ where δ_C is C-bicolinear and δ_D is D-bicolinear and they fulfill*

$$(Qq) \circ \delta_C = (iC) \circ \Delta^C \tag{147}$$

and

$$(\kappa_0' Q) \circ \delta_D = (Dj) \circ \Delta^D. \tag{148}$$

Moreover the coassociative conditions hold, that is

$$(\delta_C Q) \circ {}^C\rho_Q = (Q\delta_D) \circ \rho^D_Q \text{ and } \left(\delta_D \overline{Q}\right) \circ {}^D\rho_{\overline{Q}} = \left(\overline{Q}\delta_C\right) \circ \rho^C_{\overline{Q}}.$$

7.2. From herds to coherds.

7.3. Given an herd $\tau : Q \to QPQ$ in a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$, our purpose is to build the formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D)$ and then a coherd $\chi : Q\overline{Q}Q \to Q$ in $\mathbb{X}$.

Theorem 7.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{B} \to \mathcal{B}$ be functors. Assume that all the functors P, Q, A and B preserve equalizers. Let $u_A : \mathcal{A} \to A$ and $u_B : \mathcal{B} \to B$ be functorial monomorphisms and assume that $(\mathcal{A}, u_A) = \mathrm{Equ}_{\mathrm{Fun}}(u_A A, A u_A)$ and $(\mathcal{B}, u_B) = \mathrm{Equ}_{\mathrm{Fun}}(u_B B, B u_B)$.*

Let $\tau : Q \to QPQ$ be a functorial morphism such that

$$(QP\tau) \circ \tau = (\tau PQ) \circ \tau$$

Let $\sigma^B : PQ \to B$ be a functorial morphism such that

$$(Q\sigma^B) \circ \tau = Qu_B$$

and let $\sigma^A : QP \to A$ be a functorial morphism such that

$$(\sigma^A Q) \circ \tau = u_A Q.$$

Then there is a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D)$.

Proof. In view of Theorem 6.5 and Propositions 7.1, 7.2 a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D)$ has been constructed. $\square$

THEOREM 7.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a regular formal dual structure where $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{B} \to \mathcal{B}$ are functors that preserve equalizers. Let $\tau : Q \to QPQ$ be a pretorsor. Then there is a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D)$. Define $\chi : Q\overline{Q}Q \to Q$ by setting*

$$\chi := \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (QqQ)\,.$$

Then χ is a coherd in $\mathbb{X}$.

Proof. By Theorem 7.4 $\mathbb{X} = (\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D)$ is a formal codual structure. To show that χ is a coherd in $\mathbb{X}$, we have to prove that it satisfies the following conditions.

1) Coassociativity, in the sense that $\chi \circ \left(\chi\overline{Q}Q\right) = \chi \circ \left(Q\overline{Q}\chi\right)$. Let us compute

$$\begin{aligned}
&\chi \circ \left(Q\overline{Q}\chi\right) \\
&= \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (QqQ) \circ \left(Q\overline{Q}\chi\right) \\
&\overset{q}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (QPC\chi) \circ \left(QqQ\overline{Q}Q\right) \\
&\overset{i}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPQP\chi) \circ \left(QPiQ\overline{Q}Q\right) \circ \left(QqQ\overline{Q}Q\right) \\
&\overset{\sigma^A}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ (AQP\chi) \circ \left(\sigma^A QPQ\overline{Q}Q\right) \circ \left(QPiQ\overline{Q}Q\right) \circ \left(QqQ\overline{Q}Q\right) \\
&\overset{{}^A\mu_Q}{=} \mu_Q^B \circ \left(Q\sigma^B\right) \circ (QP\chi) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \circ \left(\sigma^A QPQ\overline{Q}Q\right) \circ \left(QPiQ\overline{Q}Q\right) \circ \left(QqQ\overline{Q}Q\right) \\
&\overset{(82)}{=} {}^A\mu_Q \circ \left(\sigma^A Q\right) \circ (QP\chi) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \circ \left(\sigma^A QPQ\overline{Q}Q\right) \circ \left(QPiQ\overline{Q}Q\right) \circ \left(QqQ\overline{Q}Q\right) \\
&\overset{\sigma^A}{=} {}^A\mu_Q \circ (A\chi) \circ \left(\sigma^A Q\overline{Q}Q\right) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \circ \left(\sigma^A QPQ\overline{Q}Q\right) \circ \left(QPiQ\overline{Q}Q\right) \circ \left(QqQ\overline{Q}Q\right).
\end{aligned}$$

We have

$$\begin{aligned}
&{}^A\mu_Q \circ (A\chi) \circ \left(\sigma^A Q\overline{Q}Q\right) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \\
&= {}^A\mu_Q \circ \left(A\mu_Q^B\right) \circ \left(A^A\mu_Q B\right) \circ \left(AAQ\sigma^B\right) \circ \left(A\sigma^A QPQ\right) \circ (AQPiQ) \circ (AQqQ) \\
&\quad \circ \left(\sigma^A Q\overline{Q}Q\right) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \\
&\overset{Q\text{bimod},\sigma^A}{=} {}^A\mu_Q \circ \left(A^A\mu_Q\right) \circ \left(AA\mu_Q^B\right) \circ \left(A\sigma^A QB\right) \circ \left(AQPQ\sigma^B\right) \circ (AQPiQ) \circ (AQqQ) \\
&\quad \circ \left(\sigma^A Q\overline{Q}Q\right) \circ \left({}^A\mu_Q PQ\overline{Q}Q\right) \\
&\overset{{}^A\mu_Q\ \text{ass}}{=} {}^A\mu_Q \circ (m_A Q) \circ \left(AA\mu_Q^B\right) \circ \left(A\sigma^A QB\right) \circ \left(AQPQ\sigma^B\right) \circ (AQPiQ) \circ (AQqQ)
\end{aligned}$$

$$\circ\left(\sigma^A Q\overline{Q}Q\right)\circ\left({}^A\mu_Q PQ\overline{Q}Q\right)$$

$$\overset{m_A}{=} {}^A\mu_Q\circ\left(A\mu_Q^B\right)\circ\left(m_A QB\right)\circ\left[\left(A\sigma^A QB\right)\circ\left(AQPQ\sigma^B\right)\circ\left(AQPiQ\right)\circ\left(AQqQ\right)\right]$$

$$\circ\left(\sigma^A Q\overline{Q}Q\right)\circ\left({}^A\mu_Q PQ\overline{Q}Q\right)$$

$$\overset{Q\text{is a bim}}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(m_A QB\right)\circ\left[\left(A\sigma^A QB\right)\circ\left(AQPQ\sigma^B\right)\circ\left(AQPiQ\right)\circ\left(AQqQ\right)\right]$$

$$\circ\left(\sigma^A Q\overline{Q}Q\right)\circ\left({}^A\mu_Q PQ\overline{Q}Q\right)$$

$$\overset{{}^A\mu_Q\text{ ass}}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left[\left(A{}^A\mu_Q B\right)\circ\left(A\sigma^A QB\right)\circ\left(AQPQ\sigma^B\right)\circ\left(AQPiQ\right)\circ\left(AQqQ\right)\right]$$

$$\circ\left(\sigma^A Q\overline{Q}Q\right)\circ\left({}^A\mu_Q PQ\overline{Q}Q\right)$$

$$\overset{(80)}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left[\left(A{}^A\mu_Q B\right)\circ\left(A\sigma^A QB\right)\circ\left(AQPQ\sigma^B\right)\circ\left(AQPiQ\right)\circ\left(AQqQ\right)\right]$$

$$\circ\left(m_A Q\overline{Q}Q\right)\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{m_A}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(m_A QB\right)\circ\left[\left(AA{}^A\mu_Q B\right)\circ\left(AA\sigma^A QB\right)\circ\left(AAQPQ\sigma^B\right)\circ\left(AAQPiQ\right)\right.$$

$$\left.\circ\left(AAQqQ\right)\right]\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{{}^A\mu_Q\text{ ass}}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(A{}^A\mu_Q B\right)\circ\left[\left(AA{}^A\mu_Q B\right)\circ\left(AA\sigma^A QB\right)\circ\left(AAQPQ\sigma^B\right)\right.$$

$$\left.\circ\left(AAQPiQ\right)\circ\left(AAQqQ\right)\right]\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{Q\text{is a bim}}{=} {}^A\mu_Q\circ\left(A\mu_Q^B\right)\circ\left(A{}^A\mu_Q B\right)\circ\left(AA{}^A\mu_Q B\right)\circ\left(AA\sigma^A QB\right)\circ\left(AAQPQ\sigma^B\right)$$

$$\circ\left(AAQPiQ\right)\circ\left(AAQqQ\right)\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{Q\text{is a bim}}{=} {}^A\mu_Q\circ\left(A{}^A\mu_Q\right)\circ\left(AA\mu_Q^B\right)\circ\left(AA{}^A\mu_Q B\right)\circ\left(AA\sigma^A QB\right)\circ\left(AAQPQ\sigma^B\right)$$

$$\circ\left(AAQPiQ\right)\circ\left(AAQqQ\right)\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{(82)}{=} {}^A\mu_Q\circ\left(A{}^A\mu_Q\right)\circ\left(AA\mu_Q^B\right)\circ\left(AA\mu_Q^B B\right)\circ\left(AAQ\sigma^B B\right)\circ\left(AAQPQ\sigma^B\right)$$

$$\circ\left(AAQPiQ\right)\circ\left(AAQqQ\right)\circ\left(A\sigma^A Q\overline{Q}Q\right)$$

$$\overset{\sigma^A}{=} {}^A\mu_Q\circ\left(A{}^A\mu_Q\right)\circ\left(A\sigma^A Q\right)\circ\left(AQP\mu_Q^B\right)\circ\left(AQP\mu_Q^B B\right)\circ\left(AQPQ\sigma^B B\right)$$

$$\circ\left(AQPQPQ\sigma^B\right)\circ\left(AQPQPiQ\right)\circ\left(AQPQqQ\right)$$

$$\overset{(82)}{=} {}^A\mu_Q\circ\left(A\mu_Q^B\right)\circ\left(AQ\sigma^B\right)\circ\left(AQP\mu_Q^B\right)\circ\left(AQP\mu_Q^B B\right)\circ\left(AQPQ\sigma^B B\right)$$

$$\circ\left(AQPQPQ\sigma^B\right)\circ\left(AQPQPiQ\right)\circ\left(AQPQqQ\right)$$

$$\overset{Q\text{is a bim}}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(AQ\sigma^B\right)\circ\left(AQP\mu_Q^B\right)\circ\left(AQP\mu_Q^B B\right)\circ\left(AQPQ\sigma^B B\right)$$

$$\circ\left(AQPQPQ\sigma^B\right)\circ\left(AQPQPiQ\right)\circ\left(AQPQqQ\right)$$

$$\overset{\mu_Q^B\text{ass}}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(AQ\sigma^B\right)\circ\left(AQP\mu_Q^B\right)\circ\left[\left(AQPQm_B\right)\circ\left(AQPQ\sigma^B B\right)\right]$$

$$\circ\left(AQPQPQ\sigma^B\right)\circ\left(AQPQPiQ\right)\circ\left(AQPQqQ\right)$$

$$\overset{(81)}{=} \mu_Q^B\circ\left({}^A\mu_Q B\right)\circ\left(AQ\sigma^B\right)\circ\left(AQP\mu_Q^B\right)\circ\left(AQPQ\sigma^B\right)\circ\left(AQPQP\mu_Q^B\right)$$

$$\circ\left(AQPQPQ\sigma^B\right)\circ\left(AQPQPiQ\right)\circ\left(AQPQqQ\right)$$

$$\overset{{}^{A}\mu_{Q}}{=} \mu_{Q}^{B} \circ \left[\left(Q\sigma^{B} \right) \circ \left(QP\mu_{Q}^{B} \right) \right] \circ \left(QPQ\sigma^{B} \right) \circ \left(QPQP\mu_{Q}^{B} \right) \circ \left(QPQPQ\sigma^{B} \right)$$
$$\circ \left(QPQPiQ \right) \circ \left(QPQqQ \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right)$$
$$\overset{(81)}{=} \mu_{Q}^{B} \circ \left(Qm_{B} \right) \circ \left(Q\sigma^{B}B \right) \circ \left(QPQ\sigma^{B} \right) \circ \left(QPQP\mu_{Q}^{B} \right) \circ \left(QPQPQ\sigma^{B} \right)$$
$$\circ \left(QPQPiQ \right) \circ \left(QPQqQ \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right)$$
$$\overset{\sigma^{B}}{=} \mu_{Q}^{B} \circ \left(Qm_{B} \right) \circ \left(QB\sigma^{B} \right) \circ \left(QBP\mu_{Q}^{B} \right) \circ \left(QBPQ\sigma^{B} \right) \circ \left(QBPiQ \right) \circ \left(QBqQ \right)$$
$$\circ \left(Q\sigma^{B}\overline{Q}Q \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right)$$
$$\overset{\mu_{Q}^{B}\text{ass}}{=} \mu_{Q}^{B} \circ \left(\mu_{Q}^{B}B \right) \circ \left(QB\sigma^{B} \right) \circ \left(QBP\mu_{Q}^{B} \right) \circ \left(QBPQ\sigma^{B} \right) \circ \left(QBPiQ \right) \circ \left(QBqQ \right)$$
$$\circ \left(Q\sigma^{B}\overline{Q}Q \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right)$$
$$\overset{\mu_{Q}^{B}}{=} \mu_{Q}^{B} \circ \left(Q\sigma^{B} \right) \circ \left(QP\mu_{Q}^{B} \right) \circ \left(QPQ\sigma^{B} \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left(Q\sigma^{B}\overline{Q}Q \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right)$$
$$\overset{{}^{A}\mu_{Q}}{=} \mu_{Q}^{B} \circ \left(Q\sigma^{B} \right) \circ \left(QP\mu_{Q}^{B} \right) \circ \left(QPQ\sigma^{B} \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left({}^{A}\mu_{Q}B\overline{Q}Q \right) \circ \left(AQ\sigma^{B}\overline{Q}Q \right)$$
$$\overset{(82)}{=} {}^{A}\mu_{Q} \circ \left(\sigma^{A}Q \right) \circ \left(QP\mu_{Q}^{B} \right) \circ \left(QPQ\sigma^{B} \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left({}^{A}\mu_{Q}B\overline{Q}Q \right) \circ \left(AQ\sigma^{B}\overline{Q}Q \right)$$
$$\overset{\sigma^{A}}{=} {}^{A}\mu_{Q} \circ \left(A\mu_{Q}^{B} \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left({}^{A}\mu_{Q}B\overline{Q}Q \right) \circ \left(AQ\sigma^{B}\overline{Q}Q \right)$$
$$\overset{Q\text{is a bim}}{=} \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q}B \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left({}^{A}\mu_{Q}B\overline{Q}Q \right) \circ \left(AQ\sigma^{B}\overline{Q}Q \right)$$

and thus we get

$$\chi \circ \left(Q\overline{Q}\chi \right) =$$
$$= {}^{A}\mu_{Q} \circ \left(A\chi \right) \circ \left(\sigma^{A}Q\overline{Q}Q \right) \circ \left({}^{A}\mu_{Q}PQ\overline{Q}Q \right) \circ \left(\sigma^{A}QPQ\overline{Q}Q \right) \circ \left(QPiQ\overline{Q}Q \right)$$
$$\circ \left(QqQ\overline{Q}Q \right)$$
$$= \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q}B \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(\mu_{Q}^{B}\overline{Q}Q \right)$$
$$\circ \left({}^{A}\mu_{Q}B\overline{Q}Q \right) \circ \left(AQ\sigma^{B}\overline{Q}Q \right) \circ \left(\sigma^{A}QPQ\overline{Q}Q \right) \circ \left(QPiQ\overline{Q}Q \right) \circ \left(QqQ\overline{Q}Q \right)$$
$$= \chi \circ \left(\chi\overline{Q}Q \right)$$

2) Counitality, in the sense that $\chi \circ (Q\delta_D) = Q\varepsilon^D$ and $\chi \circ (\delta_C Q) = \varepsilon^C Q$. We have

$$\chi \circ (Q\delta_D) =$$
$$= \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q}B \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QPiQ \right) \circ \left(QqQ \right) \circ \left(Q\delta_D \right)$$
$$\overset{(144)}{=} \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q}B \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QjPQ \right) \circ \left(Q\kappa_0'Q \right) \circ \left(Q\delta_D \right)$$
$$\overset{(148)}{=} \mu_{Q}^{B} \circ \left({}^{A}\mu_{Q}B \right) \circ \left(AQ\sigma^{B} \right) \circ \left(\sigma^{A}QPQ \right) \circ \left(QjPQ \right) \circ \left(QDj \right) \circ \left(Q\Delta^{D} \right)$$

$$
\begin{aligned}
&= \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (Qjj) \circ \left(Q\Delta^D\right)\\
&\overset{(67)}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QP\tau) \circ (Qj)\\
&\overset{\sigma^A}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(\sigma^A QB\right) \circ \left(QPQ\sigma^B\right) \circ (QP\tau) \circ (Qj)\\
&\overset{(70)}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(\sigma^A QB\right) \circ (QPQu_B) \circ (Qj)\\
&\overset{\sigma^A}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ (AQu_B) \circ \left(\sigma^A Q\right) \circ (Qj)\\
&\overset{Q\text{is a bim}}{=} {}^A\mu_Q \circ \left(A\mu_Q^B\right) \circ (AQu_B) \circ \left(\sigma^A Q\right) \circ (Qj)\\
&\overset{Q\text{is a mod}}{=} {}^A\mu_Q \circ \left(\sigma^A Q\right) \circ (Qj) \overset{(82)}{=} \mu_Q^B \circ \left(Q\sigma^B\right) \circ (Qj)\\
&\overset{(67)}{=} \mu_Q^B \circ (Qu_B) \circ \left(Q\varepsilon^D\right) \overset{Q\text{is a mod}}{=} Q\varepsilon^D.
\end{aligned}
$$

We compute

$$
\begin{aligned}
&\chi \circ (\delta_C Q) =\\
&\mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (QqQ) \circ (\delta_C Q)\\
&\overset{(147)}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (iCQ) \circ \left(\Delta^C Q\right)\\
&= \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (iiQ) \circ \left(\Delta^C Q\right)\\
&\overset{(63)}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (\tau PQ) \circ (iQ)\\
&\overset{(69)}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ (u_A QPQ) \circ (iQ)\\
&\overset{u_A}{=} \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ (u_A QB) \circ \left(Q\sigma^B\right) \circ (iQ)\\
&\overset{Q\text{is a bim}}{=} {}^A\mu_Q \circ \left(A\mu_Q^B\right) \circ (u_A QB) \circ \left(Q\sigma^B\right) \circ (iQ)\\
&\overset{u_A}{=} {}^A\mu_Q \circ (u_A Q) \circ \mu_Q^B \circ \left(Q\sigma^B\right) \circ (iQ)\\
&\overset{Q\text{is a mod}}{=} \mu_Q^B \circ \left(Q\sigma^B\right) \circ (iQ) \overset{(82)}{=} {}^A\mu_Q \circ \left(\sigma^A Q\right) \circ (iQ)\\
&\overset{(63)}{=} {}^A\mu_Q \circ (u_A Q) \circ \left(\varepsilon^C Q\right) \overset{Q\text{is a mod}}{=} \varepsilon^C Q.
\end{aligned}
$$

$\square$

7.3. **Constructing the functor $\widehat{Q}$.** Our next task is to construct a $\mathbb{B}$-$\mathbb{A}$-bimodule functor $\widehat{Q}$.

Proposition 7.6. *Within the assumptions and notations of Theorem 6.29, define a functor $\widehat{Q}$ via the coequalizer*

$$
\begin{tikzcd}
DPQP \arrow[r, shift left, "(Px)\circ(z^l P)"] \arrow[r, shift right, "(Px)\circ(z^r P)"'] & PA \arrow[r, "l"] & \hat{Q}
\end{tikzcd}
$$

Then there exists a unique functorial morphism $\nu_0' : BP \to \widehat{Q}$ such that

$$\nu_0' \circ (yP) = l \circ (Px)\,. \tag{149}$$

Moreover

$$\left(\widehat{Q},\nu_0'\right)=\mathrm{Coequ}_{\mathrm{Fun}}\left((yP)\circ\left(Pw^l\right),(yP)\circ(Pw^r)\right)$$

The functor $\widehat{Q}$ can be equipped with the structure of a $\mathbb{B}$-$\mathbb{A}$-bimodule functor $\left(\widehat{Q},\mu_{\widehat{Q}}^A,{}^B\mu_{\widehat{Q}}\right)$ where $\mu_{\widehat{Q}}^A$ and ${}^B\mu_{\widehat{Q}}$ are uniquely defined by

$$\mu_{\widehat{Q}}^A\circ(lA)=l\circ(Pm_A) \tag{150}$$

and

$${}^B\mu_{\widehat{Q}}\circ(B\nu_0')=\nu_0'\circ(m_BP)\,. \tag{151}$$

Proof. By construction we have

$$l\circ(Px)\circ\left(z^lP\right)=l\circ(Px)\circ(z^rP)\,.$$

By Lemma 2.9, we have

$$(BP,yP)=\mathrm{Coequ}_{\mathrm{Fun}}\left(z^lP,z^rP\right).$$

By the universality of coequalizers, there exists a unique functorial morphism $\nu_0':BP\to\widehat{Q}$ which fulfils (149). Let us prove that $\left(\widehat{Q},\nu_0'\right)=\mathrm{Coequ}_{\mathrm{Fun}}\left((yP)\circ\left(Pw^l\right),(yP)\circ(Pw^r)\right)$. We have

$$\nu_0'\circ(yP)\circ\left(Pw^l\right)\overset{(149)}{=}l\circ(Px)\circ\left(Pw^l\right)\overset{\text{defx}}{=}l\circ(Px)\circ(Pw^r)$$
$$\overset{(149)}{=}\nu_0'\circ(yP)\circ(Pw^r)\,.$$

Let now $\xi:BP\to X$ be a morphism such that $\xi\circ(yP)\circ\left(Pw^l\right)=\xi\circ(yP)\circ(Pw^r)$. Since P preserves coequalizers, we have

$$(PA,Px)=\mathrm{Coequ}_{\mathrm{Fun}}\left(Pw^l,Pw^r\right).$$

By universality of coequalizers there exists a unique functorial morphism $\nu_0:PA\to X$ such that

$$\nu_0\circ(Px)=\xi\circ(yP)\,. \tag{152}$$

We compute

$$\nu_0\circ(Px)\circ\left(z^lP\right)\overset{(152)}{=}\xi\circ(yP)\circ\left(z^lP\right)$$
$$=\xi\circ(yP)\circ(z^rP)\overset{(152)}{=}\nu_0\circ(Px)\circ(z^rP)\,.$$

By the universality of the coequalizer $\left(\widehat{Q},l\right)$, there exists a unique functorial morphism $\nu:\widehat{Q}\to X$ such that

$$\nu\circ l=\nu_0.$$

We compute

$$\nu\circ\nu_0'\circ(yP)\overset{(149)}{=}\nu\circ l\circ(Px)=\nu_0\circ(Px)\overset{(152)}{=}\xi\circ(yP)\,.$$

Since yP is epi, we get

$$\xi=\nu\circ\nu_0'.$$

Assume now that there is another morphism $t:\widehat{Q}\rightarrow X$ such that $\xi=t\circ\nu_0'$. Then we have

$$t\circ l\circ (Px)\overset{(149)}{=}t\circ\nu_0'\circ (yP)=\xi\circ (yP)=\nu\circ\nu_0'\circ (yP)\overset{(149)}{=}\nu\circ l\circ (Px)\,.$$

Since $l\circ (Px)$ is an epimorphism, we deduce that $t=\nu$.

(2) We want to equip $\widehat{Q}$ with the structure of a $\mathbb{B}$-$\mathbb{A}$-bimodule functor. To begin with, let us prove a number of equalities. Let us calculate

$$\begin{aligned}\chi\circ\left(Qz^{l}\right)&=\chi\circ (QP\chi)\circ (Q\delta_{D}PQ)\overset{(98)}{=}\chi\circ (\chi PQ)\circ (Q\delta_{D}PQ)\\&\overset{(105)}{=}\chi\circ\left(Q\varepsilon^{D}PQ\right)=\chi\circ (Qz^{r})\end{aligned}$$

so that

$$\chi\circ\left(Qz^{l}\right)=\chi\circ (Qz^{r})\,. \tag{153}$$

Let

$$b=m_{A}\circ (xA)\,. \tag{154}$$

Then

$$x\circ (\chi P)\overset{(102)}{=}m_{A}\circ (xx)=m_{A}\circ (xA)\circ (QPx)$$

so that

$$x\circ (\chi P)=b\circ (QPx)\,. \tag{155}$$

We have

$$\begin{aligned}(P\chi)\circ\left(z^{l}PQ\right)&=(P\chi)\circ (P\chi PQ)\circ (\delta_{D}PQPQ)\\&\overset{(98)}{=}(P\chi)\circ (PQP\chi)\circ (\delta_{D}PQPQ)\\&\overset{\delta_{D}}{=}(P\chi)\circ (\delta_{D}PQ)\circ (DP\chi)=z^{l}\circ (DP\chi)\end{aligned}$$

and hence

$$\begin{aligned}y\circ (P\chi)\circ\left(z^{l}PQ\right)&=y\circ z^{l}\circ (DP\chi)\overset{y\text{coequ}}{=}y\circ z^{r}\circ (DP\chi)=y\circ\left(\varepsilon^{D}PQ\right)\circ (DP\chi)\\&\overset{\varepsilon^{D}}{=}y\circ (P\chi)\circ\left(\varepsilon^{D}PQPQ\right)=y\circ (P\chi)\circ (z^{r}PQ)\end{aligned}$$

so that we get

$$y\circ (P\chi)\circ\left(z^{l}PQ\right)=y\circ (P\chi)\circ (z^{r}PQ)\,. \tag{156}$$

From previous equalities, it follows that

$$\begin{aligned}l\circ (Pb)\circ\left(z^{l}PA\right)\circ (DPQPx)&\overset{z^{l}}{=}l\circ (Pb)\circ (PQPx)\circ\left(z^{l}PQP\right)\\&\overset{(155)}{=}l\circ (Px)\circ (P\chi P)\circ\left(z^{l}PQP\right)\overset{(149)}{=}\nu_0'\circ (yP)\circ (P\chi P)\circ\left(z^{l}PQP\right)\\&\overset{(156)}{=}\nu_0'\circ (yP)\circ (P\chi P)\circ (z^{r}PQP)\overset{(149)}{=}l\circ (Px)\circ (P\chi P)\circ (z^{r}PQP)\\&\overset{(155)}{=}l\circ (Pb)\circ (PQPx)\circ (z^{r}PQP)\overset{z^{r}}{=}l\circ (Pb)\circ (z^{r}PA)\circ (DPQPx)\,.\end{aligned}$$

Since $DPQPx$ is an epimorphism, we obtain

$$l\circ (Pb)\circ\left(z^{l}PA\right)=l\circ (Pb)\circ (z^{r}PA) \tag{157}$$

that is

$$l \circ (Pm_A) \circ (PxA) \circ \left(z^l PA\right) = l \circ (Pm_A) \circ (PxA) \circ (z^r PA).$$

From 2.9 we have that

$$\left(\widehat{Q}A, lA\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left((PxA) \circ \left(z^l PA\right), (PxA) \circ (z^r PA)\right).$$

Hence there exists a unique functorial morphism $\mu_{\widehat{Q}}^{A} : \widehat{Q}A \to \widehat{Q}$ which satisfies (150). Now we want to prove that $\left(\widehat{Q}, \mu_{\widehat{Q}}^{A}\right)$ is a right $\mathbb{A}$-module functor. First let us prove that $\mu_{\widehat{Q}}^{A}$ is associative that is

$$\mu_{\widehat{Q}}^{A} \circ \left(\mu_{\widehat{Q}}^{A} A\right) = \mu_{\widehat{Q}}^{A} \circ \left(\widehat{Q} m_A\right).$$

We compute

$$\begin{aligned}
\mu_{\widehat{Q}}^{A} \circ \left(\mu_{\widehat{Q}}^{A} A\right) \circ (lAA) &\overset{(150)}{=} \mu_{\widehat{Q}}^{A} \circ (lA) \circ (Pm_A A) \\
&\overset{(150)}{=} l \circ (Pm_A) \circ (Pm_A A) \overset{m_A \mathrm{ass}}{=} l \circ (Pm_A) \circ (PAm_A) \\
&\overset{(150)}{=} \mu_{\widehat{Q}}^{A} \circ (lA) \circ (PAm_A) \overset{l}{=} \mu_{\widehat{Q}}^{A} \circ \left(\widehat{Q} m_A\right) \circ (lAA).
\end{aligned}$$

Since lAA is an epimorphism, we get that $\mu_{\widehat{Q}}^{A}$ is associative. Let us prove that $\mu_{\widehat{Q}}^{A}$ is unital that is

$$\mu_{\widehat{Q}}^{A} \circ \left(\widehat{Q} u_A\right) = \widehat{Q}$$

in fact

$$\mu_{\widehat{Q}}^{A} \circ \left(\widehat{Q} u_A\right) \circ l \overset{l}{=} \mu_{\widehat{Q}}^{A} \circ (lA) \circ (PAu_A) \overset{(150)}{=} l \circ (Pm_A) \circ (PAu_A) \overset{A \mathrm{monad}}{=} l$$

and since l is an epimorphism we conclude. We want to prove a series of equalities. First of all, let us prove that

$$(\chi P) \circ \left(QPw^l\right) = w^l \circ (\chi PC) \tag{158}$$

$$(\chi P) \circ (QPw^r) = w^r \circ (\chi PC) \tag{159}$$

In fact, we have

$$\begin{aligned}
(\chi P) \circ \left(QPw^l\right) &= (\chi P) \circ (QP\chi P) \circ (QPQP\delta_C) \\
&\overset{(98)}{=} (\chi P) \circ (\chi PQP) \circ (QPQP\delta_C) \\
&\overset{\chi}{=} (\chi P) \circ (QP\delta_C) \circ (\chi PC) = w^l \circ (\chi PC)
\end{aligned}$$

and

$$(\chi P) \circ (QPw^r) = (\chi P) \circ \left(QPQP\varepsilon^C\right) \overset{\chi}{=} \left(QP\varepsilon^C\right) \circ (\chi P) = w^r \circ (\chi PC).$$

From (158) we deduce that

$$\begin{aligned}
x \circ (\chi P) \circ \left(QPw^l\right) &\overset{(158)}{=} x \circ w^l \circ (\chi PC) \\
&\overset{x \mathrm{coequ}}{=} x \circ w^r \circ (\chi PC) \overset{(159)}{=} x \circ (\chi P) \circ (QPw^r)
\end{aligned}$$

and hence we get

(160) $$x \circ (\chi P) \circ \left(QPw^l\right) = x \circ (\chi P) \circ (QPw^r).$$

We observe that

$$\begin{aligned}
&\nu_0' \circ (m_B P) \circ (ByP) \circ \left(BPw^l\right) \circ (yPQPC) \\
&= \nu_0' \circ (m_B P) \circ (ByP) \circ (yPQP) \circ \left(PQPw^l\right) \\
&= \nu_0' \circ (m_B P) \circ (yyP) \circ \left(PQPw^l\right) \\
&\overset{(109)}{=} \nu_0' \circ (yP) \circ (P\chi P) \circ \left(PQPw^l\right) \\
&\overset{(149)}{=} l \circ (Px) \circ (P\chi P) \circ \left(PQPw^l\right) \\
&\overset{(160)}{=} l \circ (Px) \circ (P\chi P) \circ (PQPw^r) \\
&\overset{(149)}{=} \nu_0' \circ (yP) \circ (P\chi P) \circ (PQPw^r) \\
&\overset{(109)}{=} \nu_0' \circ (m_B P) \circ (yyP) \circ (PQPw^r) \\
&= \nu_0' \circ (m_B P) \circ (ByP) \circ (yPQP) \circ (PQPw^r) \\
&= \nu_0' \circ (m_B P) \circ (ByP) \circ (BPw^r) \circ (yPQPC)
\end{aligned}$$

since $yPQPC$ is an epimorphism, we obtain that

$$\nu_0' \circ (m_B P) \circ (ByP) \circ \left(BPw^l\right) = \nu_0' \circ (m_B P) \circ (ByP) \circ (BPw^r).$$

Since B preserves coequalizers, we have that $\left(B\widehat{Q}, B\nu_0'\right) = \mathrm{Coequ}_{\mathrm{Fun}}((ByP) \circ \left(BPw^l\right), (ByP) \circ (BPw^r))$ so that there exists a unique functorial morphism ${}^B\mu_{\widehat{Q}} : B\widehat{Q} \to \widehat{Q}$ which satisfies (151). Now we want to show that $\left(\widehat{Q}, {}^B\mu_{\widehat{Q}}\right)$ is a left $\mathbb{B}$-module functor. First let us prove that ${}^B\mu_{\widehat{Q}}$ is associative that is

$${}^B\mu_{\widehat{Q}} \circ \left(B\,{}^B\mu_{\widehat{Q}}\right) = {}^B\mu_{\widehat{Q}} \circ \left(m_B \widehat{Q}\right).$$

We have

$$\begin{aligned}
&{}^B\mu_{\widehat{Q}} \circ \left(B\,{}^B\mu_{\widehat{Q}}\right) \circ (BB\nu_0') \overset{(151)}{=} {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (Bm_B P) \\
&\overset{(151)}{=} \nu_0' \circ (m_B P) \circ (Bm_B P) \overset{m_B \mathrm{ass}}{=} \nu_0' \circ (m_B P) \circ (m_B BP) \\
&\overset{(151)}{=} {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (m_B BP) \overset{m_B}{=} {}^B\mu_{\widehat{Q}} \circ \left(m_B \widehat{Q}\right) \circ (BB\nu_0').
\end{aligned}$$

Since $BB\nu_0'$ is an epimorphism, we get that ${}^B\mu_{\widehat{Q}}$ is associative. Let us prove that ${}^B\mu_{\widehat{Q}}$ is unital that is

$${}^B\mu_{\widehat{Q}} \circ \left(u_B \widehat{Q}\right) = \widehat{Q}.$$

We calculate

$${}^B\mu_{\widehat{Q}} \circ \left(u_B \widehat{Q}\right) \circ \nu_0' = {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (u_B BP) \overset{(151)}{=} \nu_0' \circ (m_B P) \circ (u_B BP) = \nu_0'.$$

Since ν_0' is an epimorphism, we get that ${}^B\mu_{\widehat{Q}}$ is unital. Finally we have to prove the compatibility condition

$$ {}^B\mu_{\widehat{Q}} \circ \left(B\mu^A_{\widehat{Q}}\right) = \mu^A_{\widehat{Q}} \circ \left({}^B\mu_{\widehat{Q}}A\right). $$

We have

$$
\begin{aligned}
& {}^B\mu_{\widehat{Q}} \circ \left(B\mu^A_{\widehat{Q}}\right) \circ (BlA) \circ (BPxx) \circ (yPQPQP) \\
& \overset{(150)}{=} {}^B\mu_{\widehat{Q}} \circ (Bl) \circ (BPm_A) \circ (BPxx) \circ (yPQPQP) \\
& \overset{(102)}{=} {}^B\mu_{\widehat{Q}} \circ (Bl) \circ (BPx) \circ (BP\chi P) \circ (yPQPQP) \\
& \overset{(149)}{=} {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (ByP) \circ (BP\chi P) \circ (yPQPQP) \\
& \overset{y}{=} {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (ByP) \circ (yPQP) \circ (PQP\chi P) \\
& = {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (yyP) \circ (PQP\chi P) \\
& \overset{(151)}{=} \nu_0' \circ (m_BP) \circ (yyP) \circ (PQP\chi P) \\
& \overset{(109)}{=} \nu_0' \circ (yP) \circ (P\chi P) \circ (PQP\chi P) \\
& \overset{(98)}{=} \nu_0' \circ (yP) \circ (P\chi P) \circ (P\chi PQP) \\
& \overset{(149)}{=} l \circ (Px) \circ (P\chi P) \circ (P\chi PQP) \\
& \overset{(102)}{=} l \circ (Pm_A) \circ (Pxx) \circ (P\chi PQP) \\
& = l \circ (Pm_A) \circ (PxA) \circ (PQPx) \circ (P\chi PQP) \\
& \overset{\chi}{=} l \circ (Pm_A) \circ (PxA) \circ (P\chi PA) \circ (PQPQPx) \\
& \overset{(150)}{=} \mu^A_{\widehat{Q}} \circ (lA) \circ (PxA) \circ (P\chi PA) \circ (PQPQPx) \\
& \overset{(149)}{=} \mu^A_{\widehat{Q}} \circ (\nu_0'A) \circ (yPA) \circ (P\chi PA) \circ (PQPQPx) \\
& \overset{(109)}{=} \mu^A_{\widehat{Q}} \circ (\nu_0'A) \circ (m_BPA) \circ (yyPA) \circ (PQPQPx) \\
& = \mu^A_{\widehat{Q}} \circ (\nu_0'A) \circ (m_BPA) \circ (ByPA) \circ (yPQPA) \circ (PQPQPx) \\
& \overset{y}{=} \mu^A_{\widehat{Q}} \circ (\nu_0'A) \circ (m_BPA) \circ (ByPA) \circ (BPQPx) \circ (yPQPQP) \\
& \overset{(151)}{=} \mu^A_{\widehat{Q}} \circ \left({}^B\mu_{\widehat{Q}}A\right) \circ (B\nu_0'A) \circ (ByPA) \circ (BPQPx) \circ (yPQPQP) \\
& \overset{(149)}{=} \mu^A_{\widehat{Q}} \circ \left({}^B\mu_{\widehat{Q}}A\right) \circ (BlA) \circ (BPxA) \circ (BPQPx) \circ (yPQPQP) \\
& = \mu^A_{\widehat{Q}} \circ \left({}^B\mu_{\widehat{Q}}A\right) \circ (BlA) \circ (BPxx) \circ (yPQPQP)
\end{aligned}
$$

Since $(BlA) \circ (BPxx) \circ (yPQPQP)$ is an epimorphism, we conclude. Then $\left(\widehat{Q}, {}^B\mu_{\widehat{Q}}, \mu^A_{\widehat{Q}}\right)$ is a $\mathbb{B}$-$\mathbb{A}$-bimodule functor. □

PROPOSITION 7.7. *In the setting of Theorem 6.29 and Proposition 7.6, there exist two functorial morphisms* $\sigma^A : Q\widehat{Q} \to A$ *and* $\sigma^B : \widehat{Q}Q \to B$ *where* σ^A *is* A*-bilinear and* σ^B *is* B*-bilinear and they fulfill*

$$\sigma^A \circ (Ql) = m_A \circ (xA) \tag{161}$$

and

$$\sigma^B \circ (\nu_0' Q) = m_B \circ (By) . \tag{162}$$

Moreover the associative conditions hold, that is

$$ {}^A\mu_Q \circ \left(\sigma^A Q\right) = \mu_Q^B \circ \left(Q\sigma^B\right) \text{ and } {}^B\mu_{\widehat{Q}} \circ \left(\sigma^B \widehat{Q}\right) = \mu_{\widehat{Q}}^A \circ \left(\widehat{Q}\sigma^A\right).$$

Proof. First we want to prove that

$$m_A \circ (xA) \circ (QPx) \circ \left(Qz^l P\right) = m_A \circ (xA) \circ (QPx) \circ (Qz^r P) .$$

In fact we have

$$\begin{aligned} m_A \circ (xA) \circ (QPx) \circ \left(Qz^l P\right) &\overset{(102)}{=} x \circ (\chi P) \circ \left(Qz^l P\right) \\ = x \circ (\chi P) \circ (QP\chi P) \circ (Q\delta_D PQP) &\overset{(128)}{=} x \circ (\chi P) \circ (\chi PQP) \circ (Q\delta_D PQP) \\ &\overset{(130)}{=} x \circ (\chi P) \circ \left(Q\varepsilon^D PQP\right) = x \circ (\chi P) \circ (Qz^r P) \\ &\overset{(102)}{=} m_A \circ (xx) \circ (Qz^r P) = m_A \circ (xA) \circ (QPx) \circ (Qz^r P) . \end{aligned}$$

Since Q preserves coequalizers we have

$$(Q\widehat{Q}, Ql) = \mathrm{Coequ}_{\mathrm{Fun}}\left((QPx) \circ \left(Qz^l P\right), (QPx) \circ (Qz^r P)\right)$$

so that there exists a functorial morphism $\sigma^A : Q\widehat{Q} \to A$ which satisfies (161). Now we want to show that σ^A is A-bilinear that is the following equalities hold

$$\begin{aligned} \sigma^A \circ \left({}^A\mu_Q \widehat{Q}\right) &= m_A \circ \left(A\sigma^A\right) \\ \sigma^A \circ \left(Q\mu_{\widehat{Q}}^A\right) &= m_A \circ \left(\sigma^A A\right) . \end{aligned}$$

We compute

$$\begin{aligned} m_A \circ \left(A\sigma^A\right) \circ (AQl) &\overset{(161)}{=} m_A \circ (Am_A) \circ (AxA) \\ &\overset{m_A \mathrm{ass}}{=} m_A \circ (m_A A) \circ (AxA) \overset{(104)}{=} m_A \circ (xA) \circ \left({}^A\mu_Q PA\right) \\ &\overset{(161)}{=} \sigma^A \circ (Ql) \circ \left({}^A\mu_Q PA\right) \overset{{}^A\mu_Q}{=} \sigma^A \circ \left({}^A\mu_Q \widehat{Q}\right) \circ (AQl) . \end{aligned}$$

Since AQl is an epimorphism, we get that $\sigma^A \circ \left({}^A\mu_Q \widehat{Q}\right) = m_A \circ \left(A\sigma^A\right)$. We compute

$$\begin{aligned} m_A \circ \left(\sigma^A A\right) \circ (QlA) &\overset{(161)}{=} m_A \circ (m_A A) \circ (xAA) \\ &= m_A \circ (Am_A) \circ (xAA) \overset{x}{=} m_A \circ (xA) \circ (QPm_A) \\ &\overset{(161)}{=} \sigma^A \circ (Ql) \circ (QPm_A) \overset{(150)}{=} \sigma^A \circ \left(Q\mu_{\widehat{Q}}^A\right) \circ (QlA) \end{aligned}$$

Since QlA is an epimorphism, we obtain that $\sigma^A \circ \left(Q\mu^A_{\widehat{Q}}\right) = m_A \circ \left(\sigma^A A\right)$. Symmetrically,we want to define σ^B. We prove that

$$m_B \circ (By) \circ (yPQ) \circ \left(Pw^l Q\right) = m_B \circ (By) \circ (yPQ) \circ (Pw^r Q).$$

In fact, we have

$$\begin{aligned} m_B \circ (By) \circ (yPQ) \circ \left(Pw^l Q\right) &= m_B \circ (yy) \circ \left(Pw^l Q\right) \\ \overset{(109)}{=} y \circ (P\chi) \circ \left(Pw^l Q\right) &= y \circ (P\chi) \circ (P\chi PQ) \circ (PQP\delta_C Q) \\ &\overset{(128)}{=} y \circ (P\chi) \circ (PQP\chi) \circ (PQP\delta_C Q) \\ &\overset{(129)}{=} y \circ (P\chi) \circ \left(PQP\varepsilon^C Q\right) = y \circ (P\chi) \circ (Pw^r Q) \\ &\overset{(109)}{=} m_B \circ (yy) \circ (Pw^r Q) = m_B \circ (By) \circ (yPQ) \circ (Pw^r Q). \end{aligned}$$

By Lemma 2.9, we have

$$\left(\widehat{Q}Q, \nu_0' Q\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left((yPQ) \circ \left(Pw^l Q\right), (yPQ) \circ (Pw^r Q)\right)$$

so that there exists a functorial morphism $\sigma^B : \widehat{Q}Q \to B$ which satisfies (162). Now we want to show that σ^B is B-bicolinear that is the following equalities hold

$$\begin{aligned} \sigma^B \circ \left({}^B\mu_{\widehat{Q}} Q\right) &= m_B \circ \left(B\sigma^B\right) \\ m_B \circ \left(\sigma^B B\right) &= \sigma^B \circ \left(\widehat{Q}\mu^B_Q\right) \end{aligned}$$

We calculate

$$\begin{aligned} m_B \circ \left(B\sigma^B\right) \circ (B\nu_0' Q) &\overset{(162)}{=} m_B \circ (Bm_B) \circ (BBy) \\ \overset{m_B \mathrm{ass}}{=} m_B \circ (m_B B) \circ (BBy) &\overset{m_B}{=} m_B \circ (By) \circ (m_B PQ) \\ \overset{(162)}{=} \sigma^B \circ (\nu_0' Q) \circ (m_B PQ) &\overset{(151)}{=} \sigma^B \circ \left({}^B\mu_{\widehat{Q}} Q\right) \circ (B\nu_0' Q). \end{aligned}$$

Since $B\nu_0' Q$ is an epimorphism, we deduce that $\sigma^B \circ \left({}^B\mu_{\widehat{Q}} Q\right) = m_B \circ \left(B\sigma^B\right)$. We compute

$$\begin{aligned} m_B \circ \left(\sigma^B B\right) \circ (\nu_0' QB) &\overset{(162)}{=} m_B \circ (m_B B) \circ (ByB) \\ \overset{m_B \mathrm{ass}}{=} m_B \circ (Bm_B) \circ (ByB) &\overset{(108)}{=} m_B \circ (By) \circ \left(BP\mu^B_Q\right) \\ \overset{(162)}{=} \sigma^B \circ (\nu_0' Q) \circ \left(BP\mu^B_Q\right) &\overset{\nu_0'}{=} \sigma^B \circ \left(\widehat{Q}\mu^B_Q\right) \circ (\nu_0' QB). \end{aligned}$$

Since $\nu_0' QB$ is an epimorphism, we get that $m_B \circ \left(\sigma^B B\right) = \sigma^B \circ \left(\widehat{Q}\mu^B_Q\right)$. Finally we have to prove the associative conditions

$$\begin{aligned} {}^A\mu_Q \circ \left(\sigma^A Q\right) &= \mu^B_Q \circ \left(Q\sigma^B\right) \\ {}^B\mu_{\widehat{Q}} \circ \left(\sigma^B \widehat{Q}\right) &= \mu^A_{\widehat{Q}} \circ \left(\widehat{Q}\sigma^A\right). \end{aligned}$$

We calculate

$$\begin{aligned}
{}^{A}\mu_{Q}\circ\left(\sigma^{A}Q\right)\circ(QlQ)\circ(QPxQ) &\overset{(161)}{=} {}^{A}\mu_{Q}\circ(m_{A}Q)\circ(xAQ)\circ(QPxQ)\\
&= {}^{A}\mu_{Q}\circ(m_{A}Q)\circ(xxQ) \overset{(102)}{=} {}^{A}\mu_{Q}\circ(xQ)\circ(\chi PQ)\\
&\overset{(101)}{=} \chi\circ(\chi PQ) \overset{(128)}{=} \chi\circ(QP\chi) \overset{(107)}{=} \mu_{Q}^{B}\circ(Qy)\circ(QP\chi)\\
&\overset{(109)}{=} \mu_{Q}^{B}\circ(Qm_{B})\circ(Qyy) = \mu_{Q}^{B}\circ(Qm_{B})\circ(QBy)\circ(QyPQ)\\
&\overset{(162)}{=} \mu_{Q}^{B}\circ\left(Q\sigma^{B}\right)\circ(Q\nu_{0}'Q)\circ(QyPQ)\\
&\overset{(149)}{=} \mu_{Q}^{B}\circ\left(Q\sigma^{B}\right)\circ(QlQ)\circ(QPxQ)\,.
\end{aligned}$$

Since $(QlQ)\circ(QPxQ)$ is an epimorphism, we deduce that ${}^{A}\mu_{Q}\circ\left(\sigma^{A}Q\right) = \mu_{Q}^{B}\circ\left(Q\sigma^{B}\right)$.
We compute

$$\begin{aligned}
&\mu_{\widehat{Q}}^{A}\circ\left(\widehat{Q}\sigma^{A}\right)\circ\left(\nu_{0}'Q\widehat{Q}\right)\circ\left(yPQ\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{(149)}{=} \mu_{\widehat{Q}}^{A}\circ\left(\widehat{Q}\sigma^{A}\right)\circ\left(lQ\widehat{Q}\right)\circ\left(PxQ\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{l}{=} \mu_{\widehat{Q}}^{A}\circ(lA)\circ\left(PA\sigma^{A}\right)\circ\left(PxQ\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{(150)}{=} l\circ(Pm_{A})\circ\left(PA\sigma^{A}\right)\circ\left(PxQ\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{x}{=} l\circ(Pm_{A})\circ(PxA)\circ\left(PQP\sigma^{A}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{(161)}{=} l\circ(Pm_{A})\circ(PxA)\circ(PQPm_{A})\circ(PQPxA)\circ(PQPQPx)\\
&= l\circ(Pm_{A})\circ(PxA)\circ(PQPm_{A})\circ(PQPxx)\\
&\overset{(102)}{=} l\circ(Pm_{A})\circ(PxA)\circ(PQPx)\circ(PQP\chi P)\\
&= l\circ(Pm_{A})\circ(Pxx)\circ(PQP\chi P)\\
&\overset{(102)}{=} l\circ(Px)\circ(P\chi P)\circ(PQP\chi P)\\
&\overset{(128)}{=} l\circ(Px)\circ(P\chi P)\circ(P\chi PQP)\\
&\overset{(149)}{=} \nu_{0}'\circ(yP)\circ(P\chi P)\circ(P\chi PQP)\\
&\overset{(109)}{=} \nu_{0}'\circ(m_{B}P)\circ(yyP)\circ(P\chi PQP)\\
&\overset{(151)}{=} {}^{B}\mu_{\widehat{Q}}\circ(B\nu_{0}')\circ(yBP)\circ(PQyP)\circ(P\chi PQP)\\
&\overset{y}{=} {}^{B}\mu_{\widehat{Q}}\circ\left(y\widehat{Q}\right)\circ(PQ\nu_{0}')\circ(PQyP)\circ(P\chi PQP)\\
&\overset{(149)}{=} {}^{B}\mu_{\widehat{Q}}\circ\left(y\widehat{Q}\right)\circ(PQl)\circ(PQPx)\circ(P\chi PQP)\\
&\overset{\chi}{=} {}^{B}\mu_{\widehat{Q}}\circ\left(y\widehat{Q}\right)\circ\left(P\chi\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)\\
&\overset{(109)}{=} {}^{B}\mu_{\widehat{Q}}\circ\left(m_{B}\widehat{Q}\right)\circ\left(yy\widehat{Q}\right)\circ(PQPQl)\circ(PQPQPx)
\end{aligned}$$

$$= {}^{B}\mu_{\widehat{Q}} \circ \left(m_B\widehat{Q}\right) \circ \left(By\widehat{Q}\right) \circ \left(yPQ\widehat{Q}\right) \circ (PQPQl) \circ (PQPQPx)$$

$$\overset{(162)}{=} {}^{B}\mu_{\widehat{Q}} \circ \left(\sigma^B\widehat{Q}\right) \circ \left(\nu_0'Q\widehat{Q}\right) \circ \left(yPQ\widehat{Q}\right) \circ (PQPQl) \circ (PQPQPx)$$

Since $\left(\nu_0'Q\widehat{Q}\right) \circ \left(yPQ\widehat{Q}\right) \circ (PQPQl) \circ (PQPQPx)$ is an epimorphism, we get that ${}^{B}\mu_{\widehat{Q}} \circ \left(\sigma^B\widehat{Q}\right) = \mu^A_{\widehat{Q}} \circ \left(\widehat{Q}\sigma^A\right)$. □

7.4. **From coherds to herds.**

7.8. Given a coherd $\chi : QPQ \to Q$ in a formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$, our purpose is to build the formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, \widehat{Q}, Q, \sigma^A, \sigma^B)$ and then an herd $\tau : Q \to Q\widehat{Q}Q$ in $\mathbb{M}$.

THEOREM 7.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with coequalizers and let $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $C : \mathcal{A} \to \mathcal{A}$ and $D : \mathcal{B} \to \mathcal{B}$ be functors. Assume that all the functors P, Q, C and D preserve coequalizers. Let $\varepsilon^C : C \to \mathcal{A}$ and $\varepsilon^D : D \to \mathcal{B}$ be functorial epimorphisms and assume that $\left(\mathcal{A}, \varepsilon^C\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(C\varepsilon^C, \varepsilon^C C\right)$ and $\left(\mathcal{B}, \varepsilon^D\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(D\varepsilon^D, \varepsilon^D D\right)$. Let $\chi : QPQ \to Q$ be a functorial morphism such that*

$$\chi \circ (QP\chi) = \chi \circ (\chi PQ).$$

Let $\delta_C : C \to QP$ be a functorial morphism such that

$$\chi \circ (\delta_C Q) = (\varepsilon^C Q)$$

and let $\delta_D : D \to PQ$ be a functorial morphism such that

$$\chi \circ (Q\delta_D) = (Q\varepsilon^D).$$

Then there is a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, \widehat{Q}, Q, \sigma^A, \sigma^B)$.

Proof. In view of Theorem 6.29 and Propositions 7.6, 7.7 a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, \widehat{Q}, Q, \sigma^A, \sigma^B)$ has been constructed. □

THEOREM 7.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with coequalizers and let $\mathbb{X} = (\mathbb{C}, \mathbb{D}, P, Q, \delta_C, \delta_D)$ be a regular formal codual structure where $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $C : \mathcal{A} \to \mathcal{A}$ and $D : \mathcal{B} \to \mathcal{B}$ are functors that preserve coequalizers. Let $\chi : QPQ \to Q$ be a copretorsor. Then there is a formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, \widehat{Q}, Q, \sigma^A, \sigma^B)$. Define $\tau : Q \to Q\widehat{Q}Q$ by setting*

$$\tau := (QlQ) \circ (QPxQ) \circ (\delta_C QPQ) \circ (CQ\delta_D) \circ \left({}^{C}\rho_Q D\right) \circ \rho^D_Q.$$

Then τ is an herd in $\mathbb{M}$.

Proof. By Theorem 7.9, $\mathbb{M} = (\mathbb{A}, \mathbb{B}, \widehat{Q}, Q, \sigma^A, \sigma^B)$ is a formal dual structure. To show that τ is an herd in $\mathbb{M}$, we have to prove that it satisfies the following conditions.

1) Associativity, in the sense that $\left(Q\widehat{Q}\tau\right) \circ \tau = \left(\tau\widehat{Q}Q\right) \circ \tau$. We have

$$\left(Q\widehat{Q}\tau\right) \circ \tau$$

$$= \left(Q\widehat{Q}\tau\right) \circ (QlQ) \circ (QPxQ) \circ (\delta_C QPQ) \circ (CQ\delta_D) \circ \left({}^{C}\rho_Q D\right) \circ \rho^D_Q$$

$$
\begin{aligned}
&\overset{l}{=} \left(QlQ\widehat{Q}Q\right) \circ (QPA\tau) \circ (QPxQ) \circ (\delta_C QPQ) \circ (CQ\delta_D) \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D \\
&\overset{x}{=} \left(QlQ\widehat{Q}Q\right) \circ \left(QPxQ\widehat{Q}Q\right) \circ (QPQP\tau) \circ (\delta_C QPQ) \circ (CQ\delta_D) \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D \\
&\overset{\delta_C}{=} \left(QlQ\widehat{Q}Q\right) \circ \left(QPxQ\widehat{Q}Q\right) \circ \left(\delta_C QPQ\widehat{Q}Q\right) \circ (CQP\tau) \circ (CQ\delta_D) \\
&\qquad \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D \\
&\overset{{}^C\rho_Q}{=} \left(QlQ\widehat{Q}Q\right) \circ \left(QPxQ\widehat{Q}Q\right) \circ \left(\delta_C QPQ\widehat{Q}Q\right) \circ \left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ (QP\tau) \\
&\qquad \circ (Q\delta_D) \circ \rho_Q^D \\
&\overset{(127)}{=} \left(QlQ\widehat{Q}Q\right) \circ \left(QPxQ\widehat{Q}Q\right) \circ \left(\delta_C QPQ\widehat{Q}Q\right) \circ \left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ (QP\tau) \\
&\qquad \circ (\delta_C Q) \circ {}^C\rho_Q \\
&\overset{\delta_C}{=} \left(QlQ\widehat{Q}Q\right) \circ \left(QPxQ\widehat{Q}Q\right) \circ \left(\delta_C QPQ\widehat{Q}Q\right) \circ \left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ \left(\delta_C Q\widehat{Q}Q\right) \\
&\qquad \circ (C\tau) \circ {}^C\rho_Q
\end{aligned}
$$

and

$$
\begin{aligned}
&\left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ \left(\delta_C Q\widehat{Q}Q\right) \circ (C\tau) \circ {}^C\rho_Q \\
&= \left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ \left(\delta_C Q\widehat{Q}Q\right) \circ (CQlQ) \circ (CQPxQ) \circ (C\delta_C QPQ) \circ (CCQ\delta_D) \\
&\qquad \circ \left(C{}^C\rho_Q D\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q \\
&\overset{\delta_C}{=} \left({}^C\rho_Q PQ\widehat{Q}Q\right) \circ \left(\delta_C Q\widehat{Q}Q\right) \circ (CQlQ) \circ (CQPxQ) \circ (CQPQ\delta_D) \circ (C\delta_C QD) \\
&\qquad \circ \left(C{}^C\rho_Q D\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q \\
&\overset{(125)}{=} \left(C\delta_C Q\widehat{Q}Q\right) \circ \left(\Delta^C Q\widehat{Q}Q\right) \circ (CQlQ) \circ (CQPxQ) \circ (CQPQ\delta_D) \circ (C\delta_C QD) \\
&\qquad \circ \left(C{}^C\rho_Q D\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left(C{}^C\rho_Q D\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q \\
&\overset{Q\text{is a bicom}}{=} \left(CC\rho_Q^D\right) \circ \left(C{}^C\rho_Q\right) \circ {}^C\rho_Q \overset{Q\text{is a com}}{=} \left(CC\rho_Q^D\right) \circ \left(\Delta^C Q\right) \circ {}^C\rho_Q \\
&\overset{\Delta^C}{=} \left(\Delta^C QD\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q \overset{Q\text{is a bicom}}{=} \left(\Delta^C QD\right) \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D \\
&\overset{Q\text{is a com}}{=} \left(C{}^C\rho_Q D\right) \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left(C\delta_C Q\widehat{Q}Q\right) \circ \left(\Delta^C Q\widehat{Q}Q\right) \circ (CQlQ) \circ (CQPxQ) \circ (CQPQ\delta_D) \circ (C\delta_C QD) \\
&\qquad \circ \left(C{}^C\rho_Q D\right) \circ \left(C\rho_Q^D\right) \circ {}^C\rho_Q \\
&= \left(C\delta_C Q\widehat{Q}Q\right) \circ \left(\Delta^C Q\widehat{Q}Q\right) \circ (CQlQ) \circ (CQPxQ) \circ (CQPQ\delta_D) \circ (C\delta_C QD) \\
&\qquad \circ \left(C{}^C\rho_Q D\right) \circ \left({}^C\rho_Q D\right) \circ \rho_Q^D
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(127)}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ\left(\Delta^C Q\widehat{Q}Q\right)\circ(CQlQ)\circ(CQPxQ)\circ(CQPQ\delta_D)\\
&\qquad\circ(CQ\delta_D D)\circ\left(C\rho_Q^D D\right)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{\Delta^C}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ(CCQlQ)\circ(CCQPxQ)\circ(CCQPQ\delta_D)\circ(CCQ\delta_D D)\\
&\qquad\circ\left(CC\rho_Q^D D\right)\circ\left(\Delta^C QD\right)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{Q\text{is a com}}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ(CCQlQ)\circ(CCQPxQ)\circ(CCQPQ\delta_D)\circ(CCQ\delta_D D)\\
&\qquad\circ\left(CC\rho_Q^D D\right)\circ\left(C^C\rho_Q D\right)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{Q\text{is a bicom}}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ(CCQlQ)\circ(CCQPxQ)\circ(CCQPQ\delta_D)\circ(CCQ\delta_D D)\\
&\qquad\circ\left(C^C\rho_Q DD\right)\circ\left(C\rho_Q^D D\right)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{{}^C\rho_Q}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ\left(C^C\rho_Q\widehat{Q}Q\right)\circ(CQlQ)\circ(CQPxQ)\circ(CQPQ\delta_D)\circ(CQ\delta_D D)\\
&\qquad\circ\left(C\rho_Q^D D\right)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{Q\text{is a bicom}}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ\left(C^C\rho_Q\widehat{Q}Q\right)\circ(CQlQ)\circ(CQPxQ)\circ(CQPQ\delta_D)\\
&\qquad\circ(CQ\delta_D D)\circ\left({}^C\rho_Q DD\right)\circ\left(\rho_Q^D D\right)\circ\rho_Q^D\\
&\overset{{}^C\rho_Q}{=} \left(C\delta_C Q\widehat{Q}Q\right)\circ\left(C^C\rho_Q\widehat{Q}Q\right)\circ\left({}^C\rho_Q\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(QPQ\delta_D)\\
&\qquad\circ(Q\delta_D D)\circ\left(\rho_Q^D D\right)\circ\rho_Q^D\\
&\overset{(127)}{=} \left(CQ\delta_D\widehat{Q}Q\right)\circ\left(C\rho_Q^D\widehat{Q}Q\right)\circ\left({}^C\rho_Q\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(QPQ\delta_D)\\
&\qquad\circ(Q\delta_D D)\circ\left(\rho_Q^D D\right)\circ\rho_Q^D\\
&\overset{(127)}{=} \left(CQ\delta_D\widehat{Q}Q\right)\circ\left(C\rho_Q^D\widehat{Q}Q\right)\circ\left({}^C\rho_Q\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(QPQ\delta_D)\\
&\qquad\circ(\delta_C QD)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{Q\text{is a bicom}}{=} \left(CQ\delta_D\widehat{Q}Q\right)\circ\left({}^C\rho_Q D\widehat{Q}Q\right)\circ\left(\rho_Q^D\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(QPQ\delta_D)\\
&\qquad\circ(\delta_C QD)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D\\
&\overset{\delta_C}{=} \left(CQ\delta_D\widehat{Q}Q\right)\circ\left({}^C\rho_Q D\widehat{Q}Q\right)\circ\left(\rho_Q^D\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(\delta_C QPQ)\\
&\qquad\circ(CQ\delta_D)\circ\left({}^C\rho_Q D\right)\circ\rho_Q^D
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&\left(Q\widehat{Q}\tau\right)\circ\tau\\
&=\left(QlQ\widehat{Q}Q\right)\circ\left(QPxQ\widehat{Q}Q\right)\circ\left(\delta_C QPQ\widehat{Q}Q\right)\circ\left({}^C\rho_Q PQ\widehat{Q}Q\right)\circ\left(\delta_C Q\widehat{Q}Q\right)\\
&\qquad\circ(C\tau)\circ{}^C\rho_Q\\
&=\left(QlQ\widehat{Q}Q\right)\circ\left(QPxQ\widehat{Q}Q\right)\circ\left(\delta_C QPQ\widehat{Q}Q\right)\circ\left(CQ\delta_D\widehat{Q}Q\right)\circ\left({}^C\rho_Q D\widehat{Q}Q\right)
\end{aligned}
$$

$$\circ\left(\rho_Q^D\widehat{Q}Q\right)\circ(QlQ)\circ(QPxQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D$$
$$=\left(\tau\widehat{Q}Q\right)\circ\tau.$$

2) Counitality, in the sense that $\left(Q\sigma^B\right)\circ\tau=Qu_B$ and $\left(\sigma^AQ\right)\circ\tau=u_AQ$. Let us prove that

$$\left(Q\sigma^B\right)\circ\tau=Qu_B.$$

In fact, we have

$$\begin{aligned}
&\left(Q\sigma^B\right)\circ\tau\\
&=\left(Q\sigma^B\right)\circ(QlQ)\circ(QPxQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(149)}{=}\left(Q\sigma^B\right)\circ(Q\nu_0'Q)\circ(QyPQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(162)}{=}(Qm_B)\circ(QBy)\circ(QyPQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&=(Qm_B)\circ(Qyy)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(109)}{=}(Qy)\circ(QP\chi)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{\delta_C}{=}(Qy)\circ(QP\chi)\circ(QPQ\delta_D)\circ(\delta_CQD)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(130)}{=}(Qy)\circ\left(QPQ\varepsilon^D\right)\circ(\delta_CQD)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{\delta_C}{=}(Qy)\circ(\delta_CQ)\circ\left(CQ\varepsilon^D\right)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{Q\text{is a bicom}}{=}(Qy)\circ(\delta_CQ)\circ\left(CQ\varepsilon^D\right)\circ(C\rho_Q^D)\circ{}^C\rho_Q\\
&\overset{Q\text{is a com}}{=}(Qy)\circ(\delta_CQ)\circ{}^C\rho_Q\overset{(127)}{=}(Qy)\circ(Q\delta_D)\circ\rho_Q^D\\
&\overset{(110)}{=}(Qu_B)\circ\left(Q\varepsilon^D\right)\circ\rho_Q^D\overset{Q\text{is a com}}{=}Qu_B.
\end{aligned}$$

Let us prove that

$$\left(\sigma^AQ\right)\circ\tau=(u_AQ).$$

We calculate

$$\begin{aligned}
&\left(\sigma^AQ\right)\circ\tau\\
&=\left(\sigma^AQ\right)\circ(QlQ)\circ(QPxQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(161)}{=}(m_AQ)\circ(xAQ)\circ(QPxQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&=(m_AQ)\circ(xxQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(102)}{=}(xQ)\circ(\chi PQ)\circ(\delta_CQPQ)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{(129)}{=}(xQ)\circ\left(\varepsilon^CQPQ\right)\circ(CQ\delta_D)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{\varepsilon^C}{=}(xQ)\circ(Q\delta_D)\circ\left(\varepsilon^CQD\right)\circ\left({}^C\rho_QD\right)\circ\rho_Q^D\\
&\overset{Q\text{is a com}}{=}(xQ)\circ(Q\delta_D)\circ\rho_Q^D\overset{(127)}{=}(xQ)\circ(\delta_CQ)\circ{}^C\rho_Q\\
&\overset{(103)}{=}(u_AQ)\circ\left(\varepsilon^CQ\right)\circ{}^C\rho_Q\overset{Q\text{is a com}}{=}u_AQ.
\end{aligned}$$

$\square$

7.5. **Herd - Coherd - Herd.**

7.11. Let $\tau : Q \to QPQ$ be a herd for a regular formal dual structure $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B,)$ where $P : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{B} \to \mathcal{A}$, $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{B} \to \mathcal{B}$ are functors that preserve equalizers. Then, by Propositions 6.1 and 6.2, we can construct comonads $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ and $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ and functorial morphisms ${}^C\rho_Q : Q \to CQ$ and $\rho_Q^D : Q \to QD$ such that $(Q, {}^C\rho_Q, \rho_Q^D)$ is a $\mathbb{C}$-$\mathbb{D}$-bicomodule functor (see Theorem 6.5). Let $\overline{Q}$ as defined in Proposition 7.1. Then $\left(\overline{Q}, {}^D\rho_{\overline{Q}}, \rho_{\overline{Q}}^C\right)$ is a $\mathbb{D}$-$\mathbb{C}$-bicomodule functor. By Theorem 7.5, we construct a coherd $\mathbb{X} = \left(\mathbb{C}, \mathbb{D}, \overline{Q}, Q, \delta_C, \delta_D, \chi\right)$ where $\chi := \mu_Q^B \circ \left({}^A\mu_Q B\right) \circ \left(AQ\sigma^B\right) \circ \left(\sigma^A QPQ\right) \circ (QPiQ) \circ (QqQ)$. Then we can construct monads $\mathbb{A}' = (A', m_{A'}, u_{A'})$ and $\mathbb{B}' = (B', m_{B'}, u_{B'})$ following respectively Proposition 6.25 and Proposition 6.26 as the coequalizers

$$Q\overline{Q}C \underset{Q\overline{Q}\varepsilon^C}{\overset{(\chi\overline{Q})\circ(Q\overline{Q}\delta^C)}{\rightrightarrows}} Q\overline{Q} \xrightarrow{x'} A'$$

$$D\overline{Q}Q \underset{\varepsilon^D\overline{Q}Q}{\overset{(\overline{Q}\chi)\circ(\delta^D\overline{Q}Q)}{\rightrightarrows}} S\overline{Q} \xrightarrow{y'} B'$$

This means that the following hold

$$m_{A'} \circ (x'x') = x' \circ \left(\chi\overline{Q}\right) \text{ and } u_{A'} \circ \varepsilon^C = x' \circ \delta_C$$

$$m_{B'} \circ (y'y') = y' \circ \left(\overline{Q}\chi\right) \text{ and } u_{B'} \circ \varepsilon^D = y' \circ \delta_D. \tag{163}$$

Notation 7.12. *With notations of Theorem 6.5 and Proposition 7.1, let $h : \overline{Q} \to P$ be defined by setting*

$$h = {}^B\mu_P \circ \left(\sigma^B P\right) \circ (Pi) \circ q.$$

The following theorem reformulates Theorem 3.5 in [BV] in our categorical setting.

Theorem 7.13. *Let $\mathbb{M} = (\mathbb{A}, \mathbb{B}, P, Q, \sigma^A, \sigma^B)$ be a tame Morita context and let $\tau : Q \to QPQ$ be a herd for $\mathbb{M}$ such that A and B reflect equalizers and coequalizers. We denote by A' and B' the monads constructed in Claim 7.11. Then*

1) *There are functorial morphisms $\nu_A : A' \to A$ and $\nu_B : B' \to B$ such that ν_A and ν_B are morphisms of monads.*
2) *If the functorial morphism $hQ : \overline{Q}Q \to PQ$, where $h = {}^B\mu_P \circ \left(\sigma^B P\right) \circ (Pi) \circ q$, is an isomorphism, then ν_A and ν_B are isomorphisms.*
3) *If $PC = \overline{Q} \simeq \overline{Q}' = DP$ then hQ is an isomorphism and hence ν_A and ν_B are isomorphisms of monads.*

Proof. Note that, since A and B reflect equalizers, by Lemma 6.10, we have a regular herd, i.e. the assumptions $(\mathcal{A}, u_A) = \mathrm{Equ}_{\mathrm{Fun}}(u_A A, A u_A)$ and $(\mathcal{B}, u_B) = \mathrm{Equ}_{\mathrm{Fun}}(u_B B, B u_B)$ are fulfilled. We will prove only the statement for the monad B', for A' the proof is similar.

1) Consider the functorial morphism

$$\overline{\sigma}^B : \overline{Q}Q \to B$$

given by

$$\begin{aligned}\overline{\sigma}^B &= m_B \circ \left(\sigma^B\sigma^B\right) \circ (PiQ) \circ (qQ)\\ &\overset{(144)}{=} m_B \circ \left(\sigma^B\sigma^B\right) \circ (jPQ) \circ (\kappa_0'Q)\,.\end{aligned}$$

We compute

$$\begin{aligned}&\left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&\overset{q}{=} \left(\sigma^B PQ\right) \circ (PiQ) \circ (PC\chi) \circ \left(qQ\overline{Q}Q\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&\overset{i}{=} \left(\sigma^B PQ\right) \circ (PQP\chi) \circ \left(PiQ\overline{Q}Q\right) \circ \left(qQ\overline{Q}Q\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&\overset{(144)}{=} \left(\sigma^B PQ\right) \circ (PQP\chi) \circ \left(jPQ\overline{Q}Q\right) \circ \left(\kappa_0'Q\overline{Q}Q\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&\overset{(148)}{=} \left(\sigma^B PQ\right) \circ (PQP\chi) \circ \left(jPQ\overline{Q}Q\right) \circ \left(Dj\overline{Q}Q\right) \circ \left(\Delta^D\overline{Q}Q\right)\\
&\overset{(67)}{=} \left(\sigma^B PQ\right) \circ (PQP\chi) \circ \left(P\tau\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&= \left(\sigma^B PQ\right) \circ \left(PQP\mu_Q^B\right) \circ \left(PQP^A\mu_Q B\right) \circ \left(PQPAQ\sigma^B\right) \circ \left(PQP\sigma^A QPQ\right)\\
&\quad \circ (PQPQPiQ) \circ (PQPQqQ) \circ \left(P\tau\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&\overset{\sigma^A}{=} \left(\sigma^B PQ\right) \circ \left(PQP\mu_Q^B\right) \circ \left(PQP^A\mu_Q B\right) \circ \left(PQP\sigma^A QB\right) \circ \left(PQPQPQ\sigma^B\right)\\
&\quad \circ (PQPQPiQ) \circ (PQPQqQ) \circ \left(P\tau\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&\overset{(82)}{=} \left(\sigma^B PQ\right) \circ \left(PQP\mu_Q^B\right) \circ \left(PQP\mu_Q^B B\right) \circ \left(PQPQ\sigma^B B\right)\\
&\quad \circ \left(PQPQPQ\sigma^B\right) \circ (PQPQPiQ) \circ (PQPQqQ) \circ \left(P\tau\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&\overset{\sigma^B}{=} \left(BP\mu_Q^B\right) \circ \left(BP\mu_Q^B B\right) \circ \left(BPQ\sigma^B B\right) \circ \left(BPQPQ\sigma^B\right) \circ (BPQPiQ)\\
&\quad \circ (BPQqQ) \circ \left(\sigma^B PQ\overline{Q}Q\right) \circ \left(P\tau\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&\overset{\mathrm{def}D}{=} \left(BP\mu_Q^B\right) \circ \left(BP\mu_Q^B B\right) \circ \left(BPQ\sigma^B B\right) \circ \left(BPQPQ\sigma^B\right) \circ (BPQPiQ)\\
&\quad \circ (BPQqQ) \circ \left(u_B PQ\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\end{aligned}$$

and thus we obtain

$$\begin{aligned}&\left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&= \left(BP\mu_Q^B\right) \circ \left(BP\mu_Q^B B\right) \circ \left(BPQ\sigma^B B\right) \circ \left(BPQPQ\sigma^B\right) \circ (BPQPiQ)\\
&\quad \circ (BPQqQ) \circ \left(u_B PQ\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\,.\end{aligned} \tag{164}$$

Let us compute

$$\begin{aligned}&\overline{\sigma}^B \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&= m_B \circ \left(\sigma^B\sigma^B\right) \circ (PiQ) \circ (qQ) \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&= m_B \circ \left(B\sigma^B\right) \circ \left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right)\\
&\overset{(164)}{=} m_B \circ \left(B\sigma^B\right) \circ \left(BP\mu_Q^B\right) \circ \left(BP\mu_Q^B B\right) \circ \left(BPQ\sigma^B B\right)\\
&\quad \circ \left(BPQPQ\sigma^B\right) \circ (BPQPiQ) \circ (BPQqQ) \circ \left(u_B PQ\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right)\\
&\overset{u_B}{=} m_B \circ (u_B B) \circ \sigma^B \circ \left(P\mu_Q^B\right) \circ \left(P\mu_Q^B B\right) \circ \left(PQ\sigma^B B\right) \circ \left(PQPQ\sigma^B\right)\end{aligned}$$

$$
\begin{aligned}
&\circ (PQPiQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{B\text{monad}}{=} \sigma^B \circ \left(P\mu_Q^B\right) \circ \left(P\mu_Q^B B\right) \circ \left(PQ\sigma^B B\right) \circ \left(PQPQ\sigma^B\right) \circ (PQPiQ) \\
&\qquad \circ \left(j\overline{Q}Q\right) \\
&\overset{(82)}{=} \sigma^B \circ \left(P\mu_Q^B\right) \circ \left(P^A\mu_Q B\right) \circ \left(P\sigma^A QB\right) \circ \left(PQPQ\sigma^B\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{\sigma^A}{=} \sigma^B \circ \left(P\mu_Q^B\right) \circ \left(P^A\mu_Q B\right) \circ \left(PAQ\sigma^B\right) \circ \left(P\sigma^A QPQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{(81)}{=} m_B \circ \left(\sigma^B B\right) \circ \left(P^A\mu_Q B\right) \circ \left(PAQ\sigma^B\right) \circ \left(P\sigma^A QPQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{{}^A\mu_Q}{=} m_B \circ \left(\sigma^B B\right) \circ \left(PQ\sigma^B\right) \circ \left(P^A\mu_Q PQ\right) \circ \left(P\sigma^A QPQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{(82)}{=} m_B \circ \left(\sigma^B B\right) \circ \left(PQ\sigma^B\right) \circ \left(P\mu_Q^B PQ\right) \circ \left(PQ\sigma^B PQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{\sigma^B}{=} m_B \circ \left(B\sigma^B\right) \circ \left(\sigma^B PQ\right) \circ \left(P\mu_Q^B PQ\right) \circ \left(PQ\sigma^B PQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{(81)}{=} m_B \circ \left(B\sigma^B\right) \circ (m_B PQ) \circ \left(\sigma^B BPQ\right) \circ \left(PQ\sigma^B PQ\right) \circ (PQPiQ) \\
&\qquad \circ (PQqQ) \circ \left(j\overline{Q}Q\right) \\
&\overset{\sigma^B}{=} m_B \circ \left(B\sigma^B\right) \circ (m_B PQ) \circ \left(B\sigma^B PQ\right) \circ (BPiQ) \circ (BqQ) \circ \left(\sigma^B\overline{Q}Q\right) \circ \left(j\overline{Q}Q\right) \\
&\overset{(67)}{=} m_B \circ \left(B\sigma^B\right) \circ (m_B PQ) \circ \left(B\sigma^B PQ\right) \circ (BPiQ) \circ (BqQ) \circ \left(u_B\overline{Q}Q\right) \\
&\qquad \circ \left(\varepsilon^D\overline{Q}Q\right) \\
&\overset{u_B}{=} m_B \circ \left(B\sigma^B\right) \circ (m_B PQ) \circ (u_B BPQ) \circ \left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \left(\varepsilon^D\overline{Q}Q\right) \\
&\overset{B\text{monad}}{=} m_B \circ \left(B\sigma^B\right) \circ \left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \left(\varepsilon^D\overline{Q}Q\right) \\
&= \overline{\sigma}^B \circ \left(\varepsilon^D\overline{Q}Q\right)
\end{aligned}
$$

so that we get

$$
\overline{\sigma}^B \circ \left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right) = \overline{\sigma}^B \circ \left(\varepsilon^D\overline{Q}Q\right) \tag{165}
$$

and since $(B', y') = \mathrm{Coequ}_{\mathrm{Fun}}\left(\left(\overline{Q}\chi\right) \circ \left(\delta_D\overline{Q}Q\right), \varepsilon^D\overline{Q}Q\right)$ there exists a unique functorial morphism $\nu_B : B' \to B$ such that

$$
\nu_B \circ y' = \overline{\sigma}^B = m_B \circ \left(\sigma^B\sigma^B\right) \circ (PiQ) \circ (qQ)\,. \tag{166}
$$

Now we want to prove that ν_B is a morphism of monads. Let us compute

$$
m_B \circ (\nu_B\nu_B) \circ (y'y') = m_B \circ (\nu_B B) \circ (B'\nu_B) \circ (y'B') \circ \left(\overline{Q}Qy'\right)
$$

$$
\begin{aligned}
&\overset{y'}{=} m_B \circ (\nu_B B) \circ (y' B) \circ \left(\overline{Q} Q \nu_B\right) \circ \left(\overline{Q} Q y'\right) \\
&\overset{(166)}{=} m_B \circ (m_B B) \circ \left(\sigma^B \sigma^B B\right) \circ (PiQB) \circ (qQB) \circ \left(\overline{Q} Q m_B\right) \circ \left(\overline{Q} Q \sigma^B \sigma^B\right) \\
&\qquad \circ \left(\overline{Q} Q PiQ\right) \circ \left(\overline{Q} Q qQ\right) \\
&\overset{(144)}{=} m_B \circ (m_B B) \circ \left(\sigma^B \sigma^B B\right) \circ (jPQB) \circ (\kappa_0' QB) \circ \left(\overline{Q} Q m_B\right) \circ \left(\overline{Q} Q \sigma^B \sigma^B\right) \\
&\qquad \circ \left(\overline{Q} Q jPQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&= m_B \circ (m_B B) \circ \left(B \sigma^B B\right) \circ \left(\sigma^B PQB\right) \circ (jPQB) \circ (\kappa_0' QB) \circ \left(\overline{Q} Q m_B\right) \\
&\qquad \circ \left(\overline{Q} Q B \sigma^B\right) \circ \left(\overline{Q} Q \sigma^B PQ\right) \circ \left(\overline{Q} Q jPQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&\overset{(67)}{=} m_B \circ (m_B B) \circ \left(B \sigma^B B\right) \circ (u_B PQB) \circ \left(\varepsilon^D PQB\right) \circ (\kappa_0' QB) \circ \left(\overline{Q} Q m_B\right) \\
&\qquad \circ \left(\overline{Q} Q B \sigma^B\right) \circ \left(\overline{Q} Q u_B PQ\right) \circ \left(\overline{Q} Q \varepsilon^D PQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&\overset{u_B}{=} m_B \circ (m_B B) \circ (u_B BB) \circ \left(\sigma^B B\right) \circ \left(\varepsilon^D PQB\right) \circ (\kappa_0' QB) \circ \left(\overline{Q} Q m_B\right) \\
&\qquad \circ \left(\overline{Q} Q u_B B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} Q \varepsilon^D PQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&\overset{B\text{monad}}{=} m_B \circ \left(\sigma^B B\right) \circ \left(\varepsilon^D PQB\right) \circ (\kappa_0' QB) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} Q \varepsilon^D PQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_B \circ m_{B'} \circ (y'y') &\overset{(163)}{=} \nu_B \circ y' \circ \left(\overline{Q} \chi\right) = m_B \circ \left(\sigma^B \sigma^B\right) \circ (PiQ) \circ (qQ) \circ \left(\overline{Q} \chi\right) \\
&\overset{(144)}{=} m_B \circ \left(B \sigma^B\right) \circ \left(\sigma^B PQ\right) \circ (jPQ) \circ (\kappa_0' Q) \circ \left(\overline{Q} \chi\right) \\
&\overset{(67)}{=} m_B \circ \left(B \sigma^B\right) \circ (u_B PQ) \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \chi\right) \\
&\overset{u_B}{=} m_B \circ (u_B B) \circ \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \chi\right) \\
&\overset{B\text{monad}}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \chi\right) \\
&\overset{\text{def}\chi}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q}^A \mu_Q B\right) \circ \left(\overline{Q} A Q \sigma^B\right) \circ \left(\overline{Q} \sigma^A QPQ\right) \\
&\qquad \circ \left(\overline{Q} Q PiQ\right) \circ \left(\overline{Q} Q qQ\right) \\
&\overset{{}^A\mu_Q}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q}^A \mu_Q PQ\right) \circ \left(\overline{Q} \sigma^A QPQ\right) \\
&\qquad \circ \left(\overline{Q} Q PiQ\right) \circ \left(\overline{Q} Q qQ\right) \\
&\overset{(82)}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} \mu_Q^B PQ\right) \\
&\qquad \circ \left(\overline{Q} Q \sigma^B PQ\right) \circ \left(\overline{Q} Q PiQ\right) \circ \left(\overline{Q} Q qQ\right) \\
&\overset{(144)}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} \mu_Q^B PQ\right) \\
&\qquad \circ \left(\overline{Q} Q \sigma^B PQ\right) \circ \left(\overline{Q} Q jPQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&\overset{(67)}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} \mu_Q^B PQ\right) \circ \left(\overline{Q} Q u_B PQ\right) \\
&\qquad \circ \left(\overline{Q} Q \varepsilon^D PQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right) \\
&\overset{Q\text{modulefunctor}}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ (\kappa_0' Q) \circ \left(\overline{Q} \mu_Q^B\right) \circ \left(\overline{Q} Q \sigma^B\right) \circ \left(\overline{Q} Q \varepsilon^D PQ\right) \circ \left(\overline{Q} Q \kappa_0' Q\right)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\kappa_0'}{=} \sigma^B \circ \left(\varepsilon^D PQ\right) \circ \left(DP\mu_Q^B\right) \circ (\kappa_0' QB) \circ \left(\overline{Q}Q\sigma^B\right) \circ \left(\overline{Q}Q\varepsilon^D PQ\right) \circ \left(\overline{Q}Q\kappa_0' Q\right) \\
&\overset{\varepsilon^D}{=} \sigma^B \circ \left(P\mu_Q^B\right) \circ \left(\varepsilon^D PQB\right) \circ (\kappa_0' QB) \circ \left(\overline{Q}Q\sigma^B\right) \circ \left(\overline{Q}Q\varepsilon^D PQ\right) \circ \left(\overline{Q}Q\kappa_0' Q\right) \\
&\overset{(81)}{=} m_B \circ \left(\sigma^B B\right) \circ \left(\varepsilon^D PQB\right) \circ (\kappa_0' QB) \circ \left(\overline{Q}Q\sigma^B\right) \circ \left(\overline{Q}Q\varepsilon^D PQ\right) \circ \left(\overline{Q}Q\kappa_0' Q\right)
\end{aligned}
$$

so that we obtain

$$m_B \circ (\nu_B \nu_B) \circ (y' y') = \nu_B \circ m_{B'} \circ (y' y')$$

and since y' is an epimorphism we deduce that

$$m_B \circ (\nu_B \nu_B) = \nu_B \circ m_{B'}.$$

Now, let us calculate

$$
\begin{aligned}
\nu_B \circ u_{B'} \circ \varepsilon^D &\overset{(163)}{=} \nu_B \circ y' \circ \delta_D \overset{(166)}{=} m_B \circ \left(\sigma^B \sigma^B\right) \circ (PiQ) \circ (qQ) \circ \delta_D \\
&= m_B \circ \left(B\sigma^B\right) \circ \left(\sigma^B PQ\right) \circ (PiQ) \circ (qQ) \circ \delta_D \\
&\overset{(144)}{=} m_B \circ \left(B\sigma^B\right) \circ \left(\sigma^B PQ\right) \circ (jPQ) \circ (\kappa_0' Q) \circ \delta_D \\
&\overset{(67),(148)}{=} m_B \circ \left(B\sigma^B\right) \circ (u_B PQ) \circ \left(\varepsilon^D PQ\right) \circ (Dj) \circ \Delta^D \\
&\overset{u_B,\varepsilon^D}{=} m_B \circ (u_B B) \circ \sigma^B \circ j \circ \left(\varepsilon^D D\right) \circ \Delta^D \\
&\overset{B\text{monad}D\text{comonad}}{=} \sigma^B \circ j \overset{(67)}{=} u_B \circ \varepsilon^D.
\end{aligned}
$$

Since the tame condition is assumed, ${}_{BA}\sigma_{AB}^B$ is a functorial isomorphism and thus

$${}_A\sigma_A^B \overset{(97)}{=} {}_A\sigma_{AB\mathbb{B}}^B F \overset{(94)}{=} {}_{\mathbb{B}}U_{BA}\sigma_{AB\mathbb{A}}^B F$$

is also an isomorphism. By (95) we have that

$${}_A\sigma_A^B \circ (p_{PA} Q) = \sigma^B$$

and Lemma 2.6, since $p_{PA}Q$ is a regular epimorphism by construction, we deduce that σ^B is also a regular epimorphism. Thus, by Theorem 6.6, so is $B\varepsilon^D$. Since B reflects coequalizers we get that ε^D is an epimorphism and therefore we obtain

$$\nu_B \circ u_{B'} = u_B.$$

Hence ν_B is a morphism of monads.

2) Consider the following diagram

$$
\begin{tikzcd}
D\overline{Q}Q \arrow[r, shift left, "(\overline{Q}\chi)\circ(\delta^D\overline{Q}Q)"] \arrow[r, shift right, "(\varepsilon^D\overline{Q}Q)"'] \arrow[d, "DhQ"'] & \overline{Q}Q \arrow[r, "y'"] \arrow[d, "hQ"] & S' \arrow[d, dotted, "\nu_Z"] \\
DPQ \arrow[r, shift left, "(P\mu_Q^S)\circ(jS)\circ(D\sigma_S)"] \arrow[r, shift right, "(\varepsilon^D PQ)"'] & PQ \arrow[r, "\pi_Z"] & Z
\end{tikzcd}
$$

where $(Z, \pi_Z) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\left(P\mu_Q^B\right) \circ (jB) \circ \left(D\sigma^B\right), \varepsilon^D PQ\right)$. Note that

$$\left(\varepsilon^D PQ\right) \circ (DhQ) \overset{\varepsilon^D}{=} (hQ) \circ \left(\varepsilon^D \overline{Q}Q\right).$$

Now we compute

$$(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\overset{?}{=}\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ(DhQ)$$

$$\begin{aligned}
&(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\\
&=\left({}^B\mu_PQ\right)\circ\left(\sigma^BPQ\right)\circ(PiQ)\circ(qQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\\
&\overset{(164)}{=}\left({}^B\mu_PQ\right)\circ\left(BP\mu_Q^B\right)\circ\left(BP\mu_Q^BB\right)\circ\left(BPQ\sigma^BB\right)\\
&\circ\left(BPQPQ\sigma^B\right)\circ(BPQPiQ)\circ(BPQqQ)\circ\left(u_BPQ\overline{Q}Q\right)\circ\left(j\overline{Q}Q\right)\\
&\overset{Q\text{modfunctor}}{=}\left({}^B\mu_PQ\right)\circ\left(BP\mu_Q^B\right)\circ(BPQm_B)\circ\left(BPQ\sigma^BB\right)\circ\left(BPQPQ\sigma^B\right)\\
&\circ(BPQPiQ)\circ(BPQqQ)\circ\left(u_BPQ\overline{Q}Q\right)\circ\left(j\overline{Q}Q\right)\\
&\overset{u_B}{=}\left({}^B\mu_PQ\right)\circ(u_BPQ)\circ\left(P\mu_Q^B\right)\circ(PQm_B)\circ\left(PQ\sigma^BB\right)\circ\left(PQPQ\sigma^B\right)\\
&\circ(PQPiQ)\circ(PQqQ)\circ\left(j\overline{Q}Q\right)\\
&\overset{Q\text{modfunctor}}{=}\left(P\mu_Q^B\right)\circ(PQm_B)\circ\left(PQ\sigma^BB\right)\circ\left(PQPQ\sigma^B\right)\circ(PQPiQ)\\
&\circ(PQqQ)\circ\left(j\overline{Q}Q\right)\\
&\overset{\sigma^B}{=}\left(P\mu_Q^B\right)\circ(PQm_B)\circ\left(PQB\sigma^B\right)\circ\left(PQ\sigma^BPQ\right)\circ(PQPiQ)\\
&\circ(PQqQ)\circ\left(j\overline{Q}Q\right)\\
&\overset{(81)}{=}\left(P\mu_Q^B\right)\circ\left(PQ\sigma^B\right)\circ\left(PQ{}^B\mu_PQ\right)\circ\left(PQ\sigma^BPQ\right)\circ(PQPiQ)\\
&\circ(PQqQ)\circ\left(j\overline{Q}Q\right)\\
&\overset{j}{=}\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ\left(D{}^B\mu_PQ\right)\circ\left(D\sigma^BPQ\right)\circ(DPiQ)\circ(DqQ)\\
&=\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ(DhQ)
\end{aligned}$$

Then the diagram above serially commute and hence in particular

$$\begin{aligned}
\pi_Z\circ(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)&=\pi_Z\circ\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ(DhQ)\\
&\overset{\pi_Z}{=}\pi_Z\circ\left(\varepsilon^DPQ\right)\circ(DhQ)=\pi_Z\circ(hQ)\circ\left(\varepsilon^D\overline{Q}Q\right)
\end{aligned}$$

so that

$$\pi_Z\circ(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)=\pi_Z\circ(hQ)\circ\left(\varepsilon^D\overline{Q}Q\right).$$

Since $(B',y')=\mathrm{Coequ}_{\mathrm{Fun}}\left(\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right),\varepsilon^D\overline{Q}Q\right)$, by the universal property of coequalizers, there exists a unique functorial morphism $\nu_Z:B'\to Z$ such that

$$\nu_Z\circ y'=\pi_Z\circ(hQ). \tag{167}$$

We want to prove that ν_Z is an isomorphism. Since hQ is an isomorphism, there exists $(hQ)^{-1}:PQ\to\overline{Q}Q$. Note that from

$$(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)=\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ(DhQ)$$

we deduce that

$$\begin{aligned}
&(hQ)^{-1}\circ(hQ)\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\circ\left(D\,(hQ)^{-1}\right)=\\
&=(hQ)^{-1}\circ\left(P\mu_Q^B\right)\circ(jB)\circ\left(D\sigma^B\right)\circ(DhQ)\circ\left(D\,(hQ)^{-1}\right)
\end{aligned}$$

that is

$$(168)\qquad \left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\circ\left(D\left(hQ\right)^{-1}\right)=\left(hQ\right)^{-1}\circ\left(P\mu_Q^B\right)\circ\left(jB\right)\circ\left(D\sigma^B\right).$$

Similarly, from

$$\left(\varepsilon^D PQ\right)\circ\left(DhQ\right)=\left(hQ\right)\circ\left(\varepsilon^D\overline{Q}Q\right)$$

we deduce that

$$(169)\qquad \left(hQ\right)^{-1}\circ\left(\varepsilon^D PQ\right)=\left(\varepsilon^D\overline{Q}Q\right)\circ\left(D\left(hQ\right)^{-1}\right).$$

Thus we have

$$\begin{aligned} y'\circ\left(hQ\right)^{-1}\circ\left(P\mu_Q^B\right)\circ\left(jB\right)\circ\left(D\sigma^B\right) &\overset{(168)}{=} y'\circ\left(\overline{Q}\chi\right)\circ\left(\delta_D\overline{Q}Q\right)\circ\left(D\left(hQ\right)^{-1}\right)\\ &\overset{\text{def}y'}{=} y'\circ\left(\varepsilon^D\overline{Q}Q\right)\circ\left(D\left(hQ\right)^{-1}\right)\overset{(169)}{=} y'\circ\left(hQ\right)^{-1}\circ\left(\varepsilon^D PQ\right)\end{aligned}$$

so that

$$y'\circ\left(hQ\right)^{-1}\circ\left(P\mu_Q^B\right)\circ\left(jB\right)\circ\left(D\sigma^B\right)=y'\circ\left(hQ\right)^{-1}\circ\left(\varepsilon^D PQ\right).$$

Since $(Z,\pi_Z)=\mathrm{Coequ}_{\mathrm{Fun}}\left(\left(P\mu_Q^B\right)\circ\left(jB\right)\circ\left(D\sigma^B\right),\varepsilon^D PQ\right)$, by the universal property of coequalizers, there exists a unique functorial morphism $\nu_Z':Z\to B'$such that

$$(170)\qquad \nu_Z'\circ\pi_Z=y'\circ\left(hQ\right)^{-1}.$$

Now we want to prove that ν_Z' is the two-sided inverse of ν_Z. Let us compute

$$\begin{aligned} \nu_Z'\circ\nu_Z\circ y' &\overset{(167)}{=} \nu_Z'\circ\pi_Z\circ\left(hQ\right)\\ &\overset{(170)}{=} y'\circ\left(hQ\right)^{-1}\circ\left(hQ\right)=y'\end{aligned}$$

and since y' is an epimorphism we get

$$\nu_Z'\circ\nu_Z=\mathrm{Id}_{B'}.$$

Moreover

$$\nu_Z\circ\nu_Z'\circ\pi_Z\overset{(170)}{=}\nu_Z\circ y'\circ\left(hQ\right)^{-1}\overset{(167)}{=}\pi_Z\circ\left(hQ\right)\circ\left(hQ\right)^{-1}=\pi_Z$$

and since π_Z is an epimorphism we deduce that

$$\nu_Z\circ\nu_Z'=\mathrm{Id}_Z.$$

Thus ν_Z is a functorial isomorphism between $B'$and Z with inverse ν_Z'. Now we want to construct an isomorphism between B and Z. Consider the parallel pair

$$DPQ \underset{\left(\varepsilon^D PQ\right)}{\overset{\left(P\mu_Q^S\right)\circ\left(jS\right)\circ\left(D\sigma_S\right)}{\rightrightarrows}} PQ$$

and compute

$$\begin{aligned} \sigma^B\circ\left(P\mu_Q^B\right)\circ\left(jB\right)\circ\left(D\sigma^B\right) &\overset{(81),j}{=} m_B\circ\left(\sigma^B B\right)\circ\left(PQ\sigma^B\right)\circ\left(jPQ\right)\\ \overset{\sigma^B}{=} m_B\circ\left(B\sigma^B\right)\circ\left(\sigma^B PQ\right)\circ\left(jPQ\right) &\overset{(67)}{=} m_B\circ\left(B\sigma^B\right)\circ\left(u_B PQ\right)\circ\left(\varepsilon^D PQ\right)\\ \overset{u_B}{=} m_B\circ\left(u_B B\right)\circ\sigma^B\circ\left(\varepsilon^D PQ\right) &\overset{B\text{monad}}{=} \sigma^B\circ\left(\varepsilon^D PQ\right).\end{aligned}$$

Thus we obtain

$$\sigma^B \circ \left(P\mu_Q^B\right) \circ (jB) \circ \left(D\sigma^B\right) = \sigma^B \circ \left(\varepsilon^D PQ\right)$$

and since $(Z, \pi_Z) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\left(P\mu_Q^B\right) \circ (jB) \circ \left(D\sigma^B\right), \varepsilon^D PQ\right)$, by the universal property of coequalizers, there exists a unique functorial morphism $\lambda : Z \to B$ such that

$$\lambda \circ \pi_Z = \sigma^B. \tag{171}$$

Since we already proved in 1) that σ^B is a regular epimorphism, in particular we can write $\left(B, \sigma^B\right) = \mathrm{Coequ}_{\mathrm{Fun}}(\xi, \zeta)$. Let us compute

$$\begin{aligned}
\pi_Z \circ \xi \circ \left(\varepsilon^D PQ\right) &\overset{\varepsilon^D}{=} \pi_Z \circ \left(\varepsilon^D PQ\right) \circ (D\xi) \\
&\overset{\mathrm{def}\pi_Z}{=} \pi_Z \circ \left(P\mu_Q^B\right) \circ (jB) \circ \left(D\sigma^B\right) \circ (D\xi) \\
&\overset{\sigma^B\mathrm{coequ}}{=} \pi_Z \circ \left(P\mu_Q^B\right) \circ (jB) \circ \left(D\sigma^B\right) \circ (D\zeta) \\
&\overset{\mathrm{def}\pi_Z}{=} \pi_Z \circ \left(\varepsilon^D PQ\right) \circ (D\zeta) \overset{\varepsilon^D}{=} \pi_Z \circ \zeta \circ \left(\varepsilon^D PQ\right)
\end{aligned}$$

so that

$$\pi_Z \circ \xi \circ \left(\varepsilon^D PQ\right) = \pi_Z \circ \zeta \circ \left(\varepsilon^D PQ\right).$$

Since σ^B is a regular epimorphism, by Theorem 6.6, so is $B\varepsilon^D$. Since by assumption B reflects coequalizers, also ε^D is an epimorphism so that we get

$$\pi_Z \circ \xi = \pi_Z \circ \zeta. \tag{172}$$

Since $\left(B, \sigma^B\right) = \mathrm{Coequ}_{\mathrm{Fun}}(\xi, \zeta)$, by the universal property of coequalizers there exists a unique functorial morphism $\lambda' : B \to Z$ such that

$$\lambda' \circ \sigma^B = \pi_Z. \tag{173}$$

We prove that λ' is the two-sided inverse of λ. In fact

$$\lambda' \circ \lambda \circ \pi_Z \overset{(171)}{=} \lambda' \circ \sigma^B \overset{(173)}{=} \pi_Z.$$

Since π_Z is an epimorphism we deduce that

$$\lambda' \circ \lambda = \mathrm{Id}_Z.$$

Similarly

$$\lambda \circ \lambda' \circ \sigma^B \overset{(173)}{=} \lambda \circ \pi_Z \overset{(171)}{=} \sigma^B$$

and, since also σ^B is an epimorphism, we deduce that

$$\lambda \circ \lambda' = \mathrm{Id}_B.$$

We now want to prove that

$$\nu_B = \lambda \circ \nu_Z$$

i.e.

$$\lambda' \circ \nu_B = \nu_Z.$$

We compute

$$\lambda' \circ \nu_B \circ y' \overset{(109)}{=} \lambda' \circ m_B \circ \left(\sigma^B \sigma^B\right) \circ (PiQ) \circ (qQ)$$

$$\overset{(81)}{=} \lambda' \circ \sigma^B \circ \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (PiQ) \circ (qQ)$$

$$\overset{(173)}{=} \pi_Z \circ \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (PiQ) \circ (qQ) = \pi_Z \circ (hQ) \overset{(167)}{=} \nu_Z \circ y'.$$

Since y' is an epimorphism, we get

$$\lambda' \circ \nu_B = \nu_Z$$

so that we deduce that $\nu_B : B' \to B$ is an isomorphism and thus an isomorphism of monads.

3) If P preserves equalizers and $PC = \overline{Q} \overset{\kappa'}{\to} \overline{Q}' = DP$ then ν_A and ν_B are isomorphisms of monads.

By assumption we have that $q = \mathrm{Id}_{PC}$ and $q' = \mathrm{Id}_{DP}$ and κ and κ' are isomorphisms. Then we can rewrite the initial diagram as follows

$$\begin{tikzcd}
DPCQ \arrow[r, shift left, "(PC\chi)\circ(\delta^D PCQ)"] \arrow[r, shift right, "(\varepsilon^D PCQ)"'] \arrow[d, "DhQ"'] & PCQ \arrow[r, "y'"] \arrow[d, "hQ"] & S' \arrow[d, dotted, "\nu_Z"] \\
DPQ \arrow[r, shift left, "(P\mu_Q^S)\circ(jS)\circ(D\sigma_S)"] \arrow[r, shift right, "(\varepsilon^D PQ)"'] & PQ \arrow[r, "\pi_Z"] & Z
\end{tikzcd}$$

Since $(C, i) = \mathrm{Equ}_{\mathrm{Fun}} \left(\left(QP\sigma^A \right) \circ (\tau P), QPu_A \right)$, by Lemma 2.10 we also have $(CQ, iQ) = \mathrm{Equ}_{\mathrm{Fun}} \left(\left(QP\sigma^A Q \right) \circ (\tau PQ), QPu_A Q \right)$. We compute

$$\left(QP\sigma^A Q \right) \circ (\tau PQ) \circ \tau \overset{(68)}{=} \left(QP\sigma^A Q \right) \circ (QP\tau) \circ \tau \overset{(69)}{=} (QPu_A Q) \circ \tau$$

so that there exists a unique functorial morphism ${}^C\rho_Q : Q \to CQ$ such that (61) holds i.e.

$$(iQ) \circ {}^C\rho_Q = \tau$$

as constructed in Proposition 6.1. Moreover $\left(Q, {}^C\rho_Q \right)$ is a left $\mathbb{C}$-comodule by Proposition 6.1. We also get

$$(PiQ) \circ \left(P^C\rho_Q \right) = P\tau.$$

Let us compute

$$(hQ) \circ \left(P^C\rho_Q \right) = \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (PiQ) \circ (qQ) \circ \left(P^C\rho_Q \right)$$

$$\overset{q=\mathrm{Id}_{PC}}{=} \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (PiQ) \circ \left(P^C\rho_Q \right)$$

$$\overset{(61)}{=} \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (P\tau) \overset{(82)}{=} \left(\mu_P^A Q \right) \circ \left(P\sigma^A Q \right) \circ (P\tau)$$

$$\overset{(69)}{=} \left(\mu_P^A Q \right) \circ (Pu_A Q) \overset{P\text{module}}{=} PQ$$

and

$$\left(P^C\rho_Q \right) \circ (hQ) = \left(P^C\rho_Q \right) \circ \left({}^B\mu_P Q \right) \circ \left(\sigma^B PQ \right) \circ (PiQ)$$

$$\overset{(82)}{=} \left(P^C\rho_Q \right) \circ \left(\mu_P^A Q \right) \circ \left(P\sigma^A Q \right) \circ (PiQ)$$

$$\overset{(63)}{=} \left(P^C\rho_Q \right) \circ \left(\mu_P^A Q \right) \circ (Pu_A Q) \circ \left(P\varepsilon^C Q \right)$$

$$\overset{P\text{module}}{=} \left(P^C\rho_Q \right) \circ \left(P\varepsilon^C Q \right) \overset{Q\text{comodule}}{=} PQ.$$

Thus hQ is an isomorphism with inverse $P^C\rho_Q$ and we can conclude by applying 2). $\square$

8. Equivalence for (co)module categories

8.1. **Equivalence for module categories coming from copretorsor.** In this subsection we prove that, for given categories $\mathcal{A}$ and $\mathcal{B}$, under the assumptions of Theorem 6.29, there exist a monad $\mathbb{A}$ on $\mathcal{A}$ and a monad $\mathbb{B}$ on $\mathcal{B}$ such that their categories of modules are equivalent. We outline that the assumptions quoted above are satisfied in the particular case of a regular coherd.

First of all we need to define the functors ${}_AQ_B$ and ${}_B\widehat{Q}_A$ which will be used to set the equivalence between these module categories.
Using the functors Q and $\widehat{Q}$, we construct the lifting functors ${}_AQ_B : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ and ${}_B\widehat{Q}_A : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$.

Proposition 8.1. *In the setting of 6.29 there exists a functor* ${}_A(Q_B) : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ *such that* ${}_{\mathbb{A}}U_A(Q_B) = Q_B$ *where* $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$. *Moreover we have*

$$p_Q \circ \left({}^A\mu_{Q\mathbb{B}}U\right) = {}^A\mu_{Q_B} \circ (Ap_Q) \tag{174}$$

where ${}^A\mu_{Q_B} = {}_{\mathbb{A}}U\lambda_{AA}(Q_B) : AQ_B \to Q_B$.

Proof. In view of Theorem 6.29, we can apply Proposition 3.30. $\square$

8.2. In light of Proposition 8.1, a functor $Q : \mathcal{B} \to \mathcal{A}$ introduced in 6.29 induces a functor ${}_A(Q_B) : {}_{\mathbb{B}}\mathcal{B} \to {}_{\mathbb{A}}\mathcal{A}$ for the monads $\mathbb{A}$ and $\mathbb{B}$. Our next task is to prove that the $\mathbb{B}$-$\mathbb{A}$-bimodule functor $\widehat{Q}$, constructed in Proposition 7.6, induces a functor ${}_B\left(\widehat{Q}_A\right) : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ which yields the inverse of ${}_A(Q_B)$.

Proposition 8.3. *Within the assumptions and notations of Theorem 6.29, there exists a functor* ${}_B\widehat{Q}_A : {}_{\mathbb{A}}\mathcal{A} \to {}_{\mathbb{B}}\mathcal{B}$ *such that* ${}_{\mathbb{B}}U_B\widehat{Q}_A = \widehat{Q}_A$ *where* $\left(\widehat{Q}_A, p_{\widehat{Q}}\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^A_{\widehat{Q}\mathbb{A}}U, \widehat{Q}_{\mathbb{A}}U\lambda_A\right)$. *Moreover we have*

$${}^A\mu_{Q_B} \circ \left(Bp_{\widehat{Q}}\right) = p_{\widehat{Q}} \circ \left({}^B\mu_{\widehat{Q}\mathbb{A}}U\right) \tag{175}$$

where ${}^B\mu_{\widehat{Q}_A} = {}_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A : B\widehat{Q}_A \to \widehat{Q}_A$, *so that* $\left(\widehat{Q}_A, {}^B\mu_{\widehat{Q}_A}\right)$ *is an* $\mathbb{B}$*-left module functor.*

Proof. In view of Proposition 7.6, we can apply Proposition 3.30 where Q is $\widehat{Q}$ and we exchange the role of $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{A}$ and $\mathbb{B}$. $\square$

Now we want to prove the first isomorphism.
Within the assumptions and notations of Theorem 6.29, we will construct a functorial isomorphism ${}_B\widehat{Q}_{AA}Q_B \cong {}_{\mathbb{B}}\mathcal{B}$.

Lemma 8.4. *Within the assumptions and notations of Theorem 6.29 the following equality*

$$\nu_0' \circ (u_BP) = l \circ (Pu_A) \tag{176}$$

holds where ν_0' is defined in (149) .

Proof. We compute

$$
\begin{aligned}
\nu_0' \circ (u_B P) \circ \left(\varepsilon^D P\right) \circ \left(DP\varepsilon^C\right) &\overset{(110)}{=} \nu_0' \circ (yP) \circ (\delta_D P) \circ \left(DP\varepsilon^C\right) \\
&\overset{\delta_D}{=} \nu_0' \circ (yP) \circ \left(PQP\varepsilon^C\right) \circ (\delta_D PC) \\
&\overset{(149)}{=} l \circ (Px) \circ \left(PQP\varepsilon^C\right) \circ (\delta_D PC) = l \circ (Px) \circ (Pw^r) \circ (\delta_D PC) \\
&\overset{x\mathrm{coequ}}{=} l \circ (Px) \circ \left(Pw^l\right) \circ (\delta_D PC) = l \circ (Px) \circ (P\chi P) \circ (PQP\delta_C) \circ (\delta_D PC) \\
&\overset{\delta_D}{=} l \circ (Px) \circ (P\chi P) \circ (\delta_D PQP) \circ (DP\delta_C) \\
&\overset{(149)}{=} \nu_0' \circ (yP) \circ (P\chi P) \circ (\delta_D PQP) \circ (DP\delta_C) = \nu_0' \circ (yP) \circ (z^l P) \circ (DP\delta_C) \\
&\overset{y\mathrm{coequ}}{=} \nu_0' \circ (yP) \circ (z^r P) \circ (DP\delta_C) = \nu_0' \circ (yP) \circ (\varepsilon^D PQP) \circ (DP\delta_C) \\
&\overset{(149)}{=} l \circ (Px) \circ (\varepsilon^D PQP) \circ (DP\delta_C) = l \circ (Px) \circ (P\delta_C) \circ (\varepsilon^D PC) \\
&\overset{(103)}{=} l \circ (Pu_A) \circ \left(P\varepsilon^C\right) \circ (\varepsilon^D PC) \overset{\varepsilon^D}{=} l \circ (Pu_A) \circ \left(\varepsilon^D P\right) \circ \left(DP\varepsilon^C\right).
\end{aligned}
$$

Since $\left(\varepsilon^D P\right) \circ \left(DP\varepsilon^C\right)$ is an epimorphism (recall that both ε^D and ε^C are coequalizers), we conclude. $\square$

Proposition 8.5. *Within the assumptions and notations of Theorem 6.29, there exists a functorial morphism $\alpha : B_{\mathbb{B}}U \to \widehat{Q}_{AA}Q_B$ such that* $\left(\widehat{Q}_{AA}Q_B, \alpha\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(m_{B\mathbb{B}}U, B_{\mathbb{B}}U\lambda_B\right)$. *Moreover for every morphism $h : B_{\mathbb{B}}U \to X$ such that*

$$h \circ (m_{B\mathbb{B}}U) = h \circ (B_{\mathbb{B}}U\lambda_B)$$

if $\widehat{h} : \widehat{Q}_{AA}Q_B \to X$ is the unique morphism such that $\widehat{h} \circ \alpha = h$, we have that

$$\widehat{h} \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) = h \circ (y_{\mathbb{B}}U) \circ \left(P^A \mu_{Q\mathbb{B}}U\right). \tag{177}$$

Proof. Let us prove that

$$
\begin{aligned}
&\left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) \circ (Pu_A Q_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U) \\
&= \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U).
\end{aligned} \tag{178}
$$

Using Proposition 8.1, we compute

$$
\begin{aligned}
&\left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) \circ (Pu_A Q_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U) \\
&\overset{u_A}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) \circ (PA\chi_{\mathbb{B}}U) \circ (Pu_A QPQ_{\mathbb{B}}U) \\
&\overset{(101)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ (PAp_Q) \circ \left(PA^A \mu_{Q\mathbb{B}}U\right) \circ (PAxQ_{\mathbb{B}}U) \\
&\qquad \circ (Pu_A QPQ_{\mathbb{B}}U) \\
&\overset{(174)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_A Q_B) \circ \left(PA^A \mu_{Q_B}\right) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}}U) \\
&\qquad \circ (Pu_A QPQ_{\mathbb{B}}U)
\end{aligned}
$$

$$\overset{l}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ \left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}}U) \circ (Pu_AQPQ_{\mathbb{B}}U)$$

$$\overset{\text{def } {}^A\mu_{Q_B}}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ \left(\widehat{Q}_{\mathbb{A}}U\lambda_{AA}Q_B\right) \circ (lAQ_B) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}}U) \circ (Pu_AQPQ_{\mathbb{B}}U)$$

$$\overset{p_{\widehat{Q}} \text{ coeq}}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ \left(\mu^A_{\widehat{Q}}Q_B\right) \circ (lAQ_B) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}}U) \circ (Pu_AQPQ_{\mathbb{B}}U)$$

$$\overset{(150)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (Pm_AQ_B) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}}U) \circ (Pu_AQPQ_{\mathbb{B}}U)$$

$$\overset{m_A}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pm_AQ_{\mathbb{B}}U) \circ (Pu_AAQ_{\mathbb{B}}U) \circ (PxQ_{\mathbb{B}}U)$$

$$\overset{A\text{monad}}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U)$$

Now we want to prove that

$$\left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ \left(z^l{}_{\mathbb{B}}U\right)$$
$$= \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ (z^r{}_{\mathbb{B}}U).$$

We have

$$\left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ \left(z^l{}_{\mathbb{B}}U\right)$$
$$= \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U) \circ (\delta_DPQ_{\mathbb{B}}U)$$
$$\overset{(178)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U) \circ (\delta_DPQ_{\mathbb{B}}U)$$
$$\overset{x}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PxQ_B) \circ (PQPp_Q) \circ (\delta_DPQ_{\mathbb{B}}U)$$
$$\overset{\delta_D}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PxQ_B) \circ (\delta_DPQ_B) \circ (DPp_Q)$$
$$\overset{(149)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (\nu'_0Q_B) \circ (yPQ_B) \circ (\delta_DPQ_B) \circ (DPp_Q)$$
$$\overset{(110)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (\nu'_0Q_B) \circ (u_BPQ_B) \circ \left(\varepsilon^DPQ_B\right) \circ (DPp_Q)$$
$$\overset{(176)}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (Pu_AQ_B) \circ \left(\varepsilon^DPQ_B\right) \circ (DPp_Q)$$
$$\overset{\varepsilon^D}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (Pu_AQ_B) \circ (Pp_Q) \circ \left(\varepsilon^DPQ_{\mathbb{B}}U\right)$$
$$= \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (Pu_AQ_B) \circ (Pp_Q) \circ (z^r{}_{\mathbb{B}}U)$$
$$\overset{u_A}{=} \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ (z^r{}_{\mathbb{B}}U)$$

Since, by Lemma 2.10

$$(B_{\mathbb{B}}U, y_{\mathbb{B}}U) = \mathrm{Coequ}_{\mathrm{Fun}}\left(z^l{}_{\mathbb{B}}U, z^r{}_{\mathbb{B}}U\right),$$

there exists a functorial morphism $\alpha : B_{\mathbb{B}}U \to \widehat{Q}_{AA}Q_B$ such that

$$\alpha \circ (y_{\mathbb{B}}U) = (p_{\widehat{Q}A}Q_B) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U). \tag{179}$$

Now we want to prove that

$$\left(\widehat{Q}_{AA}Q_B,\alpha\right)=\mathrm{Coequ}_{\mathrm{Fun}}(m_{B\mathbb{B}}U,B_{\mathbb{B}}U\lambda_B).$$

Let us show the fork property for α, that is

$$\alpha\circ(m_{B\mathbb{B}}U)=\alpha\circ(B_{\mathbb{B}}U\lambda_B)\,. \tag{180}$$

We have

$$\begin{aligned}
\alpha\circ(B_{\mathbb{B}}U\lambda_B)\circ(yy_{\mathbb{B}}U)&=\alpha\circ(B_{\mathbb{B}}U\lambda_B)\circ(yB_{\mathbb{B}}U)\circ(PQy_{\mathbb{B}}U)\\
&\overset{y}{=}\alpha\circ(y_{\mathbb{B}}U)\circ(PQ_{\mathbb{B}}U\lambda_B)\circ(PQy_{\mathbb{B}}U)\\
&\overset{(179)}{=}\left(p_{\widehat{Q}A}Q_B\right)\circ(lQ_B)\circ(PAp_Q)\circ(Pu_AQ_{\mathbb{B}}U)\circ(PQ_{\mathbb{B}}U\lambda_B)\circ(PQy_{\mathbb{B}}U)\\
&=\left(p_{\widehat{Q}A}Q_B\right)\circ(lQ_B)\circ(Pu_AQ_B)\circ(Pp_Q)\circ(PQ_{\mathbb{B}}U\lambda_B)\circ(PQy_{\mathbb{B}}U)\\
&\overset{\mathrm{def}p_Q}{=}\left(p_{\widehat{Q}A}Q_B\right)\circ(lQ_B)\circ(Pu_AQ_B)\circ(Pp_Q)\circ\left(P\mu^B_{Q\mathbb{B}}U\right)\circ(PQy_{\mathbb{B}}U)\\
&\overset{(107)}{=}\left(p_{\widehat{Q}A}Q_B\right)\circ(lQ_B)\circ(Pu_AQ_B)\circ(Pp_Q)\circ(P\chi_{\mathbb{B}}U)\\
&\overset{u_A}{=}\left(p_{\widehat{Q}A}Q_B\right)\circ(lQ_B)\circ(PAp_Q)\circ(Pu_AQ_{\mathbb{B}}U)\circ(P\chi_{\mathbb{B}}U)\\
&\overset{(179)}{=}\alpha\circ(y_{\mathbb{B}}U)\circ(P\chi_{\mathbb{B}}U)\overset{(109)}{=}\alpha\circ(m_{B\mathbb{B}}U)\circ(yy_{\mathbb{B}}U)
\end{aligned}$$

and, since $yy_{\mathbb{B}}U$ is an epimorphism, we conclude. Now, let us consider a functorial morphism $h:B_{\mathbb{B}}U\to X$ such that $h\circ(m_{B\mathbb{B}}U)=h\circ(B_{\mathbb{B}}U\lambda_B)$. We have to show that there exists a unique functorial morphism $\widehat{h}:\widehat{Q}_{AA}Q_B\to X$ such that

$$\widehat{h}\circ\alpha=h.$$

First we will show that there exists a functorial morphism $\widehat{h}$ such that $\widehat{h}$ and h fulfill (177) i.e.

$$\widehat{h}\circ\left(p_{\widehat{Q}A}Q_B\right)\circ(l_{\mathbb{A}}U_AQ_B)\circ(PAp_Q)=h\circ(y_{\mathbb{B}}U)\circ\left(P^A\mu_{Q\mathbb{B}}U\right).$$

To do this, we need a series of equalities. First of all, let us show that

$$y\circ\left(P^A\mu_Q\right)\circ\left(PA\mu^B_Q\right)\circ(PAQy)\circ(PxQPQ)=m_B\circ(yy)\circ(P\chi PQ)\,. \tag{181}$$

In fact, we have

$$\begin{aligned}
&y\circ\left(P^A\mu_Q\right)\circ\left(PA\mu^B_Q\right)\circ(PAQy)\circ(PxQPQ)\\
&\overset{(107)}{=}y\circ\left(P^A\mu_Q\right)\circ(PA\chi)\circ(PxQPQ)\\
&\overset{x}{=}y\circ\left(P^A\mu_Q\right)\circ(PxQ)\circ(PQP\chi)\overset{(101)}{=}y\circ(P\chi)\circ(PQP\chi)\\
&\overset{(98)}{=}y\circ(P\chi)\circ(P\chi PQ)\overset{(109)}{=}m_B\circ(yy)\circ(P\chi PQ)\,.
\end{aligned}$$

Now let us prove that

$$(yy)\circ(P\chi PQ)=(yB)\circ\left(P^A\mu_QB\right)\circ(PxQB)\circ(PQPQy)\,. \tag{182}$$

In fact we have

$$\begin{aligned}(yy)\circ(P\chi PQ) &= (yB)\circ(PQy)\circ(P\chi PQ)\\ &\overset{\chi}{=} (yB)\circ(P\chi B)\circ(PQPQy)\\ &\overset{(101)}{=} (yB)\circ\left(P^{A}\mu_{Q}B\right)\circ(PxQB)\circ(PQPQy)\,.\end{aligned}$$

Therefore we deduce that

$$\begin{aligned}&y\circ\left(P^{A}\mu_{Q}\right)\circ\left(PA\mu_{Q}^{B}\right)\circ(PAQy)\circ(PxQPQ)\\ &= m_{B}\circ(yB)\circ\left(P^{A}\mu_{Q}B\right)\circ(PxQB)\circ(PQPQy)\,.\end{aligned}\tag{183}$$

Now we compute

$$\begin{aligned}&h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ\left(PA\mu_{Q\mathbb{B}}^{B}U\right)\circ(PAQy_{\mathbb{B}}U)\circ(PxQPQ_{\mathbb{B}}U)\\ &\overset{(183)}{=} h\circ(m_{B\mathbb{B}}U)\circ(yB_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q}B_{\mathbb{B}}U\right)\circ(PxQB_{\mathbb{B}}U)\circ(PQPQy_{\mathbb{B}}U)\\ &\overset{\text{assumpt}h}{=} h\circ(B_{\mathbb{B}}U\lambda_{B})\circ(yB_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q}B_{\mathbb{B}}U\right)\circ(PxQB_{\mathbb{B}}U)\circ(PQPQy_{\mathbb{B}}U)\\ &\overset{y}{=} h\circ(y_{\mathbb{B}}U)\circ(PQ_{\mathbb{B}}U\lambda_{B})\circ\left(P^{A}\mu_{Q}B_{\mathbb{B}}U\right)\circ(PxQB_{\mathbb{B}}U)\circ(PQPQy_{\mathbb{B}}U)\\ &\overset{^{A}\mu_{Q}}{=} h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ(PAQ_{\mathbb{B}}U\lambda_{B})\circ(PxQB_{\mathbb{B}}U)\circ(PQPQy_{\mathbb{B}}U)\\ &\overset{x}{=} h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ(PAQ_{\mathbb{B}}U\lambda_{B})\circ(PAQy_{\mathbb{B}}U)\circ(PxQPQ_{\mathbb{B}}U)\end{aligned}$$

Since $(PAQy_{\mathbb{B}}U)\circ(PxQPQ_{\mathbb{B}}U)$ is an epimorphism, we obtain

$$h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ\left(PA\mu_{Q\mathbb{B}}^{B}U\right) = h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ(PAQ_{\mathbb{B}}U\lambda_{B})$$

Since $(PAQ_{B},PAp_{Q}) = \mathrm{Coequ}_{\mathrm{Fun}}\left(PA\mu_{Q\mathbb{B}}^{B}U, PAQ_{\mathbb{B}}U\lambda_{B}\right)$, there exists a functorial morphism $h_{1}: PAQ_{B}\to X$ such that

$$h_{1}\circ(PAp_{Q}) = h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right).\tag{184}$$

Now we prove that

$$y\circ\left(P^{A}\mu_{Q}\right)\circ(PxQ)\circ\left(z^{l}PQ\right) = y\circ\left(P^{A}\mu_{Q}\right)\circ(PxQ)\circ(z^{r}PQ)\tag{185}$$

In fact, we have

$$\begin{aligned}&y\circ\left(P^{A}\mu_{Q}\right)\circ(PxQ)\circ\left(z^{l}PQ\right)\overset{(101)}{=} y\circ(P\chi)\circ\left(z^{l}PQ\right)\\ &\overset{(156)}{=} y\circ(P\chi)\circ(z^{r}PQ)\overset{(101)}{=} y\circ\left(P^{A}\mu_{Q}\right)\circ(PxQ)\circ(z^{r}PQ)\,.\end{aligned}$$

Using the previous equalities, we obtain

$$\begin{aligned}h_{1}\circ(PxQ_{B})\circ\left(z^{l}PQ_{B}\right)\circ(DPQPp_{Q}) &\overset{z^{l}}{=} h_{1}\circ(PxQ_{B})\circ(PQPp_{Q})\circ\left(z^{l}PQ_{\mathbb{B}}U\right)\\ &\overset{x}{=} h_{1}\circ(PAp_{Q})\circ(PxQ_{\mathbb{B}}U)\circ\left(z^{l}PQ_{\mathbb{B}}U\right)\\ &\overset{(184)}{=} h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ(PxQ_{\mathbb{B}}U)\circ\left(z^{l}PQ_{\mathbb{B}}U\right)\\ &\overset{(185)}{=} h\circ(y_{\mathbb{B}}U)\circ\left(P^{A}\mu_{Q\mathbb{B}}U\right)\circ(PxQ_{\mathbb{B}}U)\circ(z^{r}PQ_{\mathbb{B}}U)\\ &\overset{(184)}{=} h_{1}\circ(PAp_{Q})\circ(PxQ_{\mathbb{B}}U)\circ(z^{r}PQ_{\mathbb{B}}U)\\ &\overset{x}{=} h_{1}\circ(PxQ_{B})\circ(PQPp_{Q})\circ(z^{r}PQ_{\mathbb{B}}U)\end{aligned}$$

$$\overset{z^r}{=} h_1 \circ (PxQ_B) \circ (z^r PQ_B) \circ (DPQPp_Q) .$$

Since $DPQPp_Q$ is an epimorphism, we deduce that

$$h_1 \circ (PxQ_B) \circ \left(z^l PQ_B\right) = h_1 \circ (PxQ_B) \circ (z^r PQ_B) .$$

Since $\left(\widehat{Q}Q_B lQ_B\right) = \mathrm{Coequ}_{\mathrm{Fun}}((PxQ_B) \circ \left(z^l PQ_B\right), (PxQ_B) \circ (z^r PQ_B))$, there exists a functorial morphism $h_2 : \widehat{Q}Q_B \to X$ such that

$$h_2 \circ (lQ_B) = h_1. \tag{186}$$

We compute

$$\begin{aligned}
y \circ \left(P^A \mu_Q\right) \circ (PxQ) \circ (P\chi PQ) &\overset{(101)}{=} y \circ (P\chi) \circ (P\chi PQ) \\
&\overset{(98)}{=} y \circ (P\chi) \circ (PQP\chi) \\
&\overset{(101)}{=} y \circ \left(P^A \mu_Q\right) \circ (PxQ) \circ (PQP\chi) \\
&\overset{x}{=} y \circ \left(P^A \mu_Q\right) \circ (PA\chi) \circ (PxQPQ)
\end{aligned}$$

so that we get

$$y \circ \left(P^A \mu_Q\right) \circ (PxQ) \circ (P\chi PQ) = y \circ \left(P^A \mu_Q\right) \circ (PA\chi) \circ (PxQPQ) . \tag{187}$$

We also have

$$\begin{aligned}
&(lQ_B) \circ (PAp_Q) \circ (PA\chi_{\mathbb{B}} U) \circ (PxQPQ_{\mathbb{B}} U) \\
&\overset{(101)}{=} (lQ_B) \circ (PAp_Q) \circ \left(PA^A \mu_{Q\mathbb{B}} U\right) \circ (PAxQ_{\mathbb{B}} U) \circ (PxQPQ_{\mathbb{B}} U) \\
&\overset{(174)}{=} (lQ_B) \circ \left(PA^A \mu_{Q_B}\right) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}} U) \circ (PxQPQ_{\mathbb{B}} U) \\
&\overset{l}{=} \left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PAAp_Q) \circ (PAxQ_{\mathbb{B}} U) \circ (PxQPQ_{\mathbb{B}} U) \\
&\overset{x}{=} \left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PAxQ_B) \circ (PAQPp_Q) \circ (PxQPQ_{\mathbb{B}} U) \\
&\overset{x}{=} \left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PAxQ_B) \circ (PxQPQQ_B) \circ (PQPQPp_Q) \\
&= \left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PxxQ_B) \circ (PQPQPp_Q)
\end{aligned}$$

so that we get

$$\begin{aligned}
&(lQ_B) \circ (PAp_Q) \circ (PA\chi_{\mathbb{B}} U) \circ (PxQPQ_{\mathbb{B}} U) \\
= \quad &\left(\widehat{Q}^A \mu_{Q_B}\right) \circ (lAQ_B) \circ (PxxQ_B) \circ (PQPQPp_Q)
\end{aligned} \tag{188}$$

and hence we obtain

$$\begin{aligned}
&h_2 \circ \left(\mu^A_{\widehat{Q}} Q_B\right) \circ (lAQ_B) \circ (PxxQ_B) \circ (PQPQPp_Q) \\
&\overset{(150)}{=} h_2 \circ (lQ_B) \circ (Pm_A Q_B) \circ (PxxQ_B) \circ (PQPQPp_Q) \\
&\overset{(186)}{=} h_1 \circ (Pm_A Q_B) \circ (PxxQ_B) \circ (PQPQPp_Q) \\
&\overset{(102)}{=} h_1 \circ (PxQ_B) \circ (P\chi PQ_B) \circ (PQPQPp_Q)
\end{aligned}$$

$$\overset{\chi}{=} h_1 \circ (PxQ_B) \circ (PQPp_Q) \circ (P\chi PQ_{\mathbb{B}}U)$$
$$\overset{x}{=} h_1 \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U) \circ (P\chi PQ_{\mathbb{B}}U)$$
$$\overset{(184)}{=} h \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right) \circ (PxQ_{\mathbb{B}}U) \circ (P\chi PQ_{\mathbb{B}}U)$$
$$\overset{(187)}{=} h \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right) \circ (PA\chi_{\mathbb{B}}U) \circ (PxQPQ_{\mathbb{B}}U)$$
$$\overset{(184)}{=} h_1 \circ (PAp_Q) \circ (PA\chi_{\mathbb{B}}U) \circ (PxQPQ_{\mathbb{B}}U)$$
$$\overset{(186)}{=} h_2 \circ (lQ_B) \circ (PAp_Q) \circ (PA\chi_{\mathbb{B}}U) \circ (PxQPQ_{\mathbb{B}}U)$$
$$\overset{(188)}{=} h_2 \circ (\widehat{Q}^A\mu_{Q_B}) \circ (lAQ_B) \circ (PxxQ_B) \circ (PQPQPp_Q)\,.$$

Since $(lAQ_B) \circ (PxxQ_B) \circ (PQPQPp_Q)$ is an epimorphism, we deduce that

$$h_2 \circ (\widehat{Q}^A\mu_{Q_B}) = h_2 \circ \left(\mu^A_{\widehat{Q}}Q_B\right).$$

By Proposition 8.3 we have $\left(\widehat{Q}_{AA}Q_B, p_{\widehat{Q}A}Q_B\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^A_{\widehat{Q}\mathbb{A}}U_AQ_B, \widehat{Q}_{\mathbb{A}}U\lambda_{AA}Q_B\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^A_{\widehat{Q}}Q_B, \widehat{Q}^A\mu_{Q_B}\right)$ and hence we infer that there exists a functorial morphism $\widehat{h} : \widehat{Q}_{AA}Q_B \to X$ such that

$$\widehat{h} \circ \left(p_{\widehat{Q}A}Q_B\right) = h_2.$$

Hence we get

$$\widehat{h} \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q) = h_2 \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q)$$
$$= h_2 \circ (lQ_B) \circ (PAp_Q) \overset{(186)}{=} h_1 \circ (PAp_Q) \overset{(184)}{=} h \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right)$$

and hence equality (177) is proven. Now we have

$$\widehat{h} \circ \alpha \circ (y_{\mathbb{B}}U) \overset{(179)}{=} \widehat{h} \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U)$$
$$\overset{(177)}{=} h \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right) \circ (Pu_AQ_{\mathbb{B}}U) \overset{{}^A\mu_Q \text{ is unital}}{=} h \circ (y_{\mathbb{B}}U)$$

so we get

$$\widehat{h} \circ \alpha = h.$$

Let now $\widehat{h}' : \widehat{Q}_{AA}Q_B \to X$ be another functorial morphism such that $\widehat{h}' \circ \alpha = h$. Then we have

$$\widehat{h} \circ (p_{\widehat{Q}A}Q_B) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U)$$
$$\overset{(179)}{=} \widehat{h} \circ \alpha \circ (y_{\mathbb{B}}U) = h \circ (y_{\mathbb{B}}U) = \widehat{h}' \circ \alpha \circ (y_{\mathbb{B}}U)$$
$$\overset{(179)}{=} \widehat{h}' \circ (p_{\widehat{Q}A}Q_B) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U)\,.$$

Note that, by (178), since the second term is an epimorphism, $\left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U)$ is epi and so $(p_{\widehat{Q}A}Q_B) \circ (lQ_B) \circ (PAp_Q) \circ (Pu_AQ_{\mathbb{B}}U)$ is also an epimorphism. Therefore we deduce that $\widehat{h}' = \widehat{h}$. Hence we have proved that $\left(\widehat{Q}_{AA}Q_B, \alpha\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(m_{B\mathbb{B}}U, B_{\mathbb{B}}U\lambda_B\right)$. $\square$

THEOREM 8.6. *Within the assumptions and notations of Theorem 6.29, we have a functorial isomorphism* ${}_B\widehat{Q}_{AA}Q_B \cong {}_{\mathbb{B}}\mathcal{B}$.

Proof. In Proposition 8.5 we have proved the existence of a functorial morphism $\alpha : B_{\mathbb{B}}U \to \widehat{Q}_{AA}Q_B$ such that $\left(\widehat{Q}_{AA}Q_B, \alpha\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(m_{B\mathbb{B}}U, B_{\mathbb{B}}U\lambda_{B.}\right)$. By Proposition 3.13 and Proposition 3.14 also $({}_{\mathbb{B}}U, ({}_{\mathbb{B}}U\lambda_B)) = \mathrm{Coequ}_{\mathrm{Fun}}\left(m_{B\mathbb{B}}U, B_{\mathbb{B}}U\lambda_{B.}\right)$. In view of uniqueness of a coequalizer up to isomorphisms, there exists a functorial isomorphism

$$\beta : \widehat{Q}_{AA}Q_B = {}_{\mathbb{B}}U_B\widehat{Q}_{AA}Q_B \to {}_{\mathbb{B}}U \text{ such that } \beta \circ \alpha = {}_{\mathbb{B}}U\lambda_B.$$

Now since

$$({}_{\mathbb{B}}U\lambda_B) \circ (m_{B\mathbb{B}}U) = ({}_{\mathbb{B}}U\lambda_B) \circ (B_{\mathbb{B}}U\lambda_B)$$

and since $\beta \circ \alpha = {}_{\mathbb{B}}U\lambda_B$, by applying (177) where "$\widehat{h}$" $= \beta$ and "h" $= {}_{\mathbb{B}}U\lambda_B$, we deduce that

$$\beta \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q) = ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right)$$

equivalently

$$\begin{aligned} &\beta \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U) \\ &= ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}}U) \circ \left(P^A\mu_{Q\mathbb{B}}U\right) \circ (PxQ_{\mathbb{B}}U) \\ &\overset{(101)}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U) \end{aligned}$$

i.e.

$$\beta \circ \left(p_{\widehat{Q}A}Q_B\right) \circ (l_{\mathbb{A}}U_AQ_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}}U) = ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}}U) \circ (P\chi_{\mathbb{B}}U)\,. \tag{189}$$

Recall that, in view of Proposition 3.13, $({}_{\mathbb{B}}U, {}_{\mathbb{B}}U\lambda_B)$ is an $\mathbb{B}$-left module functor. Also $\left(\widehat{Q}_{AA}Q_B, {}^B\mu_{\widehat{Q}_AA}Q_B\right)$ is an $\mathbb{B}$-left module functor (see proof of Proposition 3.30 and Lemma 3.17) where ${}^B\mu_{\widehat{Q}_A} = {}_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A : B\widehat{Q}_A \to \widehat{Q}_A$.Now we want to prove that β lifts to a functorial morphism ${}_B\widehat{Q}_{AA}Q_B \to {}_{\mathbb{B}}\mathcal{B}$ i.e. that

$$\beta : \left(\widehat{Q}_{AA}Q_B, {}^B\mu_{\widehat{Q}_AA}Q_B\right) \to ({}_{\mathbb{B}}U, {}_{\mathbb{B}}U\lambda_B)$$

is a morphism of $\mathbb{B}$-left module functors. Thus we have to prove

$$({}_{\mathbb{B}}U\lambda_B) \circ (B\beta) = \beta \circ ({}^B\mu_{\widehat{Q}_AA}Q_B)$$

We calculate

$$\begin{aligned} &{}^B\mu_{\widehat{Q}} \circ (Bl) \circ (yPA) \circ (PQPx) = {}^B\mu_{\widehat{Q}} \circ \left(y\widehat{Q}\right) \circ (PQl) \circ (PQPx) \\ &\overset{(149)}{=} {}^B\mu_{\widehat{Q}} \circ \left(y\widehat{Q}\right) \circ (PQ\nu_0') \circ (PQyP) \overset{y}{=} {}^B\mu_{\widehat{Q}} \circ (B\nu_0') \circ (yBP) \circ (PQyP) \\ &\overset{(151)}{=} \nu_0' \circ (m_BP) \circ (yBP) \circ (PQyP) = \nu_0' \circ (m_BP) \circ (yyP) \\ &\overset{(109)}{=} \nu_0' \circ (yP) \circ (P\chi P) \overset{(149)}{=} l \circ (Px) \circ (P\chi P) \end{aligned}$$

so that we obtain:

$$ {}^{B}\mu_{\widehat{Q}} \circ (Bl) \circ (yPA) \circ (PQPx) = l \circ (Px) \circ (P\chi P) . \tag{190} $$

We compute

$$
\begin{aligned}
&\beta \circ ({}^{B}\mu_{\widehat{Q}_A A} Q_B) \circ \left(Bp_{\widehat{Q}A} Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (yPxp_Q) = \\
&\overset{(175)}{=} \beta \circ (p_{\widehat{Q}A} Q_B) \circ \left({}^{B}\mu_{\widehat{Q}\mathbb{A}} U_A Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (yPxp_Q) \\
&= \beta \circ (p_{\widehat{Q}A} Q_B) \circ \left({}^{B}\mu_{\widehat{Q}\mathbb{A}} U_A Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (yPAQ_B) \circ (PQPxQ_B) \\
&\qquad \circ (PQPQPp_Q) \\
&\overset{(190)}{=} \beta \circ (p_{\widehat{Q}A} Q_B) \circ (l_{\mathbb{A}} U_A Q_B) \circ (Px_{\mathbb{A}} U_A Q_B) \circ (P\chi P_{\mathbb{A}} U_A Q_B) \\
&\qquad \circ (PQPQPp_Q) \\
&\overset{\chi}{=} \beta \circ (p_{\widehat{Q}A} Q_B) \circ (l_{\mathbb{A}} U_A Q_B) \circ (Px_{\mathbb{A}} U_A Q_B) \circ (PQPp_Q) \circ (P\chi PQ_{\mathbb{B}} U) \\
&\overset{x}{=} \beta \circ (p_{\widehat{Q}A} Q_B) \circ (l_{\mathbb{A}} U_A Q_B) \circ (PAp_Q) \circ (PxQ_{\mathbb{B}} U) \circ (P\chi PQ_{\mathbb{B}} U) \\
&\overset{(189)}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}} U) \circ (P\chi_{\mathbb{B}} U) \circ (P\chi PQ_{\mathbb{B}} U) \\
&\overset{(98)}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (y_{\mathbb{B}} U) \circ (P\chi_{\mathbb{B}} U) \circ (PQP\chi_{\mathbb{B}} U) \\
&\overset{(109)}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (m_{B\mathbb{B}} U) \circ (yy_{\mathbb{B}} U) \circ (PQP\chi_{\mathbb{B}} U) \\
&\overset{{}_{\mathbb{B}}U\lambda_B \text{coequ}}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (B_{\mathbb{B}}U\lambda_B) \circ (By_{\mathbb{B}} U) \circ (yPQ_{\mathbb{B}} U) \circ (PQP\chi_{\mathbb{B}} U) \\
&\overset{y}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (B_{\mathbb{B}}U\lambda_B) \circ (By_{\mathbb{B}} U) \circ (BP\chi_{\mathbb{B}} U) \circ (yPQPQ_{\mathbb{B}} U) \\
&\overset{(189)}{=} ({}_{\mathbb{B}}U\lambda_B) \circ (B\beta) \circ \left(Bp_{\widehat{Q}A} Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (BPAp_Q) \circ (BPxQ_{\mathbb{B}} U) \\
&\qquad \circ (yPQPQ_{\mathbb{B}} U) \\
&= ({}_{\mathbb{B}}U\lambda_B) \circ (B\beta) \circ \left(Bp_{\widehat{Q}A} Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (yPxp_Q)
\end{aligned}
$$

Since $\left(Bp_{\widehat{Q}A} Q_B \right) \circ (Bl_{\mathbb{A}} U_A Q_B) \circ (yPxp_Q)$ is an epimorphism, we get that $({}_{\mathbb{B}}U\lambda_B) \circ (B\beta) = \beta \circ ({}^{B}\mu_{\widehat{Q}_A A} Q_B)$. Therefore β is a morphism of $\mathbb{B}$-left module functors and hence, in view of Proposition 3.25, it gives rise to a functorial isomorphism ${}_{B}\widehat{Q}_{AA}Q_B \cong {}_{\mathbb{B}}\mathcal{B}$. □

Now, we prove the second isomorphism.
Within the assumptions and notations of Theorem 6.29, we will construct a functorial isomorphism ${}_{A}Q_{BB}\widehat{Q}_A \cong {}_{\mathbb{A}}\mathcal{A}$.

Lemma 8.7. *Within the assumptions and notations of Theorem 6.29, there exists a functorial morphism* $\Xi : A_{\mathbb{A}} U \to Q_{BB}\widehat{Q}_A$ *uniquely determined by*

$$ \left(p_{QB}\widehat{Q}_A \right) \circ \left(Qp_{\widehat{Q}} \right) \circ (Ql_{\mathbb{A}} U) \circ (QPu_{A\mathbb{A}} U) = \Xi \circ (x_{\mathbb{A}} U) \tag{191} $$

such that

$$ \Xi \circ (m_{A\mathbb{A}} U) = \Xi \circ (A_{\mathbb{A}} U_A \lambda) . \tag{192} $$

Proof. Since $(Q_B, p_Q) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U, Q_{\mathbb{B}}U\lambda_B\right)$ we have that

$$(p_{QB}\widehat{Q}_A) \circ \left(\mu^B_{Q\mathbb{B}}U_B\widehat{Q}_A\right) = (p_{QB}\widehat{Q}_A) \circ \left(Q_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A\right) = (p_{QB}\widehat{Q}_A) \circ (Q^B\mu_{\widehat{Q}_A})$$

so that we obtain

$$\begin{aligned}
\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(\chi\widehat{Q}_{\mathbb{A}}U\right) &\overset{\chi}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(\chi\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \\
&\overset{(107)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(\mu^B_Q\widehat{Q}_A\right) \circ \left(Qy\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \\
&= (p_{QB}\widehat{Q}_A) \circ (Q^B\mu_{\widehat{Q}_A}) \circ \left(Qy\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \\
&\overset{y}{=} (p_{QB}\widehat{Q}_A) \circ (Q^B\mu_{\widehat{Q}_A}) \circ \left(QBp_{\widehat{Q}}\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \\
&\overset{(175)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right)
\end{aligned}$$

and hence we get

$$\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(\chi\widehat{Q}_{\mathbb{A}}U\right) = \left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right)\circ\left(Qy\widehat{Q}_{\mathbb{A}}U\right) \tag{193}$$

so that

$$\begin{aligned}
&\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&\overset{\chi}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(\chi\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(193)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{y}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QBl_{\mathbb{A}}U) \circ (QyPA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(190)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U).
\end{aligned}$$

Therefore we deduce that

$$\begin{aligned}
&\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&= \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U)
\end{aligned} \tag{194}$$

By using this equality we compute

$$\begin{aligned}
&\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(w^l{}_{\mathbb{A}}U\right) \circ \left(QPC\varepsilon^C{}_{\mathbb{A}}U\right) \\
&\overset{w^l}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(QP\varepsilon^C{}_{\mathbb{A}}U\right) \circ \left(w^lC_{\mathbb{A}}U\right) \\
&\overset{(103)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\delta_{C\mathbb{A}}U) \circ \left(w^lC_{\mathbb{A}}U\right) \\
&\overset{w^l}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ \left(w^lQP_{\mathbb{A}}U\right) \circ (QPC\delta_{C\mathbb{A}}U) \\
&= \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \circ (QP\delta_CQP_{\mathbb{A}}U) \\
&\qquad \circ (QPC\delta_{C\mathbb{A}}U)
\end{aligned}$$

$$\overset{(194)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \circ (QP\delta_C QP_{\mathbb{A}}U) \circ (QPC\delta_{C\mathbb{A}}U)$$

$$\overset{(99)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ \left(QP\varepsilon^C QP_{\mathbb{A}}U\right) \circ (QPC\delta_{C\mathbb{A}}U)$$

$$\overset{\varepsilon^C}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\delta_{C\mathbb{A}}U) \circ \left(QPC\varepsilon^C{}_{\mathbb{A}}U\right)$$

$$\overset{(103)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(QP\varepsilon^C{}_{\mathbb{A}}U\right) \circ \left(QPC\varepsilon^C{}_{\mathbb{A}}U\right)$$

$$= \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(w^r{}_{\mathbb{A}}U\right) \circ \left(QPC\varepsilon^C{}_{\mathbb{A}}U\right)$$

since $QPC\varepsilon^C{}_{\mathbb{A}}U$ is epi we deduce that

$$\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(w^l{}_{\mathbb{A}}U\right) = \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ \left(w^r{}_{\mathbb{A}}U\right).$$

Since $(A_{\mathbb{A}}U, x_{\mathbb{A}}U) = \mathrm{Coequ}_{\mathrm{Fun}}\left(w^l{}_{\mathbb{A}}U, w^r{}_{\mathbb{A}}U\right)$, there exists a functorial morphism $\Xi : A_{\mathbb{A}}U \to Q_{BB}\widehat{Q}_A$ such that

$$\left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) = \Xi \circ (x_{\mathbb{A}}U).$$

Let us prove that

$$\Xi \circ (m_{A\mathbb{A}}U) = \Xi \circ (A_{\mathbb{A}}U\lambda_A).$$

By the definition of $p_{\widehat{Q}}$ we have that

$$p_{\widehat{Q}} \circ \left(\mu^A_{\widehat{Q}\mathbb{A}}U\right) \circ (lA_{\mathbb{A}}U) = p_{\widehat{Q}} \circ \left(\widehat{Q}_{\mathbb{A}}U_A\lambda\right) \circ (lA_{\mathbb{A}}U)$$

so that

$$p_{\widehat{Q}} \circ (l_{\mathbb{A}}U) \circ (Pm_{A\mathbb{A}}U) \overset{(150)}{=} p_{\widehat{Q}} \circ \left(\mu^A_{\widehat{Q}\mathbb{A}}U\right) \circ (lA_{\mathbb{A}}U) \overset{p_{\widehat{Q}}\mathrm{coequ}}{=} p_{\widehat{Q}} \circ \left(\widehat{Q}_{\mathbb{A}}U_A\lambda\right) \circ (lA_{\mathbb{A}}U)$$

and hence

$$p_{\widehat{Q}} \circ (l_{\mathbb{A}}U) \circ (Pm_{A\mathbb{A}}U) = p_{\widehat{Q}} \circ \left(\widehat{Q}_{\mathbb{A}}U_A\lambda\right) \circ (lA_{\mathbb{A}}U). \tag{195}$$

We calculate

$$\Xi \circ (m_{A\mathbb{A}}U) \circ (xx_{\mathbb{A}}U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}}U\right)$$
$$\overset{(102)}{=} \Xi \circ (x_{\mathbb{A}}U) \circ (\chi P_{\mathbb{A}}U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}}U\right)$$
$$\overset{(191)}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \circ (\chi P_{\mathbb{A}}U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}}U\right)$$
$$\overset{\chi}{=} \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(\chi\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPu_{A\mathbb{A}}U)$$

$$
\begin{aligned}
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{(193)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right)\circ\left(Qy\widehat{Q}_{\mathbb{A}}U\right)\circ\left(QPQl_{\mathbb{A}}U\right)\circ\left(QPQPu_{A\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{y}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right)\circ\left(QBl_{\mathbb{A}}U\right)\circ\left(QyPA_{\mathbb{A}}U\right)\circ\left(QPQPu_{A\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{(103)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right)\circ\left(QBl_{\mathbb{A}}U\right)\circ\left(QyPA_{\mathbb{A}}U\right)\circ\left(QPQPx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\delta_{C\mathbb{A}}U\right)\\
\overset{(190)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\circ\left(QP\chi P_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\delta_{C\mathbb{A}}U\right)\\
\overset{(102)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPm_{A\mathbb{A}}U\right)\circ\left(QPxx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\delta_{C\mathbb{A}}U\right)\\
\overset{(195)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q\widehat{Q}_{\mathbb{A}}U_A\lambda\right)\circ\left(QlA_{\mathbb{A}}U\right)\circ\left(QPxA_{\mathbb{A}}U\right)\\
&\circ \left(QPQPx_{\mathbb{A}}U\right)\circ\left(QPQP\delta_{C\mathbb{A}}U\right)\\
\overset{(103)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q\widehat{Q}_{\mathbb{A}}U_A\lambda\right)\circ\left(QlA_{\mathbb{A}}U\right)\circ\left(QPxA_{\mathbb{A}}U\right)\\
&\circ \left(QPQPu_{A\mathbb{A}}U\right)\circ\left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{l,x}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPA_{\mathbb{A}}U_A\lambda\right)\circ\left(QPAu_{A\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{A\text{monad}}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\circ\left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{A\text{monad}}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPm_{A\mathbb{A}}U\right)\circ\left(QPu_AA_{\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{(195)}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Q\widehat{Q}_{\mathbb{A}}U_A\lambda\right)\circ\left(QlA_{\mathbb{A}}U\right)\circ\left(QPu_AA_{\mathbb{A}}U\right)\\
&\circ \left(QPx_{\mathbb{A}}U\right)\circ\left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{l}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPA_{\mathbb{A}}U_A\lambda\right)\circ\left(QPu_AA_{\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{u_A}{=}\;&\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ\left(Ql_{\mathbb{A}}U\right)\circ\left(QPu_{A\mathbb{A}}U\right)\circ\left(QP_{\mathbb{A}}U_A\lambda\right)\circ\left(QPx_{\mathbb{A}}U\right)\\
&\circ \left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{(191)}{=}\;&\Xi\circ\left(x_{\mathbb{A}}U\right)\circ\left(QP_{\mathbb{A}}U_A\lambda\right)\circ\left(QPx_{\mathbb{A}}U\right)\circ\left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)\\
\overset{x}{=}\;&\Xi\circ\left(A_{\mathbb{A}}U_A\lambda\right)\circ\left(xA_{\mathbb{A}}U\right)\circ\left(QPx_{\mathbb{A}}U\right)\circ\left(QPQP\varepsilon^{C}{}_{\mathbb{A}}U\right)
\end{aligned}
$$

$$= \Xi \circ (A_{\mathbb{A}} U_A \lambda) \circ (x x_{\mathbb{A}} U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}} U \right)$$

Since $(x x_{\mathbb{A}} U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}} U \right)$ is epi, we deduce (192). Note that, in particular, we have

$$\left(p_{QB} \widehat{Q}_A \right) \circ \left(Q p_{\widehat{Q}} \right) \circ (Q l_{\mathbb{A}} U) \circ (QP u_{A\mathbb{A}} U) \circ (\chi P_{\mathbb{A}} U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}} U \right) \tag{196}$$
$$= \left(p_{QB} \widehat{Q}_A \right) \circ \left(Q p_{\widehat{Q}} \right) \circ (Q l_{\mathbb{A}} U) \circ (QP x_{\mathbb{A}} U) \circ \left(QPQP\varepsilon^C{}_{\mathbb{A}} U \right)$$

and since the second term is epi, also the first is epi, and hence $\left(p_{QB} \widehat{Q}_A \right) \circ \left(Q p_{\widehat{Q}} \right) \circ (Q l_{\mathbb{A}} U) \circ (QP u_{A\mathbb{A}} U)$ is an epimorphism. $\square$

Proposition 8.8. *Within the assumptions and notations of Theorem 6.29, there exists a functorial morphism $\Xi : A_{\mathbb{A}} U \to Q_{BB} \widehat{Q}_A$ such that $\left(Q_{BB} \widehat{Q}_A, \Xi \right) = \mathrm{Coequ}_{\mathrm{Fun}} \left(m_{A\mathbb{A}} U, A_{\mathbb{A}} U_A \lambda \right)$. Moreover for every morphism k such that*

$$k \circ (m_{A\mathbb{A}} U) = k \circ (A_{\mathbb{A}} U_A \lambda)$$

if $\widehat{k} : Q_{BB} \widehat{Q}_A \to Y$ is the unique morphism such that $\widehat{k} \circ \Xi = k$, we have that

$$\widehat{k} \circ \left(p_{QB} \widehat{Q}_A \right) \circ \left(Q p_{\widehat{Q}} \right) \circ (Q l_{\mathbb{A}} U) = k \circ (m_{A\mathbb{A}} U) \circ (x A_{\mathbb{A}} U). \tag{197}$$

Proof. By Lemma 8.7 we already know that

$$\Xi \circ (m_{A\mathbb{A}} U) = \Xi \circ (A_{\mathbb{A}} U_A \lambda).$$

Now we want to prove that

$$\left(Q_{BB} \widehat{Q}_A, \Xi \right) = \mathrm{Coequ}_{\mathrm{Fun}} (m_{A\mathbb{A}} U, A_{\mathbb{A}} U_A \lambda).$$

Let $k : A_{\mathbb{A}} U \to Y$ be a functorial morphism such that

$$k \circ (m_{A\mathbb{A}} U) = k \circ (A_{\mathbb{A}} U_A \lambda) \tag{198}$$

We have to show that there exists a functorial morphism $\widehat{k} : Q_{BB} \widehat{Q}_A \to Y$ such that

$$\widehat{k} \circ \Xi = k.$$

First we will show that there exists a functorial morphism $\widehat{k}$ such that $\widehat{k}$ and k fulfil (197) i.e.

$$k \circ (m_{A\mathbb{A}} U) \circ (x A_{\mathbb{A}} U) = \widehat{k} \circ \left(p_{QB} \widehat{Q}_A \right) \circ \left(Q p_{\widehat{Q}} \right) \circ (Q l_{\mathbb{A}} U).$$

We proceed in several steps. First of all we compute

$$k \circ (m_{A\mathbb{A}} U) \circ (x A_{\mathbb{A}} U) \circ (QP x_{\mathbb{A}} U) \circ \left(Q z^l P_{\mathbb{A}} U \right)$$
$$\overset{(102)}{=} k \circ (x_{\mathbb{A}} U) \circ (\chi P_{\mathbb{A}} U) \circ \left(Q z^l P_{\mathbb{A}} U \right)$$
$$= k \circ (x_{\mathbb{A}} U) \circ (\chi P_{\mathbb{A}} U) \circ (QP \chi P_{\mathbb{A}} U) \circ (Q \delta_D PQP_{\mathbb{A}} U)$$
$$\overset{(98)}{=} k \circ (x_{\mathbb{A}} U) \circ (\chi P_{\mathbb{A}} U) \circ (\chi PQP_{\mathbb{A}} U) \circ (Q \delta_D PQP_{\mathbb{A}} U)$$
$$\overset{(105)}{=} k \circ (x_{\mathbb{A}} U) \circ (\chi P_{\mathbb{A}} U) \circ \left(Q \varepsilon^D PQP_{\mathbb{A}} U \right)$$
$$\overset{(102)}{=} k \circ (m_{A\mathbb{A}} U) \circ (x A_{\mathbb{A}} U) \circ (QP x_{\mathbb{A}} U) \circ \left(Q \varepsilon^D PQP_{\mathbb{A}} U \right)$$

$$= k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (Qz^r P_{\mathbb{A}}U)\,.$$

Since Q preserves coequalizers by assumption, by Lemma2.9 we have $\left(Q\widehat{Q}_{\mathbb{A}}U, Ql_{\mathbb{A}}U\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left((QPx_{\mathbb{A}}U \circ Qz^l P_{\mathbb{A}}U), (QPx_{\mathbb{A}}U \circ Qz^r P_{\mathbb{A}}U)\right)$, so we deduce that there exists a unique functorial morphism $k_1 : Q\widehat{Q}_{\mathbb{A}}U \to Y$ such that

$$k_1 \circ (Ql_{\mathbb{A}}U) = k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U)\,. \tag{199}$$

Then we have

$$\begin{aligned}
k_1 \circ \left(Q\mu^A_{\widehat{Q}\mathbb{A}}U\right) \circ (QlA_{\mathbb{A}}U) &\overset{(150)}{=} k_1 \circ (Ql_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \\
&\overset{(199)}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \\
&\overset{x}{=} k \circ (m_{A\mathbb{A}}U) \circ (Am_{A\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \overset{m_A\mathrm{ass}}{=} k \circ (m_{A\mathbb{A}}U) \circ (m_A A_{\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \\
&\overset{(198)}{=} k \circ (A_{\mathbb{A}}U\lambda_A) \circ (m_A A_{\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \\
&\overset{m_A}{=} k \circ (m_{A\mathbb{A}}U) \circ (AA_{\mathbb{A}}U_A\lambda) \circ (xAA_{\mathbb{A}}U) \overset{x}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPA_{\mathbb{A}}U_A\lambda) \\
&\overset{(199)}{=} k_1 \circ (Ql_{\mathbb{A}}U) \circ (QPA_{\mathbb{A}}U_A\lambda) \overset{l}{=} k_1 \circ \left(Q\widehat{Q}_{\mathbb{A}}U_A\lambda\right) \circ (QlA_{\mathbb{A}}U)
\end{aligned}$$

Since $QlA_{\mathbb{A}}U$ is epi, we get that $k_1 \circ \left(Q\mu^A_{\widehat{Q}\mathbb{A}}U\right) = k_1 \circ \left(Q\widehat{Q}_{\mathbb{A}}U\lambda_A\right)$. Since Q preserves coequalizers, $\left(Q\widehat{Q}_A, Qp_{\widehat{Q}}\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(Q\mu^A_{\widehat{Q}\mathbb{A}}U, Q\widehat{Q}_{\mathbb{A}}U\lambda_A\right)$, then there exists a unique functorial morphism $k_2 : Q\widehat{Q}_A \to Y$ such that

$$k_1 = k_2 \circ \left(Qp_{\widehat{Q}}\right). \tag{200}$$

We have

$$\begin{aligned}
&k_2 \circ \left(\mu^B_{Q\mathbb{B}}U_B\widehat{Q}_A\right) \circ \left(QBp_{\widehat{Q}}\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{y}{=} k_2 \circ \left(\mu^B_{Q\mathbb{B}}U_B\widehat{Q}_A\right) \circ \left(Qy_{\mathbb{B}}U_B\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(107)}{=} k_2 \circ \left(\chi_{\mathbb{B}}U_B\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{\chi}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&\overset{(200)}{=} k_1 \circ \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&\overset{(199)}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&\overset{(102)}{=} k \circ (x_{\mathbb{A}}U) \circ (\chi P_{\mathbb{A}}U) \circ (\chi PQP_{\mathbb{A}}U) \\
&\overset{(98)}{=} k \circ (x_{\mathbb{A}}U) \circ (\chi P_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(102)}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(199)}{=} k_1 \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(200)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U)
\end{aligned}$$

$$\begin{aligned}
&\overset{(155)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPb_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(154)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \circ (QPxAU_{\mathbb{A}}) \circ (QPQPxU_{\mathbb{A}}) \\
&= k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \circ (QPxx_{\mathbb{A}}U) \\
&\overset{(102)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(149)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Q\nu'_{0\mathbb{A}}U) \circ (QyP_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(109)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Q\nu'_{0\mathbb{A}}U) \circ (Qm_BP_{\mathbb{A}}U) \circ (QyyP_{\mathbb{A}}U) \\
&\overset{(151)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QB\nu'_0P_{\mathbb{A}}U) \circ (QyyP_{\mathbb{A}}U) \\
&= k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QB\nu'_0P_{\mathbb{A}}U) \circ (QByP_{\mathbb{A}}U) \circ (QyPQP_{\mathbb{A}}U) \\
&\overset{(149)}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QBlP_{\mathbb{A}}U) \circ (QBPx_{\mathbb{A}}U) \circ (QyPQP_{\mathbb{A}}U) \\
&\overset{y}{=} k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQlP_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(175)}{=} k_2 \circ \left(Q^B\mu_{\widehat{Q}_A}\right) \circ \left(QBp_{\widehat{Q}}\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{\mathrm{def}\,{}^A\mu_{Q_B}}{=} k_2 \circ \left(Q_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A\right) \circ \left(QBp_{\widehat{Q}}\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U)
\end{aligned}$$

Since $\left(QBp_{\widehat{Q}}\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U)$ is epi, we get

$$k_2 \circ \left(\mu^B_{Q\mathbb{B}}U_B\widehat{Q}_A\right) = k_2 \circ \left(Q_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A\right).$$

From $\left(Q_{BB}\widehat{Q}_A, p_{QB}\widehat{Q}_A\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^B_{Q\mathbb{B}}U_B\widehat{Q}_A, Q_{\mathbb{B}}U\lambda_{BB}\widehat{Q}_A\right)$ we deduce that there exists a unique functorial morphism $\widehat{k} : Q_{BB}\widehat{Q}_A \to Y$ such that

(201) $$\widehat{k} \circ \left(p_{QB}\widehat{Q}_A\right) = k_2.$$

Moreover we have

$$\begin{aligned}
\widehat{k} \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) &= k_2 \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \\
&\overset{(200)}{=} k_1 \circ (Ql_{\mathbb{A}}U) \overset{(199)}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U)
\end{aligned}$$

i.e.

(202) $$\widehat{k} \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) = k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U).$$

Now we compute

$$\begin{aligned}
\widehat{k} \circ \Xi \circ (x_{\mathbb{A}}U) &\overset{(191)}{=} \widehat{k} \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \\
&\overset{(202)}{=} k \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPu_{A\mathbb{A}}U) \overset{x}{=} k \circ (m_{A\mathbb{A}}U) \circ (Au_{A\mathbb{A}}U) \circ (x_{\mathbb{A}}U) \\
&= k \circ (x_{\mathbb{A}}U).
\end{aligned}$$

Since $x_{\mathbb{A}}U$ is epi we get that

$$\widehat{k}\circ\Xi=k.$$

Let now $\widehat{k}':Q_{BB}\widehat{Q}_A\to Y$ be another functorial morphism such that $\widehat{k}'\circ\Xi=k$. Then we have

$$\begin{aligned}&\widehat{k}\circ\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ(Ql_{\mathbb{A}}U)\circ(QPu_{A\mathbb{A}}U)\\ &\overset{(191)}{=}\widehat{k}\circ\Xi\circ(x_{\mathbb{A}}U)=k\circ(x_{\mathbb{A}}U)=\widehat{k}'\circ\Xi\circ(x_{\mathbb{A}}U)\\ &\overset{(191)}{=}\widehat{k}'\circ\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ(Ql_{\mathbb{A}}U)\circ(QPu_{A\mathbb{A}}U)\end{aligned}$$

Since we already observed from (196) that $\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ(Ql_{\mathbb{A}}U)\circ(QPu_{A\mathbb{A}}U)$ is an epimorphism, we have $\widehat{k}'=\widehat{k}$. Hence we have proved that $\left(Q_{BB}\widehat{Q}_A,\Xi\right)=\mathrm{Coequ}_{\mathrm{Fun}}(m_{A\mathbb{A}}U,A_{\mathbb{A}}U_A\lambda)$ and, in view of (202) , we get that (197)holds. $\square$

Theorem 8.9. *Within the assumptions and notations of Theorem 6.29, we have a functorial isomorphism ${}_{\mathbb{A}}\mathcal{A}\cong{}_AQ_{BB}\widehat{Q}_A$.*

Proof. In Proposition 8.8 we have proved the existence of a functorial morphism $\Xi:AU_{\mathbb{A}}\to Q_{BB}\widehat{Q}_A$ such that $\left(Q_{BB}\widehat{Q}_A,\Xi\right)=\mathrm{Coequ}_{\mathrm{Fun}}(m_{A\mathbb{A}}U,A_{\mathbb{A}}U\lambda_A)$. By Proposition 3.13 and Proposition 3.14 also $({}_{\mathbb{A}}U,{}_{\mathbb{A}}U\lambda_A)=\mathrm{Coequ}_{\mathrm{Fun}}\left((m_{A\mathbb{A}}U,A_{\mathbb{A}}U\lambda_A)\right)$, in view of uniqueness of a coequalizer up to isomorphisms, there exists a functorial isomorphism

$$\rho:{}_{\mathbb{A}}U_AQ_{BB}\widehat{Q}_A=Q_{BB}\widehat{Q}_A\to{}_{\mathbb{A}}U\text{ such that }\rho\circ\Xi={}_{\mathbb{A}}U\lambda_A.$$

Now since

$$({}_{\mathbb{A}}U\lambda_A)\circ(m_{A\mathbb{A}}U)=({}_{\mathbb{A}}U\lambda_A)\circ(A_{\mathbb{A}}U\lambda_A)$$

and since $\rho\circ\Xi={}_{\mathbb{A}}U\lambda_A$, by Proposition 8.8, we deduce that

$$\rho\circ\left(p_{QB}\widehat{Q}_A\right)\circ\left(Qp_{\widehat{Q}}\right)\circ(Ql_{\mathbb{A}}U)=({}_{\mathbb{A}}U\lambda_A)\circ(m_{A\mathbb{A}}U)\circ(xA_{\mathbb{A}}U)\tag{203}$$

Now we want to prove that ρ lifts to a functorial morphism ${}_AQ_{BB}\widehat{Q}_A\to{}_{\mathbb{A}}\mathcal{A}$ of $\mathbb{A}$-left module functors. First we observe that $({}_{\mathbb{A}}U,{}_{\mathbb{A}}U\lambda_A)$ is an $\mathbb{A}$-left module functor in view of Proposition 3.13. Also $\left(Q_{BB}\widehat{Q}_A,{}^A\mu_{Q_BB}\widehat{Q}_A\right)$ is an $\mathbb{A}$-left module functor (see proof of Proposition 3.30) where ${}^A\mu_{Q_B}={}_{\mathbb{A}}U\lambda_{AA}Q_B:AQ_B\to Q_B$. To show that ρ is morphism of $\mathbb{A}$-left module functors we have to prove

$$({}_{\mathbb{A}}U\lambda_A)\circ(A\rho)=\rho\circ({}^A\mu_{Q_BB}\widehat{Q}_A).$$

We have

$$\begin{aligned}&\rho\circ\left({}^A\mu_{Q_BB}\widehat{Q}_A\right)\circ\left(Ap_{QB}\widehat{Q}_A\right)\circ\left(xQ_{\mathbb{B}}U_B\widehat{Q}_A\right)\circ\left(QPQp_{\widehat{Q}}\right)\circ(QPQl_{\mathbb{A}}U)\circ(QPQPx_{\mathbb{A}}U)\\ &\overset{(174)}{=}\rho\circ\left(p_{QB}\widehat{Q}_A\right)\circ\left({}^A\mu_{Q\mathbb{B}}U_B\widehat{Q}_A\right)\circ\left(xQ_{\mathbb{B}}U_B\widehat{Q}_A\right)\circ\left(QPQp_{\widehat{Q}}\right)\circ(QPQl_{\mathbb{A}}U)\\ &\qquad\circ(QPQPx_{\mathbb{A}}U)\\ &\overset{(101)}{=}\rho\circ\left(p_{QB}\widehat{Q}_A\right)\circ\left(\chi_{\mathbb{B}}U_B\widehat{Q}_A\right)\circ\left(QPQp_{\widehat{Q}}\right)\circ(QPQl_{\mathbb{A}}U)\circ(QPQPx_{\mathbb{A}}U)\end{aligned}$$

$$
\begin{aligned}
&\overset{\chi}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(\chi\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(193)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ \left(Qy\widehat{Q}_{\mathbb{A}}U\right) \circ (QPQl_{\mathbb{A}}U) \\
&\qquad \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{y}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QBl_{\mathbb{A}}U) \circ (QBPx_{\mathbb{A}}U) \circ (QyPQP_{\mathbb{A}}U) \\
&\overset{(149)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q^B\mu_{\widehat{Q}\mathbb{A}}U\right) \circ (QB\nu'_{0\mathbb{A}}U) \circ (QByP_{\mathbb{A}}U) \\
&\qquad \circ (QyPQP_{\mathbb{A}}U) \\
&\overset{(151)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Q\nu'_{0\mathbb{A}}U) \circ (Qm_BP_{\mathbb{A}}U) \circ (QByP_{\mathbb{A}}U) \\
&\qquad \circ (QyPQP_{\mathbb{A}}U) \\
&= \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Q\nu'_{0\mathbb{A}}U) \circ (Qm_BP_{\mathbb{A}}U) \circ (QyyP_{\mathbb{A}}U) \\
&\overset{(109)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Q\nu'_{0\mathbb{A}}U) \circ (QyP_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(149)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPx_{\mathbb{A}}U) \circ (QP\chi P_{\mathbb{A}}U) \\
&\overset{(102)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \circ (QPxx_{\mathbb{A}}U) \\
&= \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPm_{A\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(150)}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q\mu^A_{\widehat{Q}\mathbb{A}}U\right) \circ (QlA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{\mathrm{def} p_{\widehat{Q}}}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ \left(Q\widehat{Q}_{\mathbb{A}}U\lambda_A\right) \circ (QlA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{l}{=} \rho \circ \left(p_{QB}\widehat{Q}_A\right) \circ \left(Qp_{\widehat{Q}}\right) \circ (Ql_{\mathbb{A}}U) \circ (QPA_{\mathbb{A}}U\lambda_A) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{(203)}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (m_{A\mathbb{A}}U) \circ (xA_{\mathbb{A}}U) \circ (QPA_{\mathbb{A}}U\lambda_A) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{x}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (m_{A\mathbb{A}}U) \circ (AA_{\mathbb{A}}U\lambda_A) \circ (xAA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{m_A}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (A_{\mathbb{A}}U\lambda_A) \circ (m_AA_{\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \circ (QPQPx_{\mathbb{A}}U) \\
&\overset{{}_{\mathbb{A}}U\lambda_A \text{ ass}}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (m_{A\mathbb{A}}U) \circ (m_AA_{\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \\
&\overset{m_A \text{ ass}}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (m_{A\mathbb{A}}U) \circ (Am_{A\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \\
&\overset{{}_{\mathbb{A}}U\lambda_A \text{ ass}}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (A_{\mathbb{A}}U\lambda_A) \circ (Am_{A\mathbb{A}}U) \circ (xAA_{\mathbb{A}}U) \circ (QPxA_{\mathbb{A}}U) \\
&= ({}_{\mathbb{A}}U\lambda_A) \circ (A_{\mathbb{A}}U\lambda_A) \circ (Am_{A\mathbb{A}}U) \circ (AxA_{\mathbb{A}}U) \circ (xQPA_{\mathbb{A}}U) \\
&\overset{(203)}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (A\rho) \circ \left(Ap_{QB}\widehat{Q}_A\right) \circ \left(AQp_{\widehat{Q}}\right) \circ (AQl_{\mathbb{A}}U) \circ (xQPA_{\mathbb{A}}U) \\
&\overset{x}{=} ({}_{\mathbb{A}}U\lambda_A) \circ (A\rho) \circ \left(Ap_{QB}\widehat{Q}_A\right) \circ \left(xQ\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \circ (QPQl_{\mathbb{A}}U)
\end{aligned}
$$

Since $\left(Ap_{QB}\widehat{Q}_A\right) \circ \left(xQ_{\mathbb{B}}U_B\widehat{Q}_A\right) \circ \left(QPQp_{\widehat{Q}}\right) \circ (QPQl_{\mathbb{A}}U)$ is an epimorphism, we get that $\rho \circ ({}^A\mu_{Q_BB}\widehat{Q}_A) = ({}_{\mathbb{A}}U\lambda_A) \circ (A\rho)$. Therefore ρ is a morphism of $\mathbb{A}$-left module functors and hence, in view of Proposition 3.25, it gives rise to a functorial isomorphism ${}_AQ_{BB}\widehat{Q}_A \cong {}_{\mathbb{A}}\mathcal{A}$. □

8.2. **Equivalence for comodule categories coming from pretorsor.** The results obtained in this subsection can be found in [BM].

Given categories $\mathcal{A}$ and $\mathcal{B}$, under the assumptions of Theorem 6.5, one can prove that there exist a comonad $\mathbb{C}$ on $\mathcal{A}$ and a comonad $\mathbb{D}$ on $\mathcal{B}$ such that their categories of comodules are equivalent. We outline that the assumptions quoted above are satisfied in the particular case of a regular herd.

Using the functors Q and $\overline{Q}$, we construct the functors ${}^CQ^D : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ and ${}^D\overline{Q}^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ which will be used to set the equivalence between these comodule categories.

Proposition 8.10. *In the setting of Theorem 6.5 there exists a functor* ${}^C\left(Q^D\right) : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ *such that* ${}^{\mathbb{C}}U^C\left(Q^D\right) = Q^D$ *where* $\left(Q^D, \iota^Q\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_Q^{D\mathbb{D}}U, Q^{\mathbb{D}}U\gamma^D\right)$. *Moreover we have*

$$\left({}^C\rho_Q{}^{\mathbb{D}}U\right) \circ \iota^Q = \left(C\iota^Q\right) \circ {}^C\rho_{Q^D} \tag{204}$$

where ${}^C\rho_{Q^D} = {}^{\mathbb{C}}U\gamma^{CC}\left(Q^D\right) : Q^D \to CQ^D$.

Proof. In view of Theorem 6.5, we can apply Proposition 4.29. □

8.11. In light of Proposition 8.10, a functor $Q : \mathcal{B} \to \mathcal{A}$ introduced in Theorem 6.5 induces a functor ${}^C\left(Q^D\right) : {}^{\mathbb{D}}\mathcal{B} \to {}^{\mathbb{C}}\mathcal{A}$ for the comonads $\mathbb{C}$ and $\mathbb{D}$. Our next task is to prove that the $\mathbb{D}$-$\mathbb{C}$-bicomodule functor $\overline{Q}$, constructed in Proposition 7.1, induces a functor ${}^D\left(\overline{Q}^C\right) : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ which yields the inverse of ${}^C\left(Q^D\right)$.

Proposition 8.12. *Within the assumptions and notations of Theorem 6.5, there exists a functor* ${}^D\overline{Q}^C : {}^{\mathbb{C}}\mathcal{A} \to {}^{\mathbb{D}}\mathcal{B}$ *such that* ${}^{\mathbb{D}}U^D\overline{Q}^C = \overline{Q}^C$ *where* $\left(\overline{Q}^C, \iota^{\overline{Q}}\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\rho_{\overline{Q}}^{C\mathbb{C}}U, \overline{Q}^{\mathbb{C}}U\gamma^C\right)$. *Moreover we have*

$$\left(D\iota^{\overline{Q}}\right) \circ {}^D\rho_{\overline{Q}^C} = \left({}^D\rho_{\overline{Q}}{}^{\mathbb{C}}U\right) \circ \iota^{\overline{Q}} \tag{205}$$

where ${}^D\rho_{\overline{Q}^C} = {}^{\mathbb{D}}U\gamma^{DD}\overline{Q}^C : \overline{Q}^C \to D\overline{Q}^C$ *so that* $\left(\overline{Q}^C, {}^D\rho_{\overline{Q}^C}\right)$ *is a left* $\mathbb{D}$*-comodule functor.*

Proof. In view of Proposition 7.1, we can apply Proposition 4.29 where Q is $\overline{Q}$ and we exchange the role of $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{C}$ and $\mathbb{D}$. □

Within the assumptions and notations of Theorem 6.5, one can construct functorial isomorphism ${}^D\overline{Q}^{C\,C}Q^D \cong {}^{\mathbb{D}}\mathcal{B}$ and ${}^CQ^{DD}\overline{Q}^C \cong {}^{\mathbb{C}}\mathcal{A}$. Such a result can be obtained by dualizing all the ingredients proved in details for the equivalence between module categories.

THEOREM 8.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers and let $\tau : Q \to QPQ$ be a regular pretorsor for $\Xi = \left(A, B, P, Q, \sigma^A, \sigma^B, u_A, u_B\right)$. Assume that the underlying functors P, Q, A and B preserve equalizers. Then we have functorial isomorphisms ${}^{\mathbb{D}}\mathcal{B} \cong {}^{D}\overline{Q}^{CC}Q^{D}$ and ${}^{\mathbb{C}}\mathcal{A} \cong {}^{C}Q^{DD}\overline{Q}^{C}$.*

9. EXAMPLES

LEMMA 9.1. *Let ${}_T\Sigma_R$ be a bimodule. Let $L = -\otimes_T \Sigma : Mod\text{-}T \to Mod\text{-}R$ and let $H = \mathrm{Hom}_R(\Sigma, -)$. Assume that ${}_T\Sigma$ is faithfully flat. Then the unit η of the adjunction (L, H) is a regular mono.*

Proof. It is well known (see e.g. [BM, Lemma 2.3]) that the diagram

$$L \xrightarrow{L\eta} LHL \underset{LHL\eta}{\overset{L\eta HL}{\rightrightarrows}} LHLHL$$

is a contractible (split) equalizer with respect to the functorial morphisms $(\epsilon L, \epsilon LHL)$. Since ${}_T\Sigma$ is faithfully flat we get that the diagram

$$\mathrm{Id}_{Mod\text{-}R} \xrightarrow{\eta} HL \underset{HL\eta}{\overset{\eta HL}{\rightrightarrows}} HLHL$$

is exact. $\square$

COROLLARY 9.2. *Let $\alpha : T \to A$ be a ring homomorphism and assume that ${}_TA$ is faithfully flat. Let $\gamma : A \to A \otimes_T A$ be the map defined by setting*

$$\gamma(a) = 1_A \otimes_T a - a \otimes_T 1_A.$$

Then $(T, \alpha) = \mathrm{Ker}(\gamma)$.

COROLLARY 9.3. *Let $\alpha : A \to T$ be a ring homomorphism such that T_A is faithfully flat. Then the unit of the adjunction $(-\otimes_A T, \mathrm{Hom}_T(T, -))$ is a regular monomorphism.*

Proof. Let $\eta : Mod\text{-}T \to \mathrm{Hom}_T(T, -) \otimes_A T$ be the unit of the adjunction. Then, for every $M \in Mod\text{-}T$ we have

$$\begin{array}{rcl} \eta M & : & M \to M \otimes_A T \\ x & \mapsto & x \otimes_A 1_T. \end{array}$$

We have

$$\eta M(x) = x \otimes_A 1_T = (M \otimes_A \alpha)(x \otimes_A 1_A) = (M \otimes_A \alpha) \circ \left(r_M^A\right)^{-1}(x).$$

Then, for every $M \in Mod\text{-}T$, we have

$$\begin{array}{rcl} \mathrm{Ker}(M \otimes_A \gamma) & = & (M \otimes_A A, M \otimes_A \alpha) \cong \left(M, (M \otimes_A \alpha) \circ \left(r_M^A\right)^{-1}\right) \\ & = & (M, \eta M). \end{array}$$

Hence, $(M, \eta M) = \mathrm{Ker}(M \otimes_A \gamma)$, i.e. η is a regular monomorphism. $\square$

We try here to apply the previous theory to a particular setting in which, first of all, we compute all the ingredients we need to understand the form of the herd.

9.4. Let us consider

- $R =$ associative unital algebra
- $A = R$-ring
- $C = R$-coring
- $\psi : C \otimes_R A \to A \otimes_R C$ a right entwining
- $\widetilde{C} = A \otimes_R C$ the induced A-coring
- $(\Sigma_A, \rho_\Sigma^{\widetilde{C}}) =$ right $\widetilde{C}$-comodule $=$ right entwined module.
- $T = \mathrm{End}^{\widetilde{C}}(\Sigma)$.

Note that if $T \subseteq A$ is a right C-Galois extension i.e. $\left(A_R, \rho_A^C\right)$ is a right C-comodule and the canonical Galois map

$$\begin{gathered}\mathrm{can}_C : A \otimes_T A \to A \otimes_R C \\ t \otimes_T t' \mapsto t\rho_A^C(t') = tt'_0 \otimes_R t'_1\end{gathered}$$

is an isomorphism, then we can consider the right entwining

$$\begin{array}{rcl}\psi & : & C \otimes_R A \to A \otimes_R C \\ c \otimes_R t & \mapsto & \mathrm{can}_C\left(\mathrm{can}_C^{-1}(1_A \otimes_R c)\, t\right)\end{array}$$

and hence the A-coring $A \otimes_R C$, which turns out to be a right Galois coring i.e. A is a right comodule over the A-coring $A \otimes_R C$ via $\rho_A^{\widetilde{C}}$ defined by

$$\begin{gathered}A \cong A \otimes_A A \overset{A \otimes_A \rho_A^C}{\longrightarrow} A \otimes_A A \otimes_R C \cong A \otimes_R C, \\ t \mapsto 1_A \otimes_A t_0 \otimes_R t_1.\end{gathered}$$

The coinvariants of A with respect to this coaction is still T and the canonical Galois map is

$$\begin{gathered}\mathrm{can}_{A \otimes_R C} : A \otimes_T A \to A \otimes_A A \otimes_R C \\ t \otimes_T t' \mapsto t\rho_A^{A \otimes_R C}(t') = t \otimes_A t'_0 \otimes_R t'_1 = 1 \otimes_A tt'_0 \otimes_R t'_1 = 1 \otimes_A \mathrm{can}_C(t \otimes_T t')\end{gathered}$$

so that $\mathrm{can}_{A \otimes_R C}$ is still an isomorphism. Therefore we can consider this case as a particular case of the previous one, where

$$(\Sigma_A, \rho_\Sigma^{\widetilde{C}}) = \left(A_A, \rho_A^{\widetilde{C}}\right).$$

Let A be a right Galois extension of B over the Hopf algebra H. In this case we have

$$\begin{array}{rcl}\mathcal{A} & = & Mod\text{-}R, \\ \mathcal{B} & = & Mod\text{-}B \text{ where } B = A^{co(H)} \\ \mathbb{A} & = & - \otimes_R A : \mathcal{A} = Mod\text{-}R \longrightarrow \mathcal{A} = Mod\text{-}R \\ \mathbb{B} & = & - \otimes_B A : \mathcal{B} = Mod\text{-}B \longrightarrow \mathcal{B} = Mod\text{-}B \\ Q & = & - \otimes_B A : \mathcal{B} = Mod\text{-}B \to \mathcal{A} = Mod\text{-}R \\ P & = & - \otimes_R A_B : \mathcal{A} = Mod\text{-}R \to \mathcal{B} = Mod\text{-}B \\ QP & = & - \otimes_R A \otimes_B A \overset{m_A}{\to} - \otimes_R A \\ PQ & = & - \otimes_B A \otimes_R A \overset{m_A}{\to} - \otimes_B A \\ \mathbb{C} & = & - \otimes_R H : \mathcal{A} = Mod\text{-}R \longrightarrow \mathcal{A} = Mod\text{-}R\end{array}$$

$$l \quad = \quad {}^{\widetilde{C}}\rho_L : L = - \otimes_B A \longrightarrow \widetilde{C}L = - \otimes_B \Sigma \otimes_A A \otimes_R H \cong - \otimes_B A \otimes_R C$$

A second particular case of this situation we have the one where $R = k$ **is a commutative ring,** $C = H$ **is a Hopf algebra over** k **and** A **is a right Galois extension of** $T = A^{co(H)}$.

Now let us set

$$\begin{aligned}
\mathcal{A} &= Mod\text{-}R, \\
\mathcal{B} &= Mod\text{-}T \text{ where } T = \mathrm{End}^{\widetilde{C}}(\Sigma) \\
\mathbb{A} &= - \otimes_R A : \mathcal{A} = Mod\text{-}R \longrightarrow \mathcal{A} = Mod\text{-}R \\
\mathbb{C} &= - \otimes_R C : \mathcal{A} = Mod\text{-}R \longrightarrow \mathcal{A} = Mod\text{-}R \\
\mathbb{B} &= - \otimes_T B : \mathcal{B} = Mod\text{-}T \longrightarrow \mathcal{B} = Mod\text{-}T \text{ where } B = \mathrm{Hom}_A(\Sigma_A, \Sigma_A) \\
\Psi &= - \otimes_R \psi : \mathbb{A}\mathbb{C} = - \otimes_R C \otimes_R A \longrightarrow \mathbb{C}\mathbb{A} = - \otimes_R A \otimes_R C \\
L &= - \otimes_T \Sigma : Mod\text{-}T \longrightarrow Mod\text{-}A \cong {}_{\mathbb{A}}\mathcal{A} \\
W &= \mathrm{Hom}_A(\Sigma, -) : Mod\text{-}A \longrightarrow Mod\text{-}T \\
\widetilde{\mathbb{C}} &= - \otimes_A A \otimes_R C : Mod\text{-}A \longrightarrow Mod\text{-}A \\
l &= {}^{\widetilde{C}}\rho_L : L = - \otimes_T \Sigma \longrightarrow \widetilde{C}L = - \otimes_T \Sigma \otimes_A A \otimes_R C \cong - \otimes_T \Sigma \otimes_R C
\end{aligned}$$

When Σ_A **is f.g.p.**, we set

$${}_A(\Sigma^*)_B = \mathrm{Hom}_A({}_B\Sigma,{}_A A)$$

and we consider the following formal dual structure $\mathbb{M} = (A, B, P, Q, \sigma^A, \sigma^B)$ on the categories $\mathcal{A}$ and $\mathcal{B}$.

- $\mathbb{A} = (- \otimes_R A, - \otimes_R m_A, - \otimes_R u_A)$ is a monad on $\mathcal{A} = Mod\text{-}R$
- $\mathbb{B} = (- \otimes_T B, - \otimes_T m_B, - \otimes_T u_B)$ is a monad on $\mathcal{B} = Mod\text{-}T$ where $T = \mathrm{End}^{\widetilde{C}}(\Sigma)$
- $P = - \otimes_R \Sigma_T^* : Mod\text{-}R \to Mod\text{-}T$
- $Q = {}_{\mathbb{A}}U \circ L = - \otimes_T \Sigma_R : Mod - T \to Mod - R - \otimes_T \Sigma : Mod\text{-}T \to Mod\text{-}R$
- $\sigma^A : QP \to A$ is defined by

$$\begin{aligned}
\sigma^A : QP = - \otimes_R \Sigma^* \otimes_T \Sigma &\to - \otimes_R A \\
- \otimes_R f \otimes_T x &\mapsto - \otimes_R f(x)
\end{aligned}$$

- $\sigma^B : PQ \to B$ is defined by

$$\begin{aligned}
\sigma^B : PQ = - \otimes_T \Sigma_R \otimes_R \Sigma_T^* &\to B = - \otimes_T B \cong - \otimes_T \Sigma_R \otimes_A \Sigma_T^* \\
- \otimes_T y \otimes_R \gamma &\mapsto - \otimes_T y \cdot \gamma() \simeq - \otimes_T y\gamma(x_i) \otimes_A x_i^* \\
&= - \otimes_T y \otimes_A \gamma(x_i) x_i^* = - \otimes_T y \otimes_A \gamma
\end{aligned}$$

- $\left(P : \mathcal{A} \to \mathcal{B}, {}^B\mu_P : BP \to P, \mu_P^A : PA \to P\right)$ is a bimodule functor
- $\left(Q : \mathcal{B} \to \mathcal{A}, {}^A\mu_Q : AQ \to Q, \mu_Q^B : QB \to Q\right)$ is a bimodule functor
- $\sigma^A : QP \to A$ is A-bilinear
- $\sigma^B : PQ \to B$ is B-bilinear

$$\sigma^A \circ \left({}^A\mu_Q P\right) = m_A \circ \left(A\sigma^A\right) \text{ and } \sigma^A \circ \left(Q\mu_P^A\right) = m_A \circ \left(\sigma^A A\right)$$

$$\sigma^B \circ \left({}^B\mu_P Q\right) = m_B \circ \left(B\sigma^B\right) \text{ and } \sigma^B \circ \left(P\mu_Q^B\right) = m_B \circ \left(\sigma^B B\right)$$

and the associative conditions hold

$$ {}^{A}\mu_Q \circ \left(\sigma^A Q\right) = \mu_Q^B \circ \left(Q\sigma^B\right) \text{ and } {}^{B}\mu_P \circ \left(\sigma^B P\right) = \mu_P^A \circ \left(P\sigma^A\right). $$

In fact, we compute

$$ \begin{aligned} \left[\sigma^A \circ \left({}^{A}\mu_Q P\right)\right]\left(-\otimes_R f \otimes_T x \otimes_R a\right) &= \sigma^A\left(-\otimes_R f \otimes_T xa\right) \\ &= -\otimes_R f\left(xa\right) = -\otimes_R f\left(x\right) a \end{aligned} $$

and

$$ \left[m_A \circ \left(A\sigma^A\right)\right]\left(-\otimes_R f \otimes_T x \otimes_R a\right) = m_A\left(-\otimes_R f\left(x\right) \otimes_R a\right) = -\otimes_R f\left(x\right) a $$

so that

$$ \sigma^A \circ \left({}^{A}\mu_Q P\right) = m_A \circ \left(A\sigma^A\right). $$

We compute

$$ \left[\sigma^A \circ \left(Q\mu_P^A\right)\right]\left(-\otimes_R a \otimes_R f \otimes_T x\right) = \sigma^A\left(-\otimes_R af \otimes_T x\right) = -\otimes_R af\left(x\right) $$

and

$$ \left[m_A \circ \left(\sigma^A A\right)\right]\left(-\otimes_R a \otimes_R f \otimes_T x\right) = m_A\left(-\otimes_R a \otimes_R f\left(x\right)\right) = -\otimes_R af\left(x\right) $$

so that we get

$$ \sigma^A \circ \left(Q\mu_P^A\right) = m_A \circ \left(\sigma^A A\right). $$

We compute

$$ \begin{aligned} \left[\sigma^B \circ \left({}^{B}\mu_P Q\right)\right]\left(-\otimes_T x \otimes_R f \otimes_T b\right) &= \sigma^B\left(-\otimes_T x \otimes_R fb\right) \\ &= \sigma^B\left(-\otimes_T x \otimes_R f\left(b\left(\right)\right)\right) = -\otimes_T x \cdot f\left(b\left(\right)\right) \end{aligned} $$

$$ \left[m_B \circ \left(B\sigma^B\right)\right]\left(-\otimes_T x \otimes_R f \otimes_T b\right) = m_B\left(-\otimes_T x \cdot f\left(\right) \otimes_T b\right) = -\otimes_T \left[\left(x \cdot f\left(\right)\right) \circ b\right] $$

Let us compute, for every $y \in \Sigma$ we have

$$ \left[-\otimes_T x \cdot f\left(b\left(\right)\right)\right]\left(y\right) = -\otimes_T xf\left(b\left(y\right)\right) = $$

$$ -\otimes_T \left[\left(x \cdot f\left(\right)\right) \circ b\right]\left(y\right) = -\otimes_T \left(x \cdot f\left(\right)\right)\left(b\left(y\right)\right) = -\otimes_T xf\left(b\left(y\right)\right) $$

so that we obtain

$$ \sigma^B \circ \left({}^{B}\mu_P Q\right) = m_B \circ \left(B\sigma^B\right). $$

Now we compute

$$ \left[\sigma^B \circ \left(P\mu_Q^B\right)\right]\left(-\otimes_T b \otimes_T x \otimes_R f\right) = \sigma^B\left(-\otimes_T b\left(x\right) \otimes_R f\right) = -\otimes_T b\left(x\right) \cdot f\left(\right) $$

and

$$ \left[m_B \circ \left(\sigma^B B\right)\right]\left(-\otimes_T b \otimes_T x \otimes_R f\right) = m_B\left(-\otimes_T b \otimes_T x \cdot f\left(\right)\right) = -\otimes_T \left[b \circ \left(x \cdot f\left(\right)\right)\right] $$

so that, for every $y \in \Sigma$ we have

$$ \left[-\otimes_T b\left(x\right) \cdot f\left(\right)\right]\left(y\right) = -\otimes_T b\left(x\right) f\left(y\right) $$

and

$$ \left(-\otimes_T \left[b \circ \left(x \cdot f\left(\right)\right)\right]\right)\left(y\right) = -\otimes_T b\left(x \cdot f\left(\right)\left(y\right)\right) = -\otimes_T b\left(xf\left(y\right)\right) = -\otimes_T b\left(x\right) f\left(y\right) $$

so that we get

$$ \sigma^B \circ \left(P\mu_Q^B\right) = m_B \circ \left(\sigma^B B\right). $$

Now we have

$$\left[{}^{A}\mu_{Q}\circ\left(\sigma^{A}Q\right)\right]\left(-\otimes_{T}x\otimes_{R}f\otimes_{T}y\right)={}^{A}\mu_{Q}\left(-\otimes_{T}x\otimes_{R}f\left(y\right)\right)=-\otimes_{T}xf\left(y\right)$$

and

$$\begin{aligned}\left[\mu_{Q}^{B}\circ\left(Q\sigma^{B}\right)\right]\left(-\otimes_{T}x\otimes_{R}f\otimes_{T}y\right)&=\mu_{Q}^{B}\left(-\otimes_{T}x\cdot f\left(\right)\otimes_{T}y\right)\\&=-\otimes_{T}x\cdot f\left(\right)\left(y\right)=-\otimes_{T}xf\left(y\right)\end{aligned}$$

so that

$${}^{A}\mu_{Q}\circ\left(\sigma^{A}Q\right)=\mu_{Q}^{B}\circ\left(Q\sigma^{B}\right).$$

Finally we compute

$$\left[{}^{B}\mu_{P}\circ\left(\sigma^{B}P\right)\right]\left(-\otimes_{R}f\otimes_{T}x\otimes_{R}g\right)={}^{B}\mu_{P}\left(-\otimes_{R}f\otimes_{T}x\cdot g\left(\right)\right)=-\otimes_{R}f\left(x\cdot g\left(\right)\right)$$

and

$$\left[\mu_{P}^{A}\circ\left(P\sigma^{A}\right)\right]\left(-\otimes_{R}f\otimes_{T}x\otimes_{R}g\right)=\mu_{P}^{A}\left(-\otimes_{R}f\left(x\right)\otimes_{R}g\right)=-\otimes_{R}f\left(x\right)g\left(\right)$$

so that, for every $y\in\Sigma$ we have

$$\left[-\otimes_{R}f\left(x\cdot g\left(\right)\right)\right]\left(y\right)=-\otimes_{R}f\left(xg\left(y\right)\right)=-\otimes_{R}f\left(x\right)g\left(y\right)$$

and

$$\left[-\otimes_{R}f\left(x\right)g\left(\right)\right]\left(y\right)=-\otimes_{R}f\left(x\right)g\left(y\right)$$

so that we get

$${}^{B}\mu_{P}\circ\left(\sigma^{B}P\right)=\mu_{P}^{A}\circ\left(P\sigma^{A}\right).$$

Note that, in the case ${}_{R}A$ is faithfully flat,by Corollary 9.2,

$$\begin{aligned}\left(\mathcal{A},u_{A}\right)&=\left(Mod\text{-}R,-\otimes_{R}u_{A}\right)=\operatorname{Ker}\left(-\otimes_{R}\gamma\right)\\&=\operatorname{Equ}_{\mathrm{Fun}}\left(-\otimes_{R}u_{A}\otimes_{R}A,-\otimes_{R}u_{A}\otimes_{R}A\right).\end{aligned}$$

Analogously if ${}_{T}B$ is faithfully flat we have
$(\mathcal{B},u_{B})=(Mod\text{-}T,-\otimes_{T}u_{B})=\operatorname{Equ}_{\mathrm{Fun}}\left(-\otimes_{T}u_{B}\otimes_{T}B,-\otimes_{T}u_{B}\otimes_{T}B\right)$.

Thus, in the following we will **assume that both ${}_{R}A$ and ${}_{T}B$ are faithfully flat** so that we have **a regular formal dual structure.**

The counit ϵ of the adjunction (L,W) is given by

$$\begin{aligned}\epsilon_{M}:\operatorname{Hom}_{A}\left(\Sigma,M\right)\otimes_{T}\Sigma&\longrightarrow M\\f\otimes_{T}x&\mapsto f\left(x\right)\end{aligned}$$

for each $M\in Mod\text{-}A$. Therefore we get that

$$\operatorname{can}=\left(\widetilde{\mathbb{C}}\epsilon\right)\circ\left({}^{\widetilde{C}}\rho_{L}W\right):LW=\operatorname{Hom}_{A}\left(\Sigma,-\right)\otimes_{T}\Sigma\longrightarrow\widetilde{C}=-\otimes_{A}A\otimes_{R}C_{R}$$

is defined by

$$\begin{aligned}\operatorname{can}_{M}:\operatorname{Hom}_{A}\left(\Sigma,M\right)\otimes_{T}\Sigma&\longrightarrow M\otimes_{A}A\otimes_{R}C\\\gamma\otimes_{T}x&\mapsto\gamma\left(x_{0}\right)\otimes_{R}x_{1}\end{aligned}$$

for each $M\in Mod\text{-}A$. Hence we deduce that $\left(L,{}^{\widetilde{C}}\rho_{L}\right)$ **is a left $\widetilde{\mathbb{C}}$-Galois functor if and only if $(\Sigma_{A},\rho_{\Sigma}^{\widetilde{C}})$ is a right Galois comodule.**

By Lemma 3.29, we have ${}_AQ = L = -\otimes_T \Sigma : Mod\text{-}T \to Mod\text{-}A$. We have that P_A is a right adjoint of ${}_AQ$, so that, by the uniqueness of the adjoint, we have

$$P_A = W = - \otimes_A \Sigma_T^* : Mod\text{-}A \to Mod\text{-}T.$$

Note that, by Corollary 6.22,, ${}_A\sigma_A^A$: ${}_AQP_A \to {}_{\mathbb{A}}\mathcal{A} = Mod\text{-}A$ is the counit ϵ of the adjunction $({}_AQ, P_A) = (L, W)$, i.e.

$$\begin{gathered}{}_A\sigma_A^A M = \epsilon M : {}_AQP_A M = M \otimes_A \Sigma^* \otimes_T \Sigma_A \to M \\ m \otimes_A f \otimes_T x \mapsto mf(x).\end{gathered}$$

Now, we can consider

$${}_A\mathrm{can}_A = \left(\widetilde{C}_A \sigma_A^A\right) \circ \left({}^{\widetilde{C}}\rho_L W\right) : LW \to \widetilde{C}.$$

For every $M \in Mod\text{-}A$, we have

$$\begin{gathered}{}_A\mathrm{can}_A M = \left(\widetilde{C}_A \sigma_A^A\right) \circ \left({}^{\widetilde{C}}\rho_L WM\right) : M \otimes_A \Sigma^* \otimes_T \Sigma_A \to M \otimes_R C \\ m \otimes_A f \otimes_T x \mapsto mf(x_0) \otimes_R x_1.\end{gathered}$$

Therefore we deduce that ${}_A\mathrm{can}_A = \mathrm{can}$. Recall now that, by Lemma 6.17 ${}_A\mathrm{can}_A$ is an isomorphism if and only if

$$\mathrm{can}_1 := \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) : QP \to CA$$

is an isomorphism. For every $M \in Mod\text{-}R$, we have

$$\begin{gathered}\mathrm{can}_1 M : QPM = M \otimes_R \Sigma_T^* \otimes_T \Sigma_R \longrightarrow CAM = M \otimes_R A \otimes_R C \\ m \otimes_R f \otimes_T x \mapsto \left(C\sigma^A M\right)(m \otimes_R f \otimes_T x_0 \otimes_R x_1) = m \otimes_R f(x_0) \otimes_R x_1.\end{gathered}$$

Assume now that $(\Sigma_A, \rho_\Sigma^{\widetilde{C}})$ **is a right Galois comodule**. Thus we deduce that $\left(L, {}^{\widetilde{C}}\rho_L\right)$ **is a left Galois functor** and thus $\mathrm{can}_1 := \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) : QP \to CA$ is an isomorphism. Therefore, we can consider the composite

$$\tau := \left((\mathrm{can}_1)^{-1} Q\right) \circ (Cu_A Q) \circ {}^C\rho_Q : Q \to QPQ$$

and we can apply Theorem 6.24, that implies that the functorial morphism τ **is a regular herd**. It is defined by

$$\begin{gathered}\tau : M \otimes_T \Sigma_R \longrightarrow M \otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T \Sigma_R \\ m \otimes_T x \mapsto m \otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2\end{gathered}$$

where

$$\begin{aligned} m \otimes_T x_0 \otimes_R x_1^1 \left(\left(x_1^2\right)_0\right) \otimes_T \left(x_1^2\right)_1 &= (\mathrm{can}_1 QM)\left(m \otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2\right) \\ &= \left[(\mathrm{can}_1 QM) \circ \left((\mathrm{can}_1)^{-1} QM\right) \circ (Cu_A QM) \circ \left({}^C\rho_Q M\right)\right](m \otimes_T x) \\ &= \left[(Cu_A QM) \circ \left({}^C\rho_Q M\right)\right](m \otimes_T x) \\ &= m \otimes_T x_0 \otimes_R 1_A \otimes_R x_1.\end{aligned}$$

For every $c \in C$, we denote by

$$- \otimes_R c^1 \otimes_T c^2 = (\mathrm{can}_1)^{-1}(- \otimes_R 1_A \otimes_R c)$$

so that

$$\begin{aligned} -\otimes_R 1_A \otimes_R c &= \left(\mathrm{can}_1 \circ (\mathrm{can}_1)^{-1}\right)(-\otimes_R 1_A \otimes_R c) = \mathrm{can}_1\left(-\otimes_R c^1 \otimes_T c^2\right) \\ &= -\otimes_R c^1\left(\left(c^2\right)_0\right) \otimes_R \left(c^2\right)_1 \end{aligned}$$

i.e.

$$-\otimes_R c^1\left(\left(c^2\right)_0\right) \otimes_R \left(c^2\right)_1 = -\otimes_R 1_A \otimes_R c.$$

Now, starting from a pretorsor, we want to compute the two comonads associated. First of all we compute the comonad $\mathbb{E} = \left(E, \Delta^E, \varepsilon^E\right)$ corresponding to the comonad $\mathbb{C}$ defined in Proposition 6.1. We have

$$(E, i) = \mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l, \omega^r\right)$$

where $\omega^l = \left(QP\sigma^A\right) \circ (\tau P)$ and $\omega^r = QPu_A : QP \to QPA$, i.e.

$$\begin{gathered} \omega^l : QP = -\otimes_R \Sigma_T^* \otimes_T \Sigma_R \to QPA = -\otimes_R A \otimes_R \Sigma_T^* \otimes_T \Sigma_R \\ -\otimes_R f \otimes_T x \mapsto -\otimes_R f \otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \mapsto -\otimes_R f\left(x_0\right) \otimes_R x_1^1 \otimes_T x_1^2 \end{gathered}$$

and

$$\begin{gathered} \omega^r : QP = -\otimes_R \Sigma_T^* \otimes_T \Sigma_R \to QPA = -\otimes_R A \otimes_R \Sigma_T^* \otimes_T \Sigma_R \\ -\otimes_R f \otimes_T x \mapsto -\otimes_R 1_A \otimes_R f \otimes_T x. \end{gathered}$$

We compute

$$\begin{aligned} (\mathrm{can}_1 A) \circ \omega^l &= (\mathrm{can}_1 A) \circ \left(QP\sigma^A\right) \circ (\tau P) = \left(CA\sigma^A\right) \circ (\mathrm{can}_1 QP) \circ (\tau P) \\ &= \left(CA\sigma^A\right) \circ (\mathrm{can}_1 QP) \circ \left((\mathrm{can}_1)^{-1} QP\right) \circ (Cu_A QP) \circ \left({}^C\rho_Q P\right) \\ &= \left(CA\sigma^A\right) \circ (Cu_A QP) \circ \left({}^C\rho_Q P\right) \\ &= (Cu_A A) \circ \left(C\sigma^A\right) \circ \left({}^C\rho_Q P\right) \\ &= (Cu_A A) \circ \mathrm{can}_1 \end{aligned}$$

so that we get

$$(\mathrm{can}_1 A) \circ \omega^l = (Cu_A A) \circ \mathrm{can}_1$$

Moreover

$$(\mathrm{can}_1 A) \circ \omega^r = (\mathrm{can}_1 A) \circ (QPu_A) = (CAu_A) \circ \mathrm{can}_1$$

i.e.

$$(\mathrm{can}_1 A) \circ \omega^r = (CAu_A) \circ \mathrm{can}_1.$$

Assume that the functor $C : \mathcal{A} = Mod\text{-}R \longrightarrow \mathcal{A} = Mod\text{-}R$ **preserves equalizers**. Then we know that

$$(C, (Cu_A)) = \mathrm{Equ}_{\mathrm{Fun}}\left((Cu_A A), (CAu_A)\right)$$

and thus we have

$$\left(C, \mathrm{can}_1^{-1} \circ (Cu_A)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left((Cu_A A) \circ \mathrm{can}_1, (CAu_A) \circ \mathrm{can}_1\right)$$

so that we get

$$\left(C, \mathrm{can}_1^{-1} \circ (Cu_A)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left((\mathrm{can}_1 A) \circ \omega^l, (\mathrm{can}_1 A) \circ \omega^r\right)$$

i.e.

$$\left(C, \mathrm{can}_1^{-1} \circ (Cu_A)\right) = \mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l, \omega^r\right).$$

Note that, in view of our assumptions, $(\mathcal{B}, u_B) = \mathrm{Equ}_{\mathrm{Fun}}(u_B B, B u_B)$ and hence we can apply Proposition 6.2. Now, we compute the comonad $\mathbb{D} = \left(D, \Delta^D, \varepsilon^D\right)$ defined in Proposition 6.2. We have

$$(D, j) = \mathrm{Equ}_{\mathrm{Fun}}\left(\theta^l, \theta^r\right)$$

where $\theta^l = \left(\sigma^B PQ\right) \circ (P\tau)$ and $\theta^r = u_B PQ : PQ \to BPQ = WLPQ$, i.e.

$$\begin{gathered}\theta^l : PQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \to BPQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T B \\ -\otimes_T x \otimes_R f \mapsto -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \otimes_R f \mapsto -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \cdot f\end{gathered}$$

$$\begin{array}{rcl}\theta^l & : & PQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \to WLPQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T \Sigma \otimes_A \Sigma^* \\ -\otimes_T x \otimes_R f & \mapsto & -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \otimes_R f \mapsto -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 f(x_i) \otimes_A x_i^* \\ & = & -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \otimes_A f(x_i)\, x_i^* = -\otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \otimes_A f\end{array}$$

$$\begin{gathered}\theta^r : PQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \to BPQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T B \\ -\otimes_T x \otimes_R f \mapsto -\otimes_T x \otimes_R f \otimes_T 1_B.\end{gathered}$$

$$\begin{gathered}\theta^r : PQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \to BPQ = WLPQ = -\otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T \Sigma \otimes_A \Sigma^* \\ -\otimes_T x \otimes_R f \mapsto -\otimes_T x \otimes_R f \otimes_T 1_B(x_i) \otimes_A x_i^* = -\otimes_T x \otimes_R f \otimes_T x_i \otimes_A x_i^*.\end{gathered}$$

Note that $P_A = W = -\otimes_A \Sigma^*$ and by (15) we have $P_{A\mathbb{A}}F = W_{\mathbb{A}}F = -\otimes_R A \otimes_A \Sigma^* = -\otimes_R \Sigma^* = P$. Let us consider

$$\begin{aligned}{}_A\mathrm{can}_{A\mathbb{A}}F &= \left(\widetilde{C}_A \sigma^A_{A\mathbb{A}} F\right) \circ \left({}^{\widetilde{C}}\rho_L W_{\mathbb{A}} F\right) \\ &= \left(\widetilde{C}_A \sigma^A_{A\mathbb{A}} F\right) \circ \left({}^{\widetilde{C}}\rho_L P_{A\mathbb{A}} F\right) \\ &\overset{(15)}{=} \left(\widetilde{C}_A \sigma^A_{A\mathbb{A}} F\right) \circ \left({}^{\widetilde{C}}\rho_L P\right)\end{aligned}$$

where

$${}_A\mathrm{can}_{A\mathbb{A}}F : LW_{\mathbb{A}}F = LP \longrightarrow \widetilde{C}_{\mathbb{A}}F$$

is thus defined by setting

$$\begin{gathered}{}_A\mathrm{can}_{A\mathbb{A}}F : -\otimes_R A \otimes_A \Sigma^* \otimes_T \Sigma \longrightarrow -\otimes_R A \otimes_A A \otimes_R C \cong -\otimes_R A \otimes_R C \\ -\otimes_R a \otimes_A f \otimes_T x \mapsto -\otimes_R a \otimes_A f(x_0) \otimes_R x_1 \simeq -\otimes_R a f(x_0) \otimes_R x_1\end{gathered}$$

or simply

$$\begin{gathered}{}_A\mathrm{can}_{A\mathbb{A}}F : -\otimes_R A \otimes_A \Sigma^* \otimes_T \Sigma \longrightarrow -\otimes_R A \otimes_A A \otimes_R C \cong -\otimes_R A \otimes_R C \\ -\otimes_R 1_A \otimes_A f \otimes_T x \mapsto -\otimes_R f(x_0) \otimes_R x_1.\end{gathered}$$

We have

$$W\,{}_A\mathrm{can}_{A\mathbb{A}}FQ : WLW_{\mathbb{A}}FQ = WLPQ \to W\widetilde{C}_{\mathbb{A}}FQ$$

i.e.

$$\begin{gathered}W\,{}_A\mathrm{can}_{A\mathbb{A}}FQ : -\otimes_T \Sigma_R \otimes_R A \otimes_A \Sigma_T^* \otimes_T \Sigma \otimes_A \Sigma^* \to -\otimes_T \Sigma_R \otimes_R A \otimes_A A \otimes_R C \otimes_A \Sigma^* \\ -\otimes_T x \otimes_R a \otimes_A f \otimes_T y \otimes_A g \mapsto -\otimes_T x \otimes_R a \otimes_A f(y_0) \otimes_R y_1 \otimes_A g\end{gathered}$$

$$W\,{}_A\mathrm{can}_{A\mathbb{A}}FQ : -\otimes_T \Sigma_R \otimes_R A \otimes_A \Sigma_T^* \otimes_T \Sigma \otimes_A \Sigma^* \to -\otimes_T \Sigma_R \otimes_R A \otimes_A A \otimes_R C \otimes_A \Sigma^*$$

$$- \otimes_T x \otimes_R 1_A \otimes_A f \otimes_T y \otimes_A g \mapsto - \otimes_T x \otimes_R 1_A \otimes_A f(y_0) \otimes_R y_1 \otimes_A g$$

$$\begin{gathered} W_A \mathrm{can}_{A\mathbb{A}} FQ : - \otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T \Sigma \otimes_A \Sigma^* \to - \otimes_T \Sigma_R \otimes_R A \otimes_R C \otimes_A \Sigma^* \\ - \otimes_T x \otimes_R f \otimes_T y \otimes_A g \mapsto - \otimes_T x \otimes_R f(y_0) \otimes_R y_1 \otimes_A g. \end{gathered}$$

If we apply $W_A \mathrm{can}_{A\mathbb{A}} FQ$ both to θ^l and θ^r we get the following

$$\begin{gathered} \left[(W_A \mathrm{can}_{A\mathbb{A}} FQ) \circ \theta^l \right] (- \otimes_T x \otimes_R f) \\ = (W_A \mathrm{can}_{A\mathbb{A}} FQ) \left(- \otimes_T x_0 \otimes_R x_1^1 \otimes_T x_1^2 \otimes_A f \right) \\ = - \otimes_T x_0 \otimes_R x_1^1 \left(\left(x_1^2 \right)_0 \right) \otimes_R \left(x_1^2 \right)_1 \otimes_A f \\ = - \otimes_T x_0 \otimes_R 1_A \otimes_R x_1 \otimes_A f \end{gathered}$$

and

$$\begin{gathered} \left[(W_A \mathrm{can}_{A\mathbb{A}} FQ) \circ \theta^r \right] (- \otimes_T x \otimes_R f) \\ = (W_A \mathrm{can}_{A\mathbb{A}} FQ) (- \otimes_T x \otimes_R f \otimes_T x_i \otimes_A x_i^*) \\ = - \otimes_T x \otimes_R f((x_i)_0) \otimes_R (x_i)_1 \otimes_A x_i^*. \end{gathered}$$

Since ${}_A\mathrm{can}_A$ is an isomorphism, we get that

$$(D, j) = \mathrm{Equ}_{\mathrm{Fun}} \left((W_A \mathrm{can}_{A\mathbb{A}} FQ) \circ \theta^l, (W_A \mathrm{can}_{A\mathbb{A}} FQ) \circ \theta^r \right)$$

Hence $D \subseteq PQ = - \otimes_T \Sigma \otimes_R \Sigma^*$. At this point we stop because it is not so clear what is the comonad D.

We try to compute the functor $\overline{Q}$ which does not require the comonad D, but we cannot do it as well. In fact, we have the following:

We calculate the equalizer

$$\overline{Q} \xrightarrow{q} PC \underset{(\theta^r P) \circ \left(P \mathrm{can}_1^{-1} \right) \circ (PCu_A)}{\overset{(\theta^l P) \circ \left(P \mathrm{can}_1^{-1} \right) \circ (PCu_A)}{\rightrightarrows}} BPQP$$

$$\begin{aligned} \left(\overline{Q}, q \right) &= \mathrm{Equ}_{\mathrm{Fun}} \left(\left(\theta^l P \right) \circ (Pi), (\theta^r P) \circ (Pi) \right) \\ &= \mathrm{Equ}_{\mathrm{Fun}} \left(\left(\theta^l P \right) \circ \left(P \mathrm{can}_1^{-1} \right) \circ (PCu_A), (\theta^r P) \circ \left(P \mathrm{can}_1^{-1} \right) \circ (PCu_A) \right). \end{aligned}$$

We have

$$\begin{gathered} \left(\theta^l P \right) \circ \left(P \mathrm{can}_1^{-1} \right) \circ (PCu_A) : - \otimes_R C \otimes_R \Sigma_T^* \longrightarrow - \otimes_R \Sigma_T^* \otimes_T \Sigma_R \otimes_R \Sigma_T^* \otimes_T B \\ - \otimes_R c \otimes_R f \mapsto - \otimes_R 1_A \otimes_R c \otimes_R f \mapsto - \otimes_R c^1 \otimes_T c^2 \otimes_R f \\ \mapsto - \otimes_R c^1 \otimes_T \left(c^2 \right)_0 \otimes_R \left(c^2 \right)_1^1 \otimes_T \left(c^2 \right)_1^2 \cdot f \\ = - \otimes_R c^1 \otimes_T \left(c^2 \right)_0 \otimes_R \beta^1 \otimes_T \beta^2 \cdot f \end{gathered}$$

where

$$c^1 \left(c_0^2 \right) \otimes_R c_1^2 = 1_A \otimes_R c$$

so that

$$1_A \varepsilon^C (c) = c^1 \left(c_0^2 \varepsilon^C \left(c_1^2 \right) \right) = c^1 \left(c^2 \right).$$

Moreover we have

$$\beta^1 \left(\beta_0^2 \right) \otimes_R \beta_1^2 = 1_A \otimes_R c_1^2.$$

On the other side,

$$\left(\theta^{r}P\right)\circ\left(P\mathrm{can}_{1}^{-1}\right)\circ\left(PCu_{A}\right):-\otimes_{R}C\otimes_{R}\Sigma_{T}^{*}\longrightarrow-\otimes_{R}\Sigma_{T}^{*}\otimes_{T}\Sigma_{R}\otimes_{R}\Sigma_{T}^{*}\otimes_{T}B$$
$$-\otimes_{R}c\otimes_{R}f\mapsto-\otimes_{R}1_{A}\otimes_{R}c\otimes_{R}f\mapsto-\otimes_{R}c^{1}\otimes_{T}c^{2}\otimes_{R}f$$
$$\mapsto-\otimes_{R}c^{1}\otimes_{T}c^{2}\otimes_{R}f\otimes_{T}1_{B}.$$

Maybe

$$\overline{Q}=-\otimes_{R}X$$

where

$$X=\mathrm{Ker}\left(\varphi\right)$$

$$\begin{array}{rcl}\varphi & : & C\otimes_{R}\Sigma_{T}^{*}\rightarrow\Sigma_{R}\otimes_{R}\Sigma_{T}^{*}\otimes_{T}B\\ c\otimes_{R}f & \mapsto & c^{1}\otimes_{T}\left(c^{2}\right)_{0}\otimes_{R}\beta^{1}\otimes_{T}\beta^{2}\cdot f=c^{1}\otimes_{T}c^{2}\otimes_{R}f\otimes_{T}1_{B}\end{array}$$

In case all the computations above make sense, we could write a coherd associated to the pretorsor. By Theorem 7.5, the coherd is defined by

$$\begin{array}{rcl}\chi & : & =\mu_{Q}^{B}\circ\left({}^{A}\mu_{Q}B\right)\circ\left(AQ\sigma^{B}\right)\circ\left(\sigma^{A}QPQ\right)\circ\left(QP\left[\mathrm{can}_{1}^{-1}\circ\left(Cu_{A}\right)\right]Q\right)\circ\left(QqQ\right)\\ & = & \mu_{Q}^{B}\circ\left({}^{A}\mu_{Q}B\right)\circ\left(AQ\sigma^{B}\right)\circ\left(\sigma^{A}QPQ\right)\circ\left(QP\mathrm{can}_{1}^{-1}Q\right)\circ\left(QPCu_{A}Q\right)\circ\left(QqQ\right)\end{array}$$

i.e.

$$\chi:Q\overline{Q}Q\subseteq QPCQ=-\otimes_{T}\Sigma_{R}\otimes_{R}C\otimes_{R}\Sigma_{T}^{*}\otimes_{T}\Sigma_{R}\rightarrow Q=-\otimes_{T}\Sigma_{R}$$
$$-\otimes_{T}x\otimes_{R}c\otimes_{R}f\otimes_{T}y\mapsto-\otimes_{T}x\otimes_{R}1_{A}\otimes_{R}c\otimes_{R}f\otimes_{T}y$$
$$\mapsto-\otimes_{T}x\otimes_{R}c^{1}\otimes_{T}c^{2}\otimes_{R}f\otimes_{T}y\mapsto-\otimes_{T}x\otimes_{R}c^{1}\otimes_{T}c^{2}\otimes_{R}f\left(y\right)$$
$$\mapsto-\otimes_{T}x\cdot c^{1}\otimes_{T}c^{2}\otimes_{R}f\left(y\right)\mapsto-\otimes_{T}x\cdot c^{1}\otimes_{T}c^{2}f\left(y\right)$$
$$\mapsto-\otimes_{T}\left(x\cdot c^{1}\right)\left(c^{2}f\left(y\right)\right)=-\otimes_{T}xc^{1}\left(c^{2}f\left(y\right)\right)=-\otimes_{T}xc^{1}\left(c^{2}\right)f\left(y\right)$$
$$=-\otimes_{T}x\left(1_{A}\varepsilon^{C}\left(c\right)\right)f\left(y\right)$$

where for every $x\in\Sigma_{A}$ and $h\in\Sigma^{*}$

$$x\cdot h\in B=\mathrm{Hom}_{A}\left(\Sigma_{A},\Sigma_{A}\right)\text{ is defined by setting}$$
$$\left(x\cdot h\right)\left(t\right)=xh\left(t\right)\text{ for every }t\in\Sigma.$$

In particular, we have

$$\begin{array}{rcl}x\cdot c^{1} & : & \Sigma_{A}\rightarrow\Sigma_{A}\\ t & \mapsto & xc^{1}\left(t\right).\end{array}$$

9.1. **H-Galois extension.** In order to understand better the situation of the previous example, we decide to consider a very particular case, the Schauenburg setting.

Let H be a Hopf algebra and let A/k be a right H-Galois extension. Let us recall some useful equalities related to the translation map

$$\gamma:=\mathrm{can}^{-1}\left(1_{A}\otimes-\right):H\rightarrow A\otimes A:h\rightarrow\mathrm{can}^{-1}\left(1_{A}\otimes h\right)=:h^{1}\otimes h^{2}.$$

For every $h,l\in H$, $a\in A$, we have

$$h^{1}\left(h^{2}\right)_{0}\otimes\left(h^{2}\right)_{1}=1_{A}\otimes h\tag{206}$$

$$h^{1}\otimes\left(h^{2}\right)_{0}\otimes\left(h^{2}\right)_{1}=\left(h_{1}\right)^{1}\otimes\left(h_{1}\right)^{2}\otimes h_{2}\tag{207}$$

$$
\begin{array}{lrcl}
(208) & \left(h^{1}\right)_{0} \otimes h^{2} \otimes \left(h^{1}\right)_{1} & = & (h_{2})^{1} \otimes (h_{2})^{2} \otimes S(h_{1}) \\
(209) & h^{1}h^{2} & = & \varepsilon^{H}(h)\,1_{A} \\
(210) & (hl)^{1} \otimes (hl)^{2} & = & l^{1}h^{1} \otimes h^{2}l^{2} \\
(211) & a_{0}\,(a_{1})^{1} \otimes (a_{1})^{2} & = & 1_{A} \otimes a.
\end{array}
$$

9.5. The Schauenburg situation is the particular case when $T = A^{co(H)} = k$. Hence we have

$$
\begin{array}{rcl}
\mathcal{A} & = & Mod\text{-}k, \\
\mathcal{B} & = & Mod\text{-}k \text{ where } T = \mathrm{End}^{\widetilde{C}}(\Sigma) = A^{co(H)} = k. \\
\mathbb{A} & = & - \otimes_{k} A : \mathcal{A} = Mod\text{-}k \longrightarrow \mathcal{A} = Mod\text{-}k \\
\mathbb{C} & = & - \otimes_{k} H : \mathcal{A} = Mod\text{-}k \longrightarrow \mathcal{A} = Mod\text{-}k \\
\mathbb{B} & = & - \otimes_{k} A = \mathbb{A} : \mathcal{B} = Mod\text{-}k \longrightarrow \mathcal{B} = Mod\text{-}k \text{ where } A = \mathrm{Hom}_{A}(A_{A}, A_{A}) \\
\Psi & = & - \otimes_{k} \psi : \mathbb{A}\mathbb{C} = - \otimes_{k} H \otimes_{k} A \longrightarrow \mathbb{C}\mathbb{A} = - \otimes_{k} A \otimes_{k} H \\
\psi & : & H \otimes_{k} A \to A \otimes_{k} H, h \otimes_{k} a \mapsto \mathrm{can}_{C}\left(\mathrm{can}_{C}^{-1}(1_{A} \otimes_{k} h)\,a\right) \\
L & = & - \otimes_{k} A : Mod\text{-}k \longrightarrow Mod\text{-}A \cong {}_{\mathbb{A}}\mathcal{A} \\
W & = & \mathrm{Hom}_{A}(A, -) : Mod\text{-}A \longrightarrow Mod\text{-}k \\
\widetilde{\mathbb{C}} & = & - \otimes_{A} A \otimes_{k} H \cong \otimes_{k} H : Mod\text{-}A \longrightarrow Mod\text{-}A \\
l & = & {}^{\widetilde{C}}\rho_{L} : L = - \otimes_{k} A \longrightarrow \widetilde{C}L = - \otimes_{k} A \otimes_{A} A \otimes_{k} H \cong - \otimes_{k} A \otimes_{k} H
\end{array}
$$

Let us assume that

$$
\begin{array}{rcl}
k & = & \text{commutative ring} \\
H & = & \text{Hopf algebra} \\
A/k & = & \text{faithfully flat } H\text{-Galois extension with } \rho_{A}^{H} : A \to A \otimes H \\
A^{co(H)} & = & k1_{A} \\
\mathrm{can} & : & A \otimes A \to A \otimes H \text{ defined by setting } \mathrm{can}(a \otimes b) = ab_{0} \otimes b_{1} \\
\gamma & : & H \to A \otimes A \text{ defined by setting } \gamma(h) := \mathrm{can}^{-1}(1_{A} \otimes h) = h^{1} \otimes h^{2}
\end{array}
$$

Then,

$$
\tau : A \to A \otimes A \otimes A
$$
$$
a \mapsto a_{0} \otimes \gamma(a_{1}) = a_{0} \otimes a_{1}^{1} \otimes a_{1}^{2}
$$

is a pretorsor. We want to construct the comonads C and D as in Theorem 6.5 where $\mathcal{A} = \mathcal{B} = Mod\text{-}k$ and $P = Q = - \otimes A$. First, let us consider $\omega^{l} = - \otimes_{k} \widehat{\omega^{l}}$ and $\omega^{r} = - \otimes_{k} \widehat{\omega^{r}} : - \otimes A \otimes A \to - \otimes A \otimes A \otimes A$ where

$$
\widehat{\omega^{l}} = \left(\sigma^{A} \otimes A \otimes A\right) \circ (A \otimes \tau) : A \otimes A \to A \otimes A \otimes A
$$
$$
\widehat{\omega^{l}}(a \otimes b) = ab_{0} \otimes b_{1}^{1} \otimes b_{1}^{2}
$$
$$
\widehat{\omega^{r}} = (u_{A} \otimes A \otimes A) : A \otimes A \to A \otimes A \otimes A
$$
$$
\widehat{\omega^{r}}(a \otimes b) = 1_{A} \otimes a \otimes b.
$$

Let $\omega = \omega^l - \omega^r$ and $\widehat{\omega} = \widehat{\omega^l} - \widehat{\omega^r}$. First of all we want to prove that for any k-module X we have

$$\mathrm{Ker}\left(\omega X\right) = \mathrm{Ker}\left(X \otimes \widehat{\omega}\right) = X \otimes \mathrm{Ker}\left(\widehat{\omega}\right).$$

Since A is faithfully flat over k we equivalently prove that

$$\mathrm{Ker}\left(X \otimes \widehat{\omega} \otimes A\right) = X \otimes \mathrm{Ker}\left(\widehat{\omega}\right) \otimes A.$$

Note that

$$\begin{tikzcd} A \otimes \mathrm{Ker}\left(\widehat{\omega}\right) \arrow[r] & A \otimes A \otimes A \arrow[r, shift left, "A\otimes\widehat{\omega^l}"] \arrow[r, shift right, "A\otimes\widehat{\omega^r}"'] & A \otimes A \otimes A \otimes A \end{tikzcd}$$

with respect to the map $m_A \otimes A \otimes A$ is a contractible equalizer so that also

$$\begin{tikzcd} \mathrm{Ker}\left(\widehat{\omega}\right) \arrow[r] & A \otimes A \arrow[r, shift left, "\widehat{\omega^l}"] \arrow[r, shift right, "\widehat{\omega^r}"'] & A \otimes A \otimes A \end{tikzcd}$$

is a contractible equalizer (see the dual case of [BW, Proposition 3.4 (c)]). Hence, by Proposition 2.20, it is preserved by any functor. Since $(C, i) = \mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l = - \otimes \widehat{\omega^l}, \omega^r = - \otimes \widehat{\omega^r}\right)$ we have that $C = - \otimes \widehat{C}$ where

$$\widehat{C} = \left\{\sum a^i \otimes b^i \mid \sum \left(a^i\right)\left(b^i\right)_0 \otimes \left(b^i\right)_1^1 \otimes \left(b^i\right)_1^2 = 1_A \otimes \sum a^i \otimes b^i\right\}.$$

Note that γ is a fork for $\widehat{\omega^l}$ and $\widehat{\omega^r}$, in fact

$$\left(\widehat{\omega^l} \circ \gamma\right)(h) = \widehat{\omega^l}\left(h^1 \otimes h^2\right) = h^1 h_0^2 \otimes \left(h_1^2\right)^1 \otimes \left(h_1^2\right)^2 \overset{(206)}{=} 1_A \otimes h^1 \otimes h^2$$

and

$$\left(\widehat{\omega^r} \circ \gamma\right)(h) = \widehat{\omega^r}\left(h^1 \otimes h^2\right) = 1_A \otimes h^1 \otimes h^2.$$

Since $\left(\widehat{C}, \widehat{i}\right) = \mathrm{Equ}\left(\widehat{\omega^l}, \widehat{\omega^r}\right)$, by the universal property of the equalizer, there exists a unique functorial morphism $\varphi : H \to \widehat{C}$ such that $\widehat{i} \circ \varphi = \gamma$. In our case this means that $\mathrm{Im}\gamma \subseteq C$ and hence

$$\begin{array}{rcl} \varphi & : & H \to \widehat{C} \\ h & \mapsto & h^1 \otimes h^2. \end{array}$$

We want to prove that φ is an isomorphism. We compute

$$\left[(A \otimes \varphi) \circ \mathrm{can}\right](a \otimes b) = (A \otimes \varphi)\left(ab_0 \otimes b_1\right) = ab_0 \otimes b_1^1 \otimes b_1^2.$$

Let us set

$$\psi : A \otimes C \to A \otimes A$$
$$a \otimes b \otimes d \mapsto ab \otimes d$$

and let us compute

$$\left[\psi \circ (A \otimes \varphi) \circ \mathrm{can}\right](a \otimes b) = \psi\left(ab_0 \otimes b_1^1 \otimes b_1^2\right) = ab_0 b_1^1 \otimes b_1^2 \overset{(211)}{=} a \otimes b$$

and

$$\left[(A \otimes \varphi) \circ \mathrm{can} \circ \psi\right](a \otimes b \otimes d) = \left[(A \otimes \varphi) \circ \mathrm{can}\right](ab \otimes d)$$

$$= abd_0 \otimes d_1^1 \otimes d_1^2 \overset{b\otimes d\in C}{=} a \otimes b \otimes d.$$

Therefore, $A \otimes \varphi$ is an isomorphism and since A/k is faithfully flat, also φ is an isomorphism, i.e. $\widehat{C} \cong H$.

Now we want to compute the comonad D. We have

$$\theta^l = \left(A \otimes A \otimes \sigma^B\right) \circ (\tau \otimes A) : A \otimes A \to A \otimes A \otimes A$$
$$\theta^l (a \otimes b) = a_0 \otimes a_1^1 \otimes a_1^2 b$$
$$\theta^r = A \otimes A \otimes u_B : A \otimes A \to A \otimes A \otimes A$$
$$\theta^r (a \otimes b) = a \otimes b \otimes 1_A.$$

Since $(D, j) = \mathrm{Equ}_{\mathrm{Fun}} \left(\left(A \otimes A \otimes \sigma^B\right) \circ (\tau \otimes A), A \otimes A \otimes u_B\right)$, we have

$$D = \left\{\sum a^i \otimes b^i \mid \sum \left(a^i\right)_0 \otimes \left(a^i\right)_1^1 \otimes \left(a^i\right)_1^2 b^i = \sum a^i \otimes b^i \otimes 1_A\right\}.$$

By applying $A \otimes \mathrm{can}$ to θ^l and θ^r , for every $\sum a^i \otimes b^i \in D$, we get that

$$\begin{aligned}
\left[(A \otimes \mathrm{can}) \circ \theta^l\right] \left(\sum a^i \otimes b^i\right) &= (A \otimes \mathrm{can}) \left(\sum \left(a^i\right)_0 \otimes \left(a^i\right)_1^1 \otimes \left(a^i\right)_1^2 b^i\right) \\
&= \sum \left(a^i\right)_0 \otimes \left(a^i\right)_1^1 \left(\left(a^i\right)_1^2 b^i\right)_0 \otimes \left(\left(a^i\right)_1^2 b^i\right)_1 \\
&= \sum \left(a^i\right)_0 \otimes \left(a^i\right)_1^1 \left(\left(a^i\right)_1^2\right)_0 \left(b^i\right)_0 \otimes \left(\left(a^i\right)_1^2\right)_1 \left(b^i\right)_1 \\
&\overset{(206)}{=} \sum \left(a^i\right)_0 \otimes \left(b^i\right)_0 \otimes \left(a^i\right)_1 \left(b^i\right)_1 = \rho_{A\otimes A}^H \left(\sum a^i \otimes b^i\right)
\end{aligned}$$

and

$$\left[(A \otimes \mathrm{can}) \circ \theta^r\right] \left(\sum a^i \otimes b^i\right) = (A \otimes \mathrm{can}) \left(\sum a^i \otimes b^i \otimes 1_A\right) = \sum a^i \otimes b^i \otimes 1_H.$$

Since $(A \otimes \mathrm{can})$ is an isomorphism, we get $D = \mathrm{Equ}\left(\rho_{A\otimes A}^H, A \otimes A \otimes u_H\right) = (A \otimes A)^{co(H)}$. By Theorem 6.5, Δ^D and ε^D are uniquely determined by

$$(P\tau) \circ j = (jj) \circ \Delta^D \qquad \text{and} \qquad \sigma^B \circ j = u_B \circ \varepsilon^D.$$

Let $\sum a^i \otimes b^i \in D = (A \otimes A)^{co(H)}$. Then we have

$$\begin{aligned}
(jj) \circ \Delta^D \left(\sum a^i \otimes b^i\right) &= [(\tau \otimes A) \circ j] \left(\sum a^i \otimes b^i\right) = (\tau \otimes A) \left(\sum a^i \otimes b^i\right) \\
&= \sum \left(a^i\right)_0 \otimes \left(a^i\right)_1^1 \otimes \left(a^i\right)_1^2 \otimes b^i
\end{aligned}$$

and also

$$\left(u_A \circ \varepsilon^D\right) \left(\sum a^i \otimes b^i\right) = (m_A \circ j) \left(\sum a^i \otimes b^i\right) = \sum a^i \cdot b^i$$

Since $\sum a^i \otimes b^i \in (A \otimes A)^{co(H)}$, we have

$$\sum \left(a^i\right)_0 \otimes \left(b^i\right)_0 \otimes \left(a^i\right)_1 \left(b^i\right)_1 = \sum a^i \otimes b^i \otimes 1_H$$

so that, by applying $m_A \otimes H$, since A is an H-comodule algebra, we get

$$\sum \left(a^i b^i\right)_0 \otimes \left(a^i b^i\right)_1 = \sum \left(a^i\right)_0 \left(b^i\right)_0 \otimes \left(a^i\right)_1 \left(b^i\right)_1 = \sum a^i \cdot b^i \otimes 1_H$$

i.e. $\sum a^i \cdot b^i \in A^{co(H)} = k1_A \cong k$. Note that from $m_A \circ (A \otimes u_A) \circ (r_A)^{-1} = \mathrm{Id}_A$ we get that $A \otimes u_A$ is a monomorphism and hence, since A is faithfully flat over k, also u_A is a monomorphism. We denote by $\nu = u_A^{|k1_A} : k \to k1_A$ the obvious isomorphism. Thus from

$$\left(u_A \circ \varepsilon^D\right)\left(\sum a^i \otimes b^i\right) = \sum a^i \cdot b^i$$

we get

$$\varepsilon^D\left(\sum a^i \otimes b^i\right) = v^{-1}\left(\sum a^i \cdot b^i\right).$$

Let us compute

$$\begin{aligned} \theta^l P &= A \otimes \theta^l : a \otimes b \otimes c \mapsto a \otimes b_0 \otimes b_1^1 \otimes b_1^2 c \\ \theta^r P &= A \otimes \theta^r : a \otimes b \otimes c \mapsto a \otimes b \otimes c \otimes 1_A \end{aligned}$$

i.e.

$$\begin{aligned} A \otimes \theta^l &: A \otimes A \otimes A \\ Pi &: C \otimes A \to A \otimes A \otimes A \\ h \otimes a &\mapsto h^1 \otimes h^2 \otimes a \end{aligned}$$

so that

$$\begin{aligned} \left(\theta^l P\right) \circ (Pi) &: H \otimes A \to A \otimes A \otimes A \\ h \otimes a &\mapsto h^1 \otimes \left(h^2\right)_0 \otimes \left(h^2\right)_1^1 \otimes \left(h^2\right)_1^2 a = h_1^1 \otimes h_1^2 \otimes h_2^1 \otimes h_2^2 a \\ \left(\theta^r P\right) \circ (Pi) &: H \otimes A \to A \otimes A \otimes A \\ h \otimes a &\mapsto h^1 \otimes h^2 \otimes a \otimes 1_A. \end{aligned}$$

Recall that $H \subseteq C \subseteq A \otimes A$. Such $\overline{Q} = (C \otimes A) \cap [A \otimes D] \cong$ "$(H \otimes A) \cap \left[A \otimes (A \otimes A)^{co(H)}\right]$". Note that, **assuming that** A **preserves equalizers**, for every $h \otimes a \in H \otimes A$, i.e. $h^1 \otimes h^2 \otimes a \in C \otimes A$, $h^1 \otimes h^2 \otimes a \in A \otimes (A \otimes A)^{co(H)}$ if and only if $h^1 \otimes h^2 \otimes a \in \mathrm{Equ}\left(A \otimes \rho^H_{A\otimes A}, A \otimes A \otimes A \otimes u_H\right)$ where

$$\mathrm{Equ}\left(A \otimes \rho^H_{A\otimes A}, A \otimes A \otimes A \otimes u_H\right)$$
$$= \mathrm{Equ}\left((A \otimes A \otimes A \otimes m_H) \circ (A \otimes A \otimes f \otimes H) \circ \left(A \otimes \rho^H_A \otimes \rho^H_A\right), A \otimes A \otimes A \otimes u_H\right)$$
$$= \{a \otimes b \otimes c \mid a \otimes b_0 \otimes c_0 \otimes b_1 c_1 = a \otimes b \otimes c \otimes 1_H\}$$

so that $h^1 \otimes h^2 \otimes a \in A \otimes (A \otimes A)^{co(H)} = \mathrm{Equ}\left(A \otimes \rho^H_{A\otimes A}, A \otimes A \otimes A \otimes u_H\right)$ if and only if

$$h^1 \otimes \left(h^2\right)_0 \otimes a_0 \otimes \left(h^2\right)_1 a_1 = h^1 \otimes h^2 \otimes a \otimes 1_H.$$

Let us prove that $\overline{Q} = (H \otimes A)^{co(H)}$. Now,

$$(H \otimes A)^{co(H)} = \{h \otimes a \mid h_1 \otimes a_0 \otimes h_2 a_1 = h \otimes a \otimes 1_H\}$$

where $h \in H$, $h \overset{\varphi}{\mapsto} h^1 \otimes h^2 \in C$.

1) Let $h \otimes a \in (H \otimes A)^{co(H)}$ and let us prove that $h^1 \otimes h^2 \otimes a \in A \otimes (A \otimes A)^{co(H)}$. Since $h \otimes a \in (H \otimes A)^{co(H)}$, we compute

$$h^1 \otimes \left(h^2\right)_0 \otimes a_0 \otimes \left(h^2\right)_1 a_1 \overset{(207)}{=} (h_1)^1 \otimes (h_1)^2 \otimes a_0 \otimes h_2 a_1 = h^1 \otimes h^2 \otimes a \otimes 1_H$$

so that $h^1 \otimes h^2 \otimes a \in A \otimes (A \otimes A)^{co(H)}$.
2) Now, let $h^1 \otimes h^2 \otimes a \in A \otimes (A \otimes A)^{co(H)}$, i.e. $h^1 \otimes (h^2)_0 \otimes a_0 \otimes (h^2)_1 a_1 = h^1 \otimes h^2 \otimes a \otimes 1_H$ or equivalently $(h_1)^1 \otimes (h_1)^2 \otimes a_0 \otimes h_2 a_1 = h^1 \otimes h^2 \otimes a \otimes 1_H$. By applying to this equality the map $\mathrm{can} \otimes A$ we obtain

$$\begin{aligned} 1_A \otimes h_1 \otimes a_0 \otimes h_2 a_1 &= (\mathrm{can} \otimes A)\left((h_1)^1 \otimes (h_1)^2 \otimes a_0 \otimes h_2 a_1\right) \\ &= (\mathrm{can} \otimes A)\left(h^1 \otimes h^2 \otimes a \otimes 1_H\right) = 1_A \otimes h \otimes a \otimes 1_H \end{aligned}$$

and hence

$$h_1 \otimes 1_A a_0 \otimes h_2 a_1 = h \otimes 1_A a \otimes 1_H$$

so that $h^1 \otimes h^2 \otimes a \in (H \otimes A)^{co(H)}$. Therefore we proved that

$$(\varphi \otimes A)(H \otimes A) \cap \left[A \otimes (A \otimes A)^{co(H)}\right] = (\varphi \otimes A)\left[(H \otimes A)^{co(H)}\right].$$

We can take

$$\left(\overline{Q}, q\right) = \left((H \otimes A)^{co(H)}, (\varphi \otimes A)_{|(H\otimes A)^{co(H)}}\right).$$

By Theorem 7.5, using $i \circ \varphi = \gamma$, we have that

$$\begin{aligned} \chi := m_A \circ (A \otimes m_A) \circ (m_A \otimes A \otimes A) \circ (A \otimes A \otimes A \otimes m_A) \\ \circ (A \otimes i \otimes A \otimes A) \circ \left(A \otimes (\varphi \otimes A)_{|(H\otimes A)^{co(H)}} \otimes A\right) \end{aligned}$$

$$\chi : A \otimes (H \otimes A)^{co(H)} \otimes A \to A$$

$$a \otimes h \otimes b \otimes c \mapsto a \otimes h^1 \otimes h^2 \otimes b \otimes c \mapsto \left(ah^1\right)\left(h^2 (bc)\right) = a\left(h^1 h^2\right)(bc) \overset{(209)}{=} abc\varepsilon^H(h)$$

is a coherd in $\mathbb{X} = \left(\mathbb{H}, (A \otimes A)^{co(H)}, (H \otimes A)^{co(H)}, A, \delta_H, \delta_D\right)$ where $\delta_H : H \to (H \otimes A)^{co(H)} \otimes A$ is uniquely determined by

$$\left[(\varphi \otimes A)_{|(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_H = (H \otimes i) \circ \Delta^H.$$

Since $(C, i) = \mathrm{Equ}_{\mathrm{Fun}}\left(\omega^l, \omega^r\right)$, by the universal property of the equalizer, there exists a unique functorial morphism $\varphi : H \to C$ such that $i \circ \varphi = \gamma$. In our case this means that $\mathrm{Im}\gamma \subseteq C$ and hence

$$\begin{array}{rcl} \varphi & : & H \to C \\ h & \mapsto & h^1 \otimes h^2. \end{array}$$

For every $h \in H$, we have

$$\begin{aligned} \left(\left[(\varphi \otimes A)_{|(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_C\right) &= \left[(C \otimes i) \circ \Delta^C\right] = (C \otimes i) \circ (\varphi \otimes \varphi) \circ \Delta^H \circ \varphi^{-1} \\ &= (\varphi \otimes i \circ \varphi) \circ \Delta^H \circ \varphi^{-1} = (\varphi \otimes \gamma) \circ \Delta^H \circ \varphi^{-1} \end{aligned}$$

and hence

$$\left[(\varphi \otimes A)_{|(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_C \circ \varphi = (\varphi \otimes \gamma) \circ \Delta^H = (\varphi \otimes A \otimes A) \circ (H \otimes \gamma) \circ \Delta^H$$

so that

$$\begin{aligned} &\left(\varphi^{-1} \otimes A \otimes A\right) \circ \left[(\varphi \otimes A)_{|(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_C \circ \varphi \\ &= \left(\varphi^{-1} \otimes A \otimes A\right) \circ (\varphi \otimes A \otimes A) \circ (H \otimes \gamma) \circ \Delta^H \end{aligned}$$

Now

$$(\varphi \otimes A)_{|(H\otimes A)^{co(H)}} = (\varphi \otimes A) \circ i_{(H\otimes A)^{co(H)}}$$

where $i_{(H\otimes A)^{co(H)}} : (H \otimes A)^{co(H)} \to H \otimes A$ is the canonical inclusion and hence we get

$$\begin{aligned}&\left(\varphi^{-1} \otimes A \otimes A\right) \circ [(\varphi \otimes A) \otimes A] \circ \left[i_{(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_C \circ \varphi \\ &= \left(\varphi^{-1} \otimes A \otimes A\right) \circ (\varphi \otimes A \otimes A) \circ (H \otimes \gamma) \circ \Delta^H\end{aligned}$$

i.e.

$$\left[i_{(H\otimes A)^{co(H)}} \otimes A\right] \circ \delta_C \circ \varphi = (H \otimes \gamma) \circ \Delta^H.$$

Now we have

$$\begin{aligned}\left[\left(i_{(H\otimes A)^{co(H)}} \otimes A\right) \circ \delta_C\right] \left(h^1 \otimes h^2\right) &= \left[\left(i_{(H\otimes A)^{co(H)}} \otimes A\right) \circ \delta_C \circ \varphi\right] (h) \\ &= \left((H \otimes \gamma) \circ \Delta^H\right) (h) = h_1 \otimes h_2^1 \otimes h_2^2\end{aligned}$$

i.e.

$$\left(i_{(H\otimes A)^{co(H)}} \otimes A\right) \left(\delta_C \left(h^1 \otimes h^2\right)\right) = h_1 \otimes h_2^1 \otimes h_2^2$$

Let us compute $\delta_D : D \to \overline{Q}Q = \otimes A \otimes (H \otimes A)^{co(H)}$ following Proposition 7.2 which needs to satisfy

$$(\kappa_0' Q) \circ \delta_D = (Dj) \circ \Delta^D.$$

In our case this means

$$(jPQ) \circ (\kappa_0' Q) \circ \delta_D = (jPQ) \circ (Dj) \circ \Delta^D = (jj) \circ \Delta^D = (\tau \otimes A)$$

where

$$\kappa_0' (h \otimes a) = h^1 \otimes h^2 \otimes a$$

and

$$(jj) \circ \Delta^D (a \otimes b) = [(\tau \otimes A) \circ j] (a \otimes b) = (\tau \otimes A) (a \otimes b) = a_0 \otimes a_1^1 \otimes a_1^2 \otimes b.$$

so that

$$\delta_D \left(\sum a^i \otimes b^i\right) = \sum \left(a^i\right)_0 \otimes \left(a^i\right)_1 \otimes b^i.$$

In this more specific situation we could compute both the comonads C and D and the functor $\overline{Q}$ so that we obtained a coherd. We now would like to compute the monads corresponding to the coherd following Theorem 6.29. But the computations are not straightforward and it is not clear what these new monads are.

9.2. **H-Galois coextension.** This is the most clear example of a coherd that we could give. It gives also a description of the dual case of the Morita-Takeuchi equivalence studied by Schauenburg in [Scha4]. In fact we could understand the equivalence between the module categories over the two monads constructed from the coherd.

Let $H = \left(H, m_H, u_H, \Delta^H, \varepsilon^H, S\right)$ be a Hopf algebra and let $L \subseteq H$ be a right coideal subalgebra i.e. $\Delta^H (L) \subseteq L \otimes H$. We can consider $\varepsilon^H : H \to k$ as a character so that

$$J = J_{\varepsilon^H} = \left\langle (hy)_{(1)} \varepsilon^H \left((hy)_{(2)}\right) - h_{(1)} \varepsilon^H \left(h_{(2)} y\right) \mid h \in H, y \in L\right\rangle$$

$$= \left\langle \left(h_{(1)}y_{(1)}\right) \varepsilon^{H}\left(h_{(2)}y_{(2)}\right) - h_{(1)}\varepsilon^{H}\left(h_{(2)}\right)\varepsilon^{H}\left(y\right) \mid h \in H, y \in L \right\rangle$$
$$= \left\langle h_{(1)}y_{(1)}\varepsilon^{H}\left(h_{(2)}\right)\varepsilon^{H}\left(y_{(2)}\right) - h_{(1)}\varepsilon^{H}\left(h_{(2)}\right)\varepsilon^{H}\left(y\right) \mid h \in H, y \in L \right\rangle$$
$$= \left\langle hy - h\varepsilon^{H}\left(y\right) \mid h \in H, y \in L \right\rangle = \left\langle hy' \mid h \in H, y' \in L^{+} \right\rangle = HL^{+}$$

where we denote $L^{+} = L \cap \operatorname{Ker}\left(\varepsilon^{H}\right)$. Let us prove that such J is a coideal of H (see also [BrHaj, Lemma 3.2]). In fact, since $\Delta^{H}\left(y\right) = y_{(1)} \otimes y_{(2)} \in L \otimes H$ we have

$$\Delta^{H}\left(hy - h\varepsilon^{H}\left(y\right)\right) = h_{(1)}y_{(1)} \otimes h_{(2)}y_{(2)} - h_{(1)} \otimes h_{(2)}\varepsilon^{H}\left(y\right)$$
$$= h_{(1)}\left(y_{(1)} - \varepsilon^{H}\left(y_{(1)}\right)\right) \otimes h_{(2)}y_{(2)} + h_{(1)}\varepsilon^{H}\left(y_{(1)}\right) \otimes h_{(2)}y_{(2)} - h_{(1)} \otimes h_{(2)}\varepsilon^{H}\left(y\right)$$
$$= h_{(1)}\left(y_{(1)} - \varepsilon^{H}\left(y_{(1)}\right)\right) \otimes h_{(2)}y_{(2)} + h_{(1)} \otimes h_{(2)}\varepsilon^{H}\left(y_{(1)}\right)y_{(2)} - h_{(1)} \otimes h_{(2)}\varepsilon^{H}\left(y\right)$$
$$= h_{(1)}\left(y_{(1)} - \varepsilon^{H}\left(y_{(1)}\right)\right) \otimes h_{(2)}y_{(2)} + h_{(1)} \otimes h_{(2)}\left(y - \varepsilon^{H}\left(y\right)\right) \in HL^{+} \otimes H + H \otimes HL^{+}$$

and obviously

$$\varepsilon^{H}\left(hy - h\varepsilon^{H}\left(y\right)\right) = \varepsilon^{H}\left(h\right)\varepsilon^{H}\left(y\right) - \varepsilon^{H}\left(h\right)\varepsilon^{H}\left(y\right) = 0.$$

Then we can construct the coalgebra $C := H/J = H/HL^{+}$ and we can consider the canonical projection $\pi : H \to C = H/J$ which is a coalgebra map and a left H-linear map. We can define

$$\begin{array}{rl} {}^{C}\rho_{H} := (\pi \otimes H) \circ \Delta^{H} : H \to C \otimes H & \text{and} \quad \rho_{H}^{C} := (H \otimes \pi) \circ \Delta^{H} : H \to H \otimes C \\ h \mapsto \pi\left(h_{(1)}\right) \otimes h_{(2)} & \qquad h \mapsto h_{(1)} \otimes \pi\left(h_{(2)}\right) \end{array}$$

so that H is a C-bicomodule. Note that ${}^{C}\rho_{H} : H \to C\square_{C}H$ in fact, using that π is a coalgebra map, the coassociativity and the naturality of Δ^{H}, we compute

$$\left(\Delta^{C} \otimes H\right) \circ {}^{C}\rho_{H} = \left(\Delta^{C} \otimes H\right) \circ (\pi \otimes H) \circ \Delta^{H} = (\pi \otimes \pi \otimes H) \circ \left(\Delta^{H} \otimes H\right) \circ \Delta^{H}$$
$$= (\pi \otimes \pi \otimes H) \circ \left(H \otimes \Delta^{H}\right) \circ \Delta^{H} = (C \otimes \pi \otimes H) \circ (\pi \otimes H \otimes H) \circ \left(H \otimes \Delta^{H}\right) \circ \Delta^{H}$$
$$= (C \otimes \pi \otimes H) \circ \left(C \otimes \Delta^{H}\right) \circ (\pi \otimes H) \circ \Delta^{H} = \left(C \otimes {}^{C}\rho_{H}\right) \circ {}^{C}\rho_{H}.$$

Similarly, we also have that $\rho_{H}^{C} : H \to H\square_{C}C$ in fact, using that π is a coalgebra map, the coassociativity and naturality of Δ^{H}, we have

$$\left(H \otimes \Delta^{C}\right) \circ \rho_{H}^{C} = \left(H \otimes \Delta^{C}\right) \circ (H \otimes \pi) \circ \Delta^{H} = \left(H \otimes \Delta^{C} \circ \pi\right) \circ \Delta^{H}$$
$$= \left(H \otimes (\pi \otimes \pi) \circ \Delta^{H}\right) \circ \Delta^{H} = (H \otimes \pi \otimes \pi) \circ \left(H \otimes \Delta^{H}\right) \circ \Delta^{H}$$
$$= (H \otimes \pi \otimes \pi) \circ \left(\Delta^{H} \otimes H\right) \circ \Delta^{H} = (H \otimes \pi \otimes C) \circ (H \otimes H \otimes \pi) \circ \left(\Delta^{H} \otimes H\right) \circ \Delta^{H}$$
$$= (H \otimes \pi \otimes C) \circ \left(\Delta^{H} \otimes C\right) \circ (H \otimes \pi) \circ \Delta^{H} = \left(\rho_{H}^{C} \otimes C\right) \circ \rho_{H}^{C}.$$

Now, the map

$$\begin{array}{rcl} \Delta^{H} & : & H \to H\square_{C}H \\ h & \mapsto & h_{(1)} \otimes h_{(2)} \end{array}$$

is well defined. In fact $h_{(1)_{(1)}} \otimes \pi\left(h_{(1)_{(2)}}\right) \otimes h_{(2)} = h_{(1)} \otimes \pi\left(h_{(2)_{(1)}}\right) \otimes h_{(2)_{(2)}}$. Moreover, the map $\pi : H \to C$ is a counit for H, in fact

$$\begin{array}{rcl} \left[(\pi\square_{C}H) \circ \Delta^{H}\right](h) & = & \pi\left(h_{(1)}\right) \otimes h_{(2)} = {}^{C}\rho_{H}\left(h\right) \simeq h \\ \left[(H\square_{C}\pi) \circ \Delta^{H}\right](h) & = & h_{(1)} \otimes \pi\left(h_{(2)}\right) = \rho_{H}^{C}\left(h\right) \simeq h \end{array}$$

$$\left[\left(\varepsilon^{C}\otimes H\right)\circ(\pi\otimes H)\circ\Delta^{H}\right](h)=\left(\varepsilon^{C}\pi\right)\left(h_{(1)}\right)\otimes h_{(2)}=\varepsilon^{H}\left(h_{(1)}\right)\otimes h_{(2)}\simeq h$$
$$\left[\left(H\otimes\varepsilon^{C}\right)\circ(H\otimes\pi)\circ\Delta^{H}\right](h)=h_{(1)}\otimes\left(\varepsilon^{C}\pi\right)\left(h_{(2)}\right)=h_{(1)}\otimes\varepsilon^{H}\left(h_{(2)}\right)\simeq h$$

so that H is a C-coring. Moreover, H has a right L-module structure

$$\begin{aligned}\mu_{H}^{L} &: H\otimes L\to H\\ h\otimes b &\mapsto m_{H}(h\otimes b)=hb\end{aligned}$$

which is left C-colinear i.e.

$$\pi\left(h_{(1)}b_{(1)}\right)\otimes h_{(2)}b_{(2)}=\pi\left(h_{1}\right)\otimes h_{(2)}b \tag{212}$$

(see [BrHaj, Lemma 3.3]), so that $\left[\left(H\otimes\mu_{H}^{L}\right)\circ\left(\Delta^{H}\otimes L\right)\right](H\otimes L)\subseteq H\square_{C}H$. **Assume that H is a right L-Galois coextension over C**, that is

$$\begin{gathered}\mathrm{cocan}=\left(H\otimes\mu_{H}^{L}\right)\circ\left(\Delta^{H}\otimes L\right):H\otimes L\to H\square_{C}H\\ h\otimes b\mapsto h_{(1)}\otimes h_{(2)}b\end{gathered}$$

is an isomorphism and assume also that H is **flat over** k. In particular, **if H_{L} is faithfully flat**, we know that ${}^{co(C)}H=L$ (see [Schn2, Lemma 1.3 (2)] and [BrWi, 34.2 p. 343]) where we denote

$${}^{co(C)}H=\left\{h\in H\mid{}^{C}\rho_{H}(h)=\pi\left(1_{H}\right)\otimes h\right\}.$$

In this case, we can also define the inverse of the cocanonical map, i.e.

$$\begin{gathered}\mathrm{cocan}^{-1}:H\square_{C}H\to H\otimes L\\ \sum h^{i}\otimes g^{i}\mapsto\sum h_{(1)}^{i}\otimes S\left(h_{(2)}^{i}\right)g^{i}.\end{gathered}$$

For every $\sum h^{i}\otimes g^{i}\in H\square_{C}H$, we have $\sum h_{(1)}^{i}\otimes\pi\left(h_{(2)}^{i}\right)\otimes g^{i}=\sum h^{i}\otimes\pi\left(g_{(1)}^{i}\right)\otimes g_{(2)}^{i}$. By means of the left H-linearity of π and of this equality we have

$$\begin{aligned}&\sum h_{(1)}^{i}\otimes\pi\left(S\left(h_{(3)}^{i}\right)g_{(1)}^{i}\right)\otimes S\left(h_{(2)}^{i}\right)g_{(2)}^{i}=\sum h_{(1)}^{i}\otimes S\left(h_{(3)}^{i}\right)\pi\left(g_{(1)}^{i}\right)\otimes S\left(h_{(2)}^{i}\right)g_{(2)}^{i}\\ &=\sum h_{(1)}^{i}\otimes S\left(h_{(3)}^{i}\right)\pi\left(h_{(4)}^{i}\right)\otimes S\left(h_{(2)}^{i}\right)g^{i}=\sum h_{(1)}^{i}\otimes\pi\left(S\left(h_{(3)}^{i}\right)h_{(4)}^{i}\right)\otimes S\left(h_{(2)}^{i}\right)g^{i}\\ &=\sum h_{(1)}^{i}\otimes\pi\left(1_{H}\right)\otimes S\left(h_{(2)}^{i}\right)g^{i}\end{aligned}$$

so that

$$\sum h_{(1)}^{i}\otimes S\left(h_{(2)}^{i}\right)g^{i}\in\operatorname{Ker}\left(H\otimes\left[{}^{C}\rho_{H}-\pi\left(1_{H}\right)\otimes(-)\right]\right)=H\otimes{}^{co(C)}H=H\otimes L$$

where in the first equality we have used that H is flat over k. Therefore cocan^{-1} is a well-defined map. Note that, by applying $\varepsilon^{H}\otimes L$ to this element, we also deduce that, for every $\sum h^{i}\otimes g^{i}\in H\square_{C}H$, we have

$$\sum S\left(h^{i}\right)g^{i}\in L. \tag{213}$$

Now, let k be a commutative ring, let H be a k-Hopf algebra and let $L\subseteq H$ be a right coideal subalgebra. Assume that H is a right L-Galois coextension over the coalgebra $C=H/HL^{+}$, assume that H_{L} is faithfully flat, so that ${}^{co(C)}H=L$, assume that H_{k} is faithfully flat and assume that

H^C **is faithfully coflat. Assume also** ${}^C H$ **coflat.** Then we can consider the following formal codual structure $\mathbb{X} = (\mathbb{C}, \mathbb{D}, Q, P, \delta_C, \delta_D)$ where

$$\begin{array}{rcl} \mathcal{A} & = & Mod\text{-}k \\ \mathcal{B} & = & Comod\text{-}C \\ \mathbb{C} & = & \left(-\otimes H, -\otimes \Delta^H, -\otimes \varepsilon^H\right) : \mathcal{A} = Mod\text{-}k \longrightarrow \mathcal{A} = Mod\text{-}k \\ \mathbb{D} & = & \left(-\square_C H, -\square_C \Delta^H, -\square_C \varepsilon^{{}^C H^C}\right) : \mathcal{B} = Comod\text{-}C \longrightarrow \mathcal{B} = Comod\text{-}C \\ Q & = & -\square_C H : \mathcal{B} = Comod\text{-}C \to \mathcal{A} = Mod\text{-}k \\ P & = & -\otimes H^C : \mathcal{A} = Mod\text{-}k \to \mathcal{B} = Comod\text{-}C \\ \delta_C & : & = -\otimes \Delta^H : C = -\otimes H \to QP = -\otimes H \square_C H \\ \delta_D & : & = -\square_C \Delta^H : D = -\square_C H \to PQ = -\square_C H \otimes H. \end{array}$$

Now, for every $\sum_i \sum_j k^{i,j} \otimes h^{i,j} \otimes g^i \in (H \otimes H) \square_C H$, we have that

$$\sum_i \sum_j k^{i,j} \otimes h^{i,j}_{(1)} \otimes \pi\left(h^{i,j}_{(2)}\right) \otimes g^i = \sum_i \sum_j k^{i,j} \otimes h^{i,j} \otimes \pi\left(g^i_{(1)}\right) \otimes g^i_{(2)}. \tag{214}$$

We want to prove that $\sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}\right) g^i \in H \otimes L$. We compute, using the left H-linearity of π and (214)

$$\begin{aligned} &\sum_i \sum_j k^{i,j} \otimes \pi\left(S\left(h^{i,j}\right)_{(1)} g^i_{(1)}\right) \otimes S\left(h^{i,j}\right)_{(2)} g^i_{(2)} \\ &= \sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}_{(2)}\right) \pi\left(g^i_{(1)}\right) \otimes S\left(h^{i,j}_{(1)}\right) g^i_{(2)} \\ &= \sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}_{(2)}\right) \pi\left(h^{i,j}_{(3)}\right) \otimes S\left(h^{i,j}_{(1)}\right) g^i \\ &= \sum_i \sum_j k^{i,j} \otimes \pi\left(S\left(h^{i,j}_{(2)}\right) h^{i,j}_{(3)}\right) \otimes S\left(h^{i,j}_{(1)}\right) g^i \\ &= \sum_i \sum_j k^{i,j} \otimes \pi\left(1_H\right) \otimes S\left(h^{i,j}\right) g^i \end{aligned}$$

so that we get

$$\sum_i \sum_j k^{i,j} \otimes \pi\left(S\left(h^{i,j}\right)_{(1)} g^i_{(1)}\right) \otimes S\left(h^{i,j}\right)_{(2)} g^i_{(2)} = \sum_i \sum_j k^{i,j} \otimes \pi\left(1_H\right) \otimes S\left(h^{i,j}\right) g^i$$

which means

$$\sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}\right) g^i \in \operatorname{Ker}\left(H \otimes \left[{}^C\rho_H - \pi\left(1_H\right) \otimes (-)\right]\right) = H \otimes {}^{co(C)} H = H \otimes L$$

i.e.

$$\sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}\right) g^i \in H \otimes L. \tag{215}$$

Similarly, for every $\sum_i \sum_j l^{i,j} \otimes h^{i,j} \otimes g^i \in (L \otimes H) \square_C H$, we have that

$$\sum_i \sum_j l^{i,j} \otimes S\left(h^{i,j}\right) g^i \in \operatorname{Ker}\left(L \otimes \left[{}^C\rho_H - \pi\left(1_H\right) \otimes (-)\right]\right) = L \otimes {}^{co(C)} H = L \otimes L$$

i.e.

$$\sum_i \sum_j l^{i,j} \otimes S\left(h^{i,j}\right) g^i \in L \otimes L \tag{216}$$

so that, since L is a subalgebra of H we get that, for every $\sum_i \sum_j l^{i,j} \otimes h^{i,j} \otimes g^i \in (L \otimes H) \square_C H$,

$$\sum_i \sum_j l^{i,j} S\left(h^{i,j}\right) g^i \in L. \tag{217}$$

Let us consider the following map

$$\begin{gathered}\widehat{\chi} : (H \otimes H) \square_C H \to H \\ \sum_i \sum_j k^{i,j} \otimes h^{i,j} \otimes g^i \mapsto \sum_i \sum_j k^{i,j} S\left(h^{i,j}\right) g^i\end{gathered}$$

which is left C-colinear. In fact, in view of (215), we have that $\sum_i \sum_j k^{i,j} \otimes S\left(h^{i,j}\right) g^i \in H \otimes L$ and by (212), we get that

$$\sum_i \sum_j \pi\left(k^{i,j}_{(1)} \left[S\left(h^{i,j}\right) g^i\right]_{(1)}\right) \otimes k^{i,j}_{(2)} \left[S\left(h^{i,j}\right) g^i\right]_{(2)} = \sum_i \sum_j \pi\left(k^{i,j}_{(1)}\right) \otimes k^{i,j}_{(2)} S\left(h^{i,j}\right) g^i.$$

Therefore, we can define the coherd $\chi = -\square_C \widehat{\chi}$ given by

$$\begin{gathered}\chi : QPQ = -\square_C H \otimes H \square_C H \to Q = -\square_C H \\ -\square_C \sum_i \sum_j k^{i,j} \otimes h^{i,j} \otimes g^i \mapsto -\square_C \sum_i \sum_j k^{i,j} S\left(h^{i,j}\right) g^i.\end{gathered}$$

Let us prove the properties of χ. We have

$$\begin{aligned}&[\chi \circ (QP\chi)]\left(-\square_C k \otimes \sum h^i \otimes g^i \otimes l^j \otimes n^j\right) \\ &= [\chi \circ (\chi \otimes H \square_C H)]\left(-\square_C k \otimes \sum h^i \otimes g^i \otimes l^j \otimes n^j\right) \\ &= \chi\left(-\square_C \sum kS\left(h^i\right) g^i \otimes l^j \otimes n^j\right) = -\square_C \sum \left(kS\left(h^i\right) g^i\right) S\left(l^j\right) n^j\end{aligned}$$

and

$$\begin{aligned}&[\chi \circ (\chi PQ)]\left(-\square_C k \otimes \sum h^i \otimes g^i \otimes l^j \otimes n^j\right) \\ &= [\chi \circ (-\square_C H \otimes H \square_C \chi)]\left(k \otimes \sum h^i \otimes g^i \otimes l^j \otimes n^j\right) \\ &= \chi\left(k \otimes \sum h^i \otimes g^i S\left(l^j\right) n^j\right) = -\square_C \sum kS\left(h^i\right)\left(g^i S\left(l^j\right) n^j\right)\end{aligned}$$

so that χ is coassociative. Moreover, we have

$$\begin{aligned}&[\chi \circ (\delta_C Q)](k \otimes h) = [\chi \circ (-\square_C H \otimes \delta_C)](-\square_C k \otimes h) = \chi\left(-\square_C k \otimes h_{(1)} \otimes h_{(2)}\right) \\ &= -\square_C k S\left(h_{(1)}\right) h_{(2)} = k\varepsilon^H(h) = \left(-\square_C H \otimes \varepsilon^H\right)(k \otimes h) = \left(\varepsilon^C Q\right)(k \otimes h)\end{aligned}$$

and

$$\begin{aligned}&[\chi \circ (Q\delta_D)]\left(-\square_C \sum k^i \otimes h^i\right) = [\chi \circ (\delta_D \square_C H)]\left(-\square_C \sum k^i \otimes h^i\right) \\ &= \chi\left(-\square_C \sum k^i_{(1)} \otimes k^i_{(2)} \otimes h^i\right) = -\square_C \sum k^i_{(1)} S\left(k^i_{(2)}\right) h^i \\ &= \sum \varepsilon^H\left(k^i\right) 1_H h^i = \sum \varepsilon^H\left(k^i\right) h^i = -\square_C \sum \left(\varepsilon^C \circ \pi\right)\left(k^i\right) h^i \\ &= -\square_C \sum \varepsilon^C\left(\pi\left(k^i\right)\right) h^i \simeq -\square_C \sum \pi\left(k^i\right) \square_C h^i \\ &= -\square_C \sum \varepsilon^{C^{H^C}}\left(k^i\right) \square_C h^i = \left(\varepsilon^{C^{H^C}} \square_C H\right)\left(-\square_C \sum k^i \otimes h^i\right)\end{aligned}$$

$$= \left(Q\varepsilon^{C_{H^C}}\right)\left(-\square_C \sum k^i \otimes h^i\right)$$

so that the counitality conditions are also satisfied, i.e. χ is really a coherd. Since H_k is faithfully flat, we have that $\left(k, \varepsilon^H\right) = \mathrm{Coequ}_{\textit{Mod-k}}\left(H \otimes \varepsilon^H, \varepsilon^H \otimes H\right)$ and since H^C is faithfully coflat, by [Schn1, Proposition 1.1], we also have that

$$\begin{aligned}(C, \pi) &= \left(C, \varepsilon^{C_{H^C}}\right) = \mathrm{Coequ}_{\textit{Comod-C}}\left(H\square_C\varepsilon^{C_{H^C}}, \varepsilon^{C_{H^C}}\square_C H\right)\\ &= \mathrm{Coequ}_{\textit{Comod-C}}\left(H\square_C\pi, \pi\square_C H\right)\end{aligned}$$

so that $\mathbb{X}$ is a regular formal codual structure and thus χ is a regular coherd. Following Theorem 6.29, we calculate the monad

$$(A, x) = \mathrm{Coequ}_{\mathrm{Fun}}\left(w^l, w^r\right)$$

where $w^l = (\chi P) \circ (QP\delta_C)$ and $w^r = QP\varepsilon^C : QPC \to QP$. In our case

$$w^l : - \otimes H \otimes H\square_C H \to - \otimes H\square_C H$$
$$- \otimes \sum\nolimits_i \sum\nolimits_j k^{i,j} \otimes \sum h^{i,j} \otimes g^i \mapsto - \otimes \sum\nolimits_i \sum\nolimits_j k^{i,j}_{(1)} \otimes k^{i,j}_{(2)} S\left(h^i, j\right) g^i$$

and

$$w^r : - \otimes H \otimes H\square_C H \to - \otimes H\square_C H$$
$$- \otimes \sum\nolimits_i \sum\nolimits_j k^{i,j} \otimes \sum h^{i,j} \otimes g^i \mapsto - \otimes \sum\nolimits_i \sum\nolimits_j \varepsilon^H\left(k^{i,j}\right) h^{i,j} \otimes g^i.$$

Assume now that k is a field, so that everything is flat over k. Hence, for every $X \in \textit{Mod-k}$

$$\begin{aligned}AX &= \frac{X \otimes H\square_C H}{\mathrm{Im}\left(X \otimes w^l - X \otimes w^r\right)} = \frac{X \otimes H\square_C H}{\mathrm{Im}\left(X \otimes \left(w^l - w^r\right)\right)}\\ &= \frac{X \otimes H\square_C H}{X \otimes \mathrm{Im}\left(w^l - w^r\right)} = X \otimes \frac{H\square_C H}{\mathrm{Im}\left(w^l - w^r\right)}\end{aligned}$$

and thus

$$A = - \otimes \frac{H\square_C H}{I_w}$$

where $I_w = \left\langle \sum_i \sum_j k^{i,j}_{(1)} \otimes k^{i,j}_{(2)} S\left(h^i, j\right) g^i - \sum_i \sum_j \varepsilon^H\left(k^{i,j}\right) h^{i,j} \otimes g^i \right\rangle$. In the sequel, given elements $\sum_i h^i \otimes g^i \in H\square_C H$, we will use the notation

$$\left[\sum\nolimits_i h^i \otimes g^i\right]_A = \sum\nolimits_i h^i \otimes g^i + I_w.$$

We will prove that this new monad A on the category $\textit{Mod-k}$ is isomorphic to the monad coming from the algebra L. Consider the following map

$$\varphi : \frac{H\square_C H}{I_w} \longrightarrow L$$
$$\left[\sum\nolimits_i h^i \otimes g^i\right]_A \mapsto \sum\nolimits_i S\left(h^i\right) g^i$$

which is well-defined by (213), i.e. $\sum S\left(h^i\right) g^i \in L$. Note that, since $L = {}^{co(C)}H$, for every $b \in L$, we have

$$1_H \otimes \pi\left(1_H\right) \otimes b = 1_H \otimes \sum \pi\left(b_1\right) \otimes b_2.$$

The inverse of this map is given by

$$\begin{gathered}\varphi^{-1} : L \longrightarrow \frac{H \square_C H}{I_w} \\ b \mapsto \left[1_H \otimes b\right]_A.\end{gathered}$$

In fact we have $\left(\varphi^{-1} \circ \varphi\right)\left(\left[\sum_i h^i \otimes g^i\right]_A\right) = \varphi^{-1}\left(\sum_i S\left(h^i\right) g^i\right) = \left[1_H \otimes \sum_i S\left(h^i\right) g^i\right]_A = \left[\sum_i h^i \otimes g^i\right]_A$ by definition of I_w and $\left(\varphi \circ \varphi^{-1}\right)(b) = \varphi\left(\left[1_H \otimes b\right]_A\right) = S\left(1_H\right) b = b$ and thus φ is bijective so that

$$A = - \otimes \frac{H \square_C H}{I_w} \simeq - \otimes L : Mod\text{-}k \to Mod\text{-}k.$$

The functorial morphisms m_A and u_A of the monad A are uniquely determined by

$$x \circ (\chi P) = m_A \circ (xx) \qquad \text{and} \qquad x \circ \delta_C = u_A \circ \varepsilon^C$$

where $x : H \square_C H \to \frac{H \square_C H}{I_w}$ denotes the canonical projection. In our case we have

$$\begin{aligned}
m_{\frac{H \square_C H}{I_w}}\left(\left[\sum\nolimits_i h^i \otimes g^i\right]_A \otimes \left[\sum\nolimits_j k^j \otimes l^j\right]_A\right) &= \left[\sum\nolimits_{i,j} h^i \otimes g^i S\left(k^j\right) l^j\right]_A \\
= \left[\sum\nolimits_{i,j} 1_H \otimes S\left(h^i\right) g^i S\left(k^j\right) l^j\right]_A &= \varphi^{-1}\left(\sum\nolimits_{i,j} S\left(h^i\right) g^i S\left(k^j\right) l^j\right) \\
&= \varphi^{-1}\left(m_L\left(\sum\nolimits_i S\left(h^i\right) g^i \otimes \sum\nolimits_j S\left(k^j\right) l^j\right)\right) \\
&= \left(\varphi^{-1} \circ m_L \circ (\varphi \otimes \varphi)\right)\left(\left[\sum\nolimits_i h^i \otimes g^i\right]_A \otimes \left[\sum\nolimits_j k^j \otimes l^j\right]_A\right)
\end{aligned}$$

from which we deduce that

$$\varphi \circ m_{\frac{H \square_C H}{I_w}} = m_L \circ (\varphi \otimes \varphi).$$

Moreover

$$u_{\frac{H \square_C H}{I_w}}\left(\varepsilon^H(h)\right) = \left(x \circ \delta_C\right)(h) = \left[h_{(1)} \otimes h_{(2)}\right]_A$$

so that

$$\begin{aligned}
u_{\frac{H \square_C H}{I_w}}\left(1_k\right) &= u_{\frac{H \square_C H}{I_w}}\left(\varepsilon^H\left(1_H\right)\right) = \left(x \circ \delta_C\right)\left(1_H\right) = \left[1_{H(1)} \otimes 1_{H(2)}\right]_A = \left[1_H \otimes 1_H\right]_A \\
&= \left[1_H \otimes 1_L\right]_A = \varphi^{-1}\left(1_L\right) = \left(\varphi^{-1} \circ u_L\right)\left(1_k\right)
\end{aligned}$$

from which we deduce that

$$\varphi \circ u_{\frac{H \square_C H}{I_w}} = u_L.$$

The two relations obtained say that $\varphi : \frac{H \square_C H}{I_w} \longrightarrow L$ is an algebra isomorphism so that

$$A = - \otimes \frac{H \square_C H}{I_w} \simeq - \otimes L \text{ as monads.}$$

Following Theorem 6.29, we now calculate the monad

$$(E,y)=\mathrm{Coequ}_{\mathrm{Fun}}\left(z^l,z^r\right)$$

where $z^l=(P\chi)\circ(\delta_D PQ)$ and $z^r=\varepsilon^D PQ:DPQ\to PQ$. In our case, let us consider

$$\widehat{z^l}:(H\otimes H)\,\square_C H\longrightarrow H\otimes H$$

$$\sum_i\sum_j k^{i,j}\otimes h^{i,j}\otimes g^i\mapsto\sum_i\sum_j k^{i,j}S\left(h^{i,j}\right)g^i_{(1)}\otimes g^i_{(2)}$$

and let us prove that $\widehat{z^l}$ it is left C-colinear. By (215) we have that $\sum_i\sum_j k^{i,j}\otimes S\left(h^{i,j}\right)g^i\in H\otimes L$ so that, in view of (212), we have

$$\begin{aligned}&\sum_i\sum_j\pi\left(k^{i,j}_{(1)}\left[S\left(h^{i,j}\right)g^i_{(1)}\right]_{(1)}\right)\otimes k^{i,j}_{(2)}\left[S\left(h^{i,j}\right)g^i_{(1)}\right]_{(2)}\otimes g^i_{(3)}\\&=\sum_i\sum_j\pi\left(k^{i,j}_{(1)}\right)\otimes k^{i,j}_{(2)}S\left(h^{i,j}\right)g^i_{(1)}\otimes g^i_{(2)}.\end{aligned}$$

Hence

$$z^l:-\square_C H\otimes H\square_C H\longrightarrow-\square_C H\otimes H$$

$$-\square_C\sum_i\sum_j k^{i,j}\otimes h^{i,j}\otimes g^i\mapsto-\square_C\sum_i\sum_j k^{i,j}S\left(h^{i,j}\right)g^i_{(1)}\otimes g^i_{(2)}$$

and

$$z^r:-\square_C H\otimes H\square_C H\longrightarrow-\square_C H\otimes H$$

$$-\square_C\sum_i\sum_j k^{i,j}\otimes h^{i,j}\otimes g^i\mapsto-\square_C\sum_i\sum_j k^{i,j}\otimes h^{i,j}\varepsilon^H\left(g^i\right)$$

are well-defined. For every $\left(X,\rho^C_X\right)\in\mathit{Comod}\text{-}C$ we have

$$E\left(X,\rho^C_X\right)=\frac{X\square_C H\otimes H}{\mathrm{Im}\left(X\square_C z^l-X\square_C z^r\right)}=\frac{X\square_C H\otimes H}{\mathrm{Im}\left(X\square_C\left(z^l-z^r\right)\right)}$$

so that

$$E\left(X,\rho^C_X\right)=\frac{X\square_C H\otimes H}{I_{X\square_C z}}$$

where

$$I_{X\square_C z}=\left\langle\begin{array}{c}\sum_{i,j}x^j\otimes k^jS\left(h^i\right)g^i_{(1)}\otimes g^i_{(2)}-\sum_{i,j}x^j\otimes k^j\otimes h^i\varepsilon^H\left(g^i\right)\\ \mid\sum_j x^j\otimes k^j,\sum_i h^i\otimes g^i\in H\square_C H\end{array}\right\rangle.$$

Recall (see [BrHaj, Theorem 3.5]) that, associated to the cocanonical map, we have a unique canonical entwining structure given by

$$\psi=(\widehat{\tau}\otimes H)\circ\left(H\otimes\Delta^H\right)\circ\mathrm{cocan}:H\otimes L\longrightarrow L\otimes H$$

$$h\otimes y\mapsto y_{(1)}\otimes hy_{(2)}$$

where $\widehat{\tau}=\left(\varepsilon^H\otimes L\right)\circ\mathrm{cocan}^{-1}:H\square_C H\longrightarrow L$ is the cotranslation map. Since cocan is an isomorphism, in order to understand better the monad E we first compose with the isomorphism $H\otimes\mathrm{cocan}$ and we compute for every $h,g\in H$, $y\in L$, $\left(z^l\circ(H\otimes\mathrm{cocan})\right)(h\otimes g\otimes y)=hy_{(1)}\otimes gy_{(2)}$ and $\left(z^r\circ(H\otimes\mathrm{cocan})\right)(h\otimes g\otimes y)=h\otimes g\varepsilon^H(y)$. Let

$$i:(H\otimes H)\,L^+\to(H\otimes H)$$

denote the canonical inclusion. Then i is a left C-comodule map, in fact, for every $\sum_i (h_i \otimes g_i)\left(l_i - \varepsilon^H(l_i)\right) = \sum_i h_i l_{i(1)} \otimes g_i l_{i(2)} - h_i \otimes g_i \varepsilon^H(l_i) \in (H \otimes H) L^+$, since $\Delta^H(l_i) = l_{i(1)} \otimes l_{i(2)} \in L \otimes H$, we have

$$\begin{aligned}
&\sum_i \pi\left(h_{i(1)} l_{i(1)}\right) \otimes h_{i(2)} l_{i(2)} \otimes g_i l_{i(3)} - \pi\left(h_{i(1)}\right) \otimes h_{i(2)} \otimes g_i \varepsilon^H(l_i) \\
&= \sum_i \pi\left(h_{i(1)}\right) \otimes h_{i(2)} l_{i(1)} \otimes g_i l_{i(2)} - \pi\left(h_{i(1)}\right) \otimes h_{i(2)} \otimes g_i \varepsilon^H(l_i) \\
&= \sum_i \pi\left(h_{i(1)}\right) \otimes \left(h_{i(2)} \otimes g_i\right)\left(l_i - \varepsilon^H(l_i)\right) \in C \otimes (H \otimes H) L^+.
\end{aligned}$$

Hence, for every $\left(X, \rho_X^C\right) \in$ *Comod-C*, we can consider the map

$$X \square_C i : X \square_C (H \otimes H) L^+ \to X \square_C (H \otimes H).$$

so that, for every $\left(X, \rho_X^C\right) \in$ *Comod-C*, we have

$$E\left(X, \rho_X^C\right) = \frac{X \square_C H \otimes H}{I_{X \square_C z}} = \frac{X \square_C H \otimes H}{I_{X \square_C L}}$$

where

$$\begin{aligned}
I_{X \square_C L} &= \left\langle \begin{array}{c} \sum_i x^i \otimes h^i y_{(1)} \otimes g y_{(2)} - \sum_i x^i \otimes h^i \otimes g \varepsilon^H(y) \\ \mid \sum_i x^i \otimes h^i \otimes g \otimes y \in X \square_C H \otimes H \otimes L \end{array} \right\rangle \\
&= X \square_C \left[(H \otimes H) L^+\right].
\end{aligned}$$

Let $p : H \otimes H \to \frac{H \otimes H}{(H \otimes H) L^+}$ be the canonical projection and let us assume that i is **left C-copure** i.e. for every $\left(X, \rho_X^C\right) \in$ *Comod-C*, the sequence

$$0 \to X \square_C (H \otimes H) L^+ \xrightarrow{X \square_C i} X \square_C (H \otimes H) \xrightarrow{X \square_C p} X \square_C \frac{H \otimes H}{(H \otimes H) L^+} \to 0$$

is exact. In this case we get that, for every $\left(X, \rho_X^C\right) \in$ *Comod-C*,

$$E\left(X, \rho_X^C\right) \cong X \square_C \frac{H \otimes H}{(H \otimes H) L^+} = X \square_C (H \otimes H)_L$$

where $(H \otimes H)_L$ denotes the invariants with respect to the algebra L. In the sequel, given $h^i, k^i \in H$ we will use the notation

$$\left[\sum_i h^i \otimes k^i\right]_E = \sum_i h^i \otimes k^i + (H \otimes H) L^+$$

Let us denote $E := - \square_C (H \otimes H)_L$ and let us consider multiplication and unit of E. Following Theorem 6.29, they are uniquely determined by

$$m_E \circ (yy) = y \circ (P\chi) \qquad \text{and} \qquad y \circ \delta_D = u_E \circ \varepsilon^D$$

i.e.

$$m_E = - \square_C \widehat{m_E} \qquad \text{and} \qquad u_E = - \square_C \widehat{u_E}$$

where

$$\widehat{m_E} : \frac{H \otimes H}{(H \otimes H) L^+} \square_C \frac{H \otimes H}{(H \otimes H) L^+} \longrightarrow \frac{H \otimes H}{(H \otimes H) L^+} \text{ and } \widehat{u_E} : C \longrightarrow \frac{H \otimes H}{(H \otimes H) L^+}$$

given by

$$\widehat{m_E}\left(\sum\sum\sum \left[k^{i,j} \otimes h^{i,j}\right]_E \square_C \left[g^{i,s} \otimes l^{i,s}\right]_E\right) = \left[\sum k^{i,j} S\left(h^{i,j}\right) g^{i,s} \otimes l^{i,s}\right]_E$$

$$\widehat{u_E}\left(\varepsilon^{C^{H^C}}(h)\right) = \widehat{u_E}(\pi(h)) = \left[h_{(1)} \otimes h_{(2)}\right]_E.$$

Let us check that $\widehat{m_E}$ is a well-defined map. Let us consider

$$\begin{aligned}\overline{f} : H \otimes H &\to H \\ h \otimes k &\mapsto hS(k).\end{aligned}$$

For every $(h \otimes k) \cdot l \in (H \otimes H) L^+$, we have

$$\overline{f}\left[\left(hl_{(1)} \otimes kl_{(2)}\right) - (h \otimes k)\varepsilon(l)\right] = hl_{(1)}S\left(l_{(2)}\right)S(k) - hS(k)\varepsilon(l) = 0$$

so that $\overline{f}$ induces a morphism

$$\begin{aligned}f : \frac{H \otimes H}{(H \otimes H) L^+} &\to H \\ [h \otimes k] &\mapsto hS(k).\end{aligned}$$

Now, let us consider the composite

$$\frac{H \otimes H}{(H \otimes H) L^+} \otimes H \otimes H \overset{f \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H}{\longrightarrow} H \otimes H \otimes H \overset{m_H \otimes \mathrm{Id}_H}{\longrightarrow} H \otimes H \overset{p}{\longrightarrow} \frac{H \otimes H}{(H \otimes H) L^+}$$

where p denotes the canonical projection. Note that

$$[p \circ (m_H \otimes H)]\left(H \otimes (H \otimes H) L^+\right) = 0$$

in fact, for every $x \in H$ and $(h \otimes k) \cdot l \in (H \otimes H) L^+$ we have

$$x \otimes \left[\left(hl_{(1)} \otimes kl_{(2)}\right) - (h \otimes k\varepsilon(l))\right] = x \otimes \left(hl_{(1)} \otimes kl_{(2)}\right) - x \otimes (h \otimes k\varepsilon(l))$$

and thus

$$xhl_{(1)} \otimes kl_{(2)} - xh \otimes k\varepsilon(l) = (xh \otimes k)(l - \varepsilon(l)) \in (H \otimes H) L^+.$$

Therefore, the above composite map induces the map

$$\begin{array}{rcl}\dfrac{H \otimes H}{(H \otimes H) L^+} \otimes \dfrac{H \otimes H}{(H \otimes H) L^+} & \to & \dfrac{H \otimes H}{(H \otimes H) L^+} \\ [k \otimes h] \otimes [g \otimes l] & \mapsto & [kS(h) g \otimes l]\end{array}$$

which is well-defined and hence also the map

$$\begin{aligned}\widehat{m_E} : \frac{H \otimes H}{(H \otimes H) L^+} \square_C \frac{H \otimes H}{(H \otimes H) L^+} &\to \frac{H \otimes H}{(H \otimes H) L^+} \\ \sum \left[k \otimes h^i\right]_E \otimes \left[g^i \otimes l\right]_E &\mapsto \left[\sum kS\left(h^i\right) g^i \otimes l\right]_E.\end{aligned}$$

is well defined. Observe that, by using (213) and (212) we have

$$\sum \pi\left(k_{(1)}\left(S\left(h^i\right) g^i\right)_{(1)}\right) \otimes k_{(2)}\left(S\left(h^i\right) g^i\right)_{(2)} = \sum \pi\left(k_{(1)}\right) \otimes k_{(2)}\left(S\left(h^i\right) g^i\right)$$

so that the maps

$$\sum \left[k \otimes h^i\right]_E \otimes \left[g^i \otimes l\right]_E \mapsto \left[\sum kS\left(h^i\right) g^i \otimes l\right]_E,$$

$$\sum k \otimes \left[h^i \otimes g^i\right]_A \mapsto \sum kS\left(h^i\right) g^i$$

and

$$\sum \left[k \otimes h^i\right]_E \otimes g^i \mapsto \sum kS\left(h^i\right) g^i$$

are left C-colinear and hence $\widehat{m_E}$ is also left colinear. Therefore the map

$$m_E = -\square_C \widehat{m_E} : -\square_C \frac{H\otimes H}{(H\otimes H)\,L^+} \square_C \frac{H\otimes H}{(H\otimes H)\,L^+} \to \frac{H\otimes H}{(H\otimes H)\,L^+}$$

is well-defined. Moreover, $Q = -\square_C H$ can be equipped with the structure of an $\mathbb{A}$-$\mathbb{B}$-bimodule functor, i.e. in our setting, with a well-defined structure of L-E-bimodule functor given by ${}^A\mu_Q = -\square_C \widehat{{}^A\mu_Q}$ and $\mu_Q^E = -\square_C \widehat{\mu_Q^E}$ where

$$\begin{gathered} \widehat{{}^A\mu_Q} : H\otimes L \longrightarrow H \\ k\otimes l \mapsto kl \end{gathered}$$

$$\begin{gathered} \widehat{\mu_Q^E} : \frac{H\otimes H}{(H\otimes H)\,L^+} \square_C H \longrightarrow H \\ \sum\nolimits_i \sum\nolimits_j \left[k^{i,j}\otimes h^{i,j}\right]_E \otimes g^i \mapsto \sum\nolimits_i \sum\nolimits_j k^{i,j} S\left(h^{i,j}\right) g^i . \end{gathered}$$

Similarly one can prove that ${}^A\mu_Q$ and μ_Q^E are well-defined. Let us calculate the coequalizer $\left(\widehat{Q}, l\right) = \mathrm{Coequ}_{\mathrm{Fun}}\left((Px)\circ\left(z^l P\right), (Px)\circ(z^r P)\right)$ defined in Proposition 7.6

$$\begin{aligned} \widehat{Q} &= \frac{-\otimes A\otimes H}{-\otimes \mathrm{Im}\left((x\otimes H)\circ(H\square_C z^l) - (x\otimes H)\circ(H\square_C z^r)\right)} \\ &= \frac{-\otimes \frac{H\square_C H}{I_w}\otimes H}{-\otimes \mathrm{Im}\left((x\otimes H)\circ(H\square_C z^l) - (x\otimes H)\circ(H\square_C z^r)\right)}. \end{aligned}$$

Since we are in the case when cocan is an isomorphism, we equivalently calculate, for every $\sum h^i\otimes g^i \in H\square_C H$, $k\in H$ and $t\in L$,

$$\begin{gathered} \left((x\otimes H)\circ\left(H\square_C z^l\right)\circ(H\square_C H\otimes \mathrm{cocan})\right)\left(\sum h^i\otimes g^i\otimes k\otimes t\right) \\ = \left[\sum h^i\otimes g^i t_{(1)}\right]\otimes k t_{(2)} \end{gathered}$$

and

$$\begin{gathered} \left((x\otimes H)\circ(H\square_C z^r)\circ(H\square_C H\otimes \mathrm{cocan})\right)\left(\sum h^i\otimes g^i\otimes k\otimes t\right) \\ = \left[\sum h^i\otimes g^i\right]\otimes k\varepsilon^H(t). \end{gathered}$$

Having in mind that also φ is an isomorphism, we also compute

$$\begin{gathered} (\varphi\otimes H)\left((x\otimes H)\circ\left(H\square_C z^l\right)\circ(H\square_C H\otimes \mathrm{cocan})\right)\left(\sum h^i\otimes g^i\otimes k\otimes t\right) \\ = (\varphi\otimes H)\left[\sum h^i\otimes g^i t_{(1)}\right]\otimes k t_{(2)} = \sum S\left(h^i\right) g^i t_{(1)}\otimes k t_{(2)} \end{gathered}$$

and

$$\begin{gathered} (\varphi\otimes H)\left((x\otimes H)\circ(H\square_C z^r)\circ(H\square_C H\otimes \mathrm{cocan})\right)\left(\sum h^i\otimes g^i\otimes k\otimes t\right) \\ = (\varphi\otimes H)\left[\sum h^i\otimes g^i\right]\otimes k\varepsilon^H(t) = \sum S\left(h^i\right) g^i\otimes k\varepsilon^H(t). \end{gathered}$$

Let

$$\alpha_l = (\varphi \otimes H)\left((x \otimes H) \circ \left(H\square_C z^l\right) \circ (H\square_C H \otimes \mathrm{cocan})\right)$$

and

$$\alpha_r = (\varphi \otimes H)\left((x \otimes H) \circ (H\square_C z^r) \circ (H\square_C H \otimes \mathrm{cocan})\right).$$

Then, for every $\sum h^i \otimes g^i \in H\square_C H$, $k \in H$ and $t \in L$,

$$\begin{aligned}(\alpha_l - \alpha_r)\left(\sum h^i \otimes g^i \otimes k \otimes t\right) &= \sum S\left(h^i\right) g^i t_{(1)} \otimes k t_{(2)} - \sum S\left(h^i\right) g^i \otimes k\varepsilon^H(t) \\ &= \left[\sum S\left(h^i\right) g^i \otimes k\right] \cdot t - \left[\sum S\left(h^i\right) g^i \otimes k\right] \cdot \varepsilon^H(t) 1_L \\ &= \left[\sum S\left(h^i\right) g^i \otimes k\right] \cdot \left(t - \varepsilon^H(t) 1_L\right)\end{aligned}$$

so that we get

$$\mathrm{Im}\,(\alpha_l - \alpha_r) = (L \otimes H) L^+$$

and hence the isomorphism $\varphi : A = \frac{H\square_C H}{I_w} \longrightarrow L$ induces an isomorphism

$$\begin{aligned}\left(\widehat{Q}, l\right) &= \mathrm{Coequ}_{\mathrm{Fun}}\left((Px) \circ \left(z^l P\right), (Px) \circ (z^r P)\right) \\ &\cong - \otimes \mathrm{Coequ}\,(\alpha_l, \alpha_r) = - \otimes \frac{L \otimes H}{(L \otimes H) L^+}.\end{aligned}$$

In the sequel, given elements $l^i \in L$ and $h^i \in H$ we will use the notation

$$\left[\sum l^i \otimes h^i\right]_{\widehat{Q}} = \sum l^i \otimes h^i + (L \otimes H) L^+.$$

Following Proposition 7.6, the functor $\widehat{Q}$ can be equipped with the structure of a $\mathbb{B}$-$\mathbb{A}$-bimodule functor, i.e. in our setting, with a structure of E-L-bimodule functor. In particular ${}^E\mu_{\widehat{Q}} = - \otimes \widehat{{}^E\mu_{\widehat{Q}}} : E\widehat{Q} \to \widehat{Q}$ and $\mu^L_{\widehat{Q}} = - \otimes \widehat{\mu^L_{\widehat{Q}}} : \widehat{Q}A \to \widehat{Q}$ where

$$\begin{gathered}\widehat{{}^E\mu_{\widehat{Q}}} : \frac{L \otimes H}{(L \otimes H) L^+} \square_C \frac{H \otimes H}{(H \otimes H) L^+} \longrightarrow \frac{L \otimes H}{(L \otimes H) L^+} \\ \sum_i \sum_j \sum_s \left[l^{i,j} \otimes k^{i,j}\right]_{\widehat{Q}} \square_C \left[h^{i,s} \otimes t^{i,s}\right]_E \mapsto \sum_i \sum_j \sum_s \left[l^{i,j} S\left(k^{i,j}\right) h^{i,s} \otimes t^{i,s}\right]_{\widehat{Q}}\end{gathered}$$

and

$$\begin{gathered}\widehat{\mu^L_{\widehat{Q}}} : L \otimes \frac{L \otimes H}{(L \otimes H) L^+} \longrightarrow \frac{L \otimes H}{(L \otimes H) L^+} \\ y \otimes [y' \otimes h]_{\widehat{Q}} \mapsto [yy' \otimes h]_{\widehat{Q}}.\end{gathered}$$

Such a bimodule functor $\widehat{Q}$ is the one giving rise, together with the functor Q, to the equivalence of the categories of modules over the monads $A \simeq L$ and $B = E$ constructed in the above Subsection 8.1 (in particular see Theorems 8.6 and 8.9). More explicitly,

$${}_L Q_E = - \otimes_E H_L : {}_{\mathbb{E}}\mathcal{B} = {}_{\mathbb{E}}\left(\mathit{Comod}\text{-}C\right) \to {}_{\mathbb{L}}\mathcal{A} = {}_{\mathbb{L}}\left(\mathit{Mod}\text{-}k\right) = \mathit{Mod}\text{-}L$$

and

$${}_E\widehat{Q}_L = - \otimes_L \frac{L \otimes H}{(L \otimes H) L^+} : {}_{\mathbb{L}}\mathcal{A} = \mathit{Mod}\text{-}L \to {}_{\mathbb{E}}\mathcal{B} = {}_{\mathbb{E}}\left(\mathit{Comod}\text{-}C\right).$$

Now we will give details of the isomorphisms associated to the equivalence of categories. Given a right E-module functor F we will denote simply by $-\otimes_E F$ the functor defined by

$$\mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^E_{F\mathbb{E}}U, F_{\mathbb{E}}U\lambda_E\right).$$

Let us consider the functor

$${}_E\widehat{Q}_{LL}Q_E = -\otimes_E H_L \otimes_L \frac{L\otimes H}{(L\otimes H)\,L^+} : {}_{\mathbb{E}}\mathcal{B} = {}_{\mathbb{E}}\left(Comod\text{-}C\right) \to {}_{\mathbb{E}}\mathcal{B} = {}_{\mathbb{E}}\left(Comod\text{-}C\right).$$

We want to prove that ${}_E\widehat{Q}_{LL}Q_E$ is functorially isomorphic to $\mathrm{Id}_{{}_{\mathbb{E}}\mathcal{B}}$. Now, for any $\left(X, {}^E\mu_X\right) \in {}_{\mathbb{E}}\mathcal{B}$ we have

$$\begin{aligned}\left(X, {}^E\mu_X\right)\otimes_E E &= \mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^E_{E\mathbb{E}}U\left(X, {}^E\mu_X\right), E_{\mathbb{E}}U\lambda_E\left(X, {}^E\mu_X\right)\right)\\ &= \mathrm{Coequ}_{\mathrm{Fun}}\left(m_E X, E^E\mu_X\right) \overset{3.14}{=} \left(X, {}^E\mu_X\right).\end{aligned}$$

Thus to this aim it is enough to construct an isomorphism of left E-modules $\widehat{\beta} : H_L \otimes_L \frac{L\otimes H}{(L\otimes H)L^+} \to \frac{H\otimes H}{(H\otimes H)L^+}$. This will imply that $\beta = -\square_C\widehat{\beta} : \widehat{Q}_{LL}Q = -\square_C H_L \otimes_L \frac{L\otimes H}{(L\otimes H)L^+} \to E = -\square_C \frac{H\otimes H}{(H\otimes H)L^+}$ gives rise to a functorial isomorphism ${}_E\widehat{Q}_{LL}Q_E \simeq \mathrm{Id}_{{}_{\mathbb{E}}\mathcal{B}}$. We want to show that $\widehat{\beta}$ is the following morphism

$$\begin{aligned}\widehat{\beta} : H\otimes_L \frac{L\otimes H}{(L\otimes H)\,L^+} &\to \frac{H\otimes H}{(H\otimes H)\,L^+}\\ h\otimes_L [x\otimes h']_{\widehat{Q}} &\mapsto [hx\otimes h']_E.\end{aligned}$$

First we have to prove that it is a well-defined map. Let us consider the map

$$\begin{aligned}\overline{\beta} : H\otimes L\otimes H &\to \frac{H\otimes H}{(H\otimes H)\,L^+}\\ h\otimes x\otimes h' &\mapsto [hx\otimes h']_E.\end{aligned}$$

For every $(x\otimes h')\cdot\left(t-\varepsilon^H(t)\right) \in (L\otimes H)\,L^+$ we have

$$\begin{aligned}\overline{\beta}\left(h\otimes\left[(x\otimes h')\cdot\left(t-\varepsilon^H(t)\right)\right]\right) &= \overline{\beta}\left(h\otimes xt_{(1)}\otimes h't_{(2)} - h\otimes x\otimes h'\varepsilon^H(t)\right)\\ &= hxt_{(1)}\otimes h't_{(2)} - hx\otimes h'\varepsilon^H(t) \in (H\otimes H)\,L^+\end{aligned}$$

so that $\overline{\beta}$ factors through $\overline{\overline{\beta}} : H\otimes \frac{L\otimes H}{(L\otimes H)L^+} \to \frac{H\otimes H}{(H\otimes H)L^+}$. Moreover, for every $l\in L$, we have

$$\overline{\overline{\beta}}\,(hl\otimes x\otimes h') = [(hl)\,x\otimes h']_E = [h\,(lx)\otimes h']_E = \overline{\overline{\beta}}\,(h\otimes lx\otimes h')$$

so that $\overline{\overline{\beta}}$ is also L-balanced and gives rise to the map $\widehat{\beta} : H\otimes_L \frac{L\otimes H}{(L\otimes H)L^+} \to \frac{H\otimes H}{(H\otimes H)L^+}$. The inverse of $\widehat{\beta}$ is given by

$$\begin{aligned}\widehat{\theta} : \frac{H\otimes H}{(H\otimes H)\,L^+} &\to H\otimes_L \frac{L\otimes H}{(L\otimes H)\,L^+}\\ [x\otimes y]_E &\mapsto x\otimes_L [1_L\otimes y]_{\widehat{Q}}.\end{aligned}$$

This map is well-defined, in fact, let us consider the map $\overline{\theta}:H\otimes H\to H\otimes_L \frac{L\otimes H}{(L\otimes H)L^+}$defined by setting

$$\overline{\theta}\left(x\otimes y\right)=x\otimes_L\left[1_L\otimes y\right]_{\widehat{Q}}.$$

For every $(h\otimes g)\cdot\left(t-\varepsilon^H(t)\right)\in(H\otimes H)L^+$, we have $(h\otimes g)\cdot\left(t-\varepsilon^H(t)\right)=ht_{(1)}\otimes gt_{(2)}-h\otimes g\varepsilon^H(t)$ and using that $\Delta(L)\subseteq L\otimes H$, we compute

$$\begin{aligned}
& ht_{(1)}\otimes_L\left(1_L\otimes gt_{(2)}\right)-h\otimes_L\left(1_L\otimes g\varepsilon^H(t)\right)\\
&= h\otimes_L t_{(1)}\cdot\left(1_L\otimes gt_{(2)}\right)-h\otimes_L\left(1_L\otimes g\varepsilon^H(t)\right)\\
&= h\otimes_L\left(t_{(1)}\otimes gt_{(2)}\right)-h\otimes_L\left(1_L\otimes g\varepsilon^H(t)\right)\\
&= h\otimes_L\left(t_{(1)}\otimes gt_{(2)}-1_L\otimes g\varepsilon^H(t)\right)\\
&= h\otimes_L\left((1_L\otimes g)\cdot t-(1_L\otimes g)\,\varepsilon^H(t)\right)\\
&= h\otimes_L\left((1_L\otimes g)\cdot\left(t-\varepsilon^H(t)\right)\right)\in H\otimes_L(L\otimes H)L^+
\end{aligned}$$

so that $\overline{\theta}$ factors through $\frac{H\otimes H}{(H\otimes H)L^+}\to H\otimes_L\frac{L\otimes H}{(L\otimes H)L^+}$ giving rise to the map $\widehat{\theta}$. We compute, using definition of $\widehat{Q}_L=\otimes_L\widehat{Q}$ and $\mu^A_{\widehat{Q}}$

$$\begin{aligned}
\left(\widehat{\theta}\circ\widehat{\beta}\right)\left(h\otimes_L\left[x\otimes h'\right]_{\widehat{Q}}\right) &= \widehat{\theta}\left(\left[hx\otimes h'\right]_E\right)=hx\otimes_L\left[1_L\otimes h'\right]_{\widehat{Q}}=h\otimes_L x\cdot\left[1_L\otimes h'\right]_{\widehat{Q}}\\
&= h\otimes_L\left[x1_L\otimes h'\right]_{\widehat{Q}}=h\otimes_L\left[x\otimes h'\right]_{\widehat{Q}}
\end{aligned}$$

and

$$\left(\widehat{\beta}\circ\widehat{\theta}\right)\left(\left[x\otimes y\right]_E\right)=\widehat{\beta}\left(x\otimes_L\left[1_L\otimes y\right]_{\widehat{Q}}\right)=\left[x\otimes y\right]_E.$$

Let us show that $\widehat{\beta}$ is an isomorphism of left E-modules. Using definition of $\widehat{\mu^E_Q}$, (215) i.e. $\sum_i\sum_j k^{i,j}\otimes S\left(h^{i,j}\right)g^i\in H\otimes L$, definition of $\widehat{m_E}$ we compute

$$\begin{aligned}
&\widehat{\beta}\left(\sum\nolimits_i\sum\nolimits_j\left[k^{i,j}\otimes h^{i,j}\right]_E\cdot g^i\otimes_L\left[x\otimes h'\right]_{\widehat{Q}}\right)=\widehat{\beta}\left(\sum\nolimits_i\sum\nolimits_j k^{i,j}S\left(h^{i,j}\right)g^i\otimes_L\left[x\otimes h'\right]_{\widehat{Q}}\right)\\
&=\widehat{\beta}\left(\sum\nolimits_i\sum\nolimits_j k^{i,j}\otimes_L S\left(h^{i,j}\right)g^i\cdot\left[x\otimes h'\right]_{\widehat{Q}}\right)=\widehat{\beta}\left(\sum\nolimits_i\sum\nolimits_j k^{i,j}\otimes_L\left[S\left(h^{i,j}\right)g^ix\otimes h'\right]_{\widehat{Q}}\right)\\
&\qquad=\left[\sum\nolimits_i\sum\nolimits_j k^{i,j}S\left(h^{i,j}\right)g^ix\otimes h'\right]_E=\sum\nolimits_i\sum\nolimits_j\left[k^{i,j}S\left(h^{i,j}\right)g^ix\otimes h'\right]_E\\
&=\sum\nolimits_i\sum\nolimits_j\left[k^{i,j}\otimes h^{i,j}\right]_E\cdot\left[g^ix\otimes h'\right]_E=\sum\nolimits_i\sum\nolimits_j\left[k^{i,j}\otimes h^{i,j}\right]_E\cdot\widehat{\beta}\left(g^i\otimes_L\left[x\otimes h'\right]_{\widehat{Q}}\right).
\end{aligned}$$

Similarly we want to understand the other isomorphism. Given a right L-module functor G we will denote simply by $-\otimes_L G$ the functor defined by

$$\mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^L_{G\mathbb{L}}U,G_{\mathbb{L}}U\lambda_L\right).$$

Let us consider the functor

$${}_LQ_{EE}\widehat{Q}_L=-\otimes_L\frac{L\otimes H}{(L\otimes H)L^+}\otimes_E H:{}_{\mathbb{L}}\mathcal{A}=Mod\text{-}L\to{}_{\mathbb{L}}\mathcal{A}=Mod\text{-}L.$$

We want to prove that ${}_LQ_{EE}\widehat{Q}_L$ is functorially isomorphic to $\mathrm{Id}_{{}_{\mathbb{L}}\mathcal{A}}$. Now, for any $\left(X,{}^L\mu_X\right)\in{}_{\mathbb{L}}\mathcal{A}$ we have

$$\left(X,{}^L\mu_X\right)\otimes_L L=\mathrm{Coequ}_{\mathrm{Fun}}\left(\mu^L_{L\mathbb{L}}U\left(X,{}^L\mu_X\right),L_{\mathbb{L}}U\lambda_L\left(X,{}^L\mu_X\right)\right)$$

$$= \quad \mathrm{Coequ}_{\mathrm{Fun}}\left(m_L X, L^L\mu_X\right) \overset{3.14}{=} \left(X, {}^L\mu_X\right).$$

Thus to this aim it is enough to construct an isomorphism of left L-modules $\widehat{\zeta}$: $\frac{L\otimes H}{(L\otimes H)L^+} \otimes_E H \to L$. This will imply that $\zeta = -\otimes\widehat{\zeta} : Q_{EE}\widehat{Q} = -\otimes \frac{L\otimes H}{(L\otimes H)L^+} \otimes_E H \to -\otimes L$ gives rise to a functorial isomorphism ${}_L Q_{EE}\widehat{Q}_L \cong \mathrm{Id}_{\mathbb{L}\mathcal{A}}$. We want to show that $\widehat{\zeta}$ is the following morphism

$$\widehat{\zeta} : \frac{L\otimes H}{(L\otimes H)\,L^+} \otimes_E H \to L$$
$$\sum\nolimits_i \sum\nolimits_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i \mapsto \sum\nolimits_i \sum\nolimits_j l^{i,j} S\left(h^{i,j}\right) g^i.$$

Let us consider

$$\overline{\zeta} : L \otimes H \square_C H \to L$$
$$\sum\nolimits_i \sum\nolimits_j l^{i,j} \otimes h^{i,j} \otimes g^i \mapsto \sum\nolimits_i \sum\nolimits_j l^{i,j} S\left(h^{i,j}\right) g^i$$

and let us prove that it is well-defined. By (217) we get that $\sum_i \sum_j l^{i,j} S\left(h^{i,j}\right) g^i \in L$. Now **we use that** ${}^C H$ **is coflat**. Let us prove that

$$\overline{\zeta}\left[(L\otimes H)\, L^+ \square_C H\right] = 0.$$

Let $\sum_i z^i \otimes h^i \in (L\otimes H)\, L^+ \square_C H$ where, for each i, $z^i \in (L\otimes H)\, L^+$. This means that there exist elements $w^{i,j} \in L\otimes H$ and elements $t^{i,j} \in L^+$ such that

$$z^i = \sum\nolimits_j w^{i,j} \cdot t^{i,j}$$

Since $w^{i,j} \in L\otimes H$ there exist $l^{i,j,k} \in L$ and $g^{i,j,k} \in H$ such that

$$w^{i,j} = \sum\nolimits_k l^{i,j,k} \otimes g^{i,j,k}.$$

Hence we have

$$\begin{aligned}
\sum\nolimits_i z^i \otimes h^i &= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k \left(l^{i,j,k} \otimes g^{i,j,k}\right) \cdot t^{i,j} \otimes h^i \\
&= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k \left[l^{i,j,k} t^{i,j}_{(1)} \otimes g^{i,j,k} t^{i,j}_{(2)} - \left(l^{i,j,k} \otimes g^{i,j,k}\right) \varepsilon_H\left(t^{i,j}\right)\right] \otimes h^i \\
&= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k l^{i,j,k} t^{i,j}_{(1)} \otimes g^{i,j,k} t^{i,j}_{(2)} \otimes h^i - \left(l^{i,j,k} \otimes g^{i,j,k}\right) \varepsilon_H\left(t^{i,j}\right) \otimes h^i
\end{aligned}$$

so that

$$\begin{aligned}
\overline{\zeta}\left(\sum\nolimits_i z^i \otimes h^i\right) &= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k l^{i,j,k} t^{i,j}_{(1)} S\left(g^{i,j,k} t^{i,j}_{(2)}\right) h^i - \left(l^{i,j,k} S\left(g^{i,j,k}\right)\right) \varepsilon_H\left(t^{i,j}\right) h^i \\
&= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k l^{i,j,k} t^{i,j}_{(1)} S\left(t^{i,j}_{(2)}\right) S\left(g^{i,j,k}\right) h^i - \left(l^{i,j,k} S\left(g^{i,j,k}\right)\right) \varepsilon_H\left(t^{i,j}\right) h^i \\
&= \sum\nolimits_i \sum\nolimits_j \sum\nolimits_k l^{i,j,k} \varepsilon_H\left(t^{i,j}\right) S\left(g^{i,j,k}\right) h^i - \left(l^{i,j,k} S\left(g^{i,j,k}\right)\right) \varepsilon_H\left(t^{i,j}\right) h^i = 0.
\end{aligned}$$

hence we have a well defined map $\overline{\overline{\zeta}} : \frac{L\otimes H}{(L\otimes H)L^+} \square_C H \to L$ defined by setting

$$\overline{\overline{\zeta}}\left(\sum\nolimits_i \sum\nolimits_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes g^i\right) = \sum\nolimits_i \sum\nolimits_j l^{i,j} S\left(h^{i,j}\right) g^i.$$

We now have to prove that this map induces a map $\zeta : \frac{L\otimes H}{(L\otimes H)L^{+}} \otimes_E H \to L$. Let $e \in E$. We have to prove that

$$\overline{\overline{\zeta}}\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \cdot e \otimes g^i\right) = \overline{\overline{\zeta}}\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes e \cdot g^i\right).$$

Since $e \in E$, there exist $x^k, y^k \in H$ such that $e = \left[\sum_k x^k \otimes y^k\right]_E$. Hence we have to prove that

$$\begin{aligned}&\overline{\overline{\zeta}}\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \cdot \left[\sum_k x^k \otimes y^k\right]_E \otimes g^i\right)\\ &= \overline{\overline{\zeta}}\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes \left[\sum_k x^k \otimes y^k\right]_E \cdot g^i\right)\end{aligned}$$

and by using definition of ${}^E\mu_{\widehat{Q}}$ and μ_Q^E we have to prove that

$$\begin{aligned}&\overline{\overline{\zeta}}\left(\sum_i \sum_j \sum_k \left[l^{i,j} S\left(h^{i,j}\right) x^k \otimes y^k\right]_{\widehat{Q}} \otimes g^i\right)\\ = \ &\overline{\overline{\zeta}}\left(\sum_i \sum_j \sum_k \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes x^k S\left(y^k\right) g^i\right)\end{aligned}$$

i.e.

$$\sum_i \sum_j \sum_k l^{i,j} S\left(h^{i,j}\right) x^k S\left(y^k\right) g^i = \sum_i \sum_j \sum_k l^{i,j} S\left(h^{i,j}\right) x^k S\left(y^k\right) g^i$$

which is true, so that we can conclude that the map $\widehat{\zeta} : \frac{L\otimes H}{(L\otimes H)L^{+}} \otimes_E H \to L$ is well-defined. Now, we want to prove that $\widehat{\zeta}$ is bijective. The inverse of $\widehat{\zeta}$ is given by

$$\begin{aligned}\Xi : L &\to \frac{L \otimes H}{(L \otimes H)\, L^{+}} \otimes_E H\\ l &\mapsto [1_H \otimes 1_H]_{\widehat{Q}} \otimes_E l.\end{aligned}$$

Now we compute

$$\begin{aligned}\left(\Xi \circ \widehat{\zeta}\right)\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i\right) &= \Xi\left(\sum_i \sum_j l^{i,j} S\left(h^{i,j}\right) g^i\right)\\ &= [1_H \otimes 1_H]_{\widehat{Q}} \otimes_E \sum_i \sum_j l^{i,j} S\left(h^{i,j}\right) g^i\end{aligned}$$

$$= \sum_i \sum_j [1_H \otimes 1_H]_{\widehat{Q}} \otimes_E \left[l^{i,j} \otimes h^{i,j}\right]_E \cdot g^i = \sum_i \sum_j [1_H \otimes 1_H]_{\widehat{Q}} \left[l^{i,j} \otimes h^{i,j}\right]_E \otimes_E g^i$$

$$= \sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i$$

and

$$\left(\widehat{\zeta} \circ \Xi\right)(l) = \zeta\left([1_H \otimes 1_H]_{\widehat{Q}} \otimes_E l\right) = 1_H S\left(1_H\right) l = l.$$

Let us show that $\widehat{\zeta}$ is an isomorphism of left L-modules. Let $a \in L$ and let us consider

$$\begin{aligned}\widehat{\zeta}\left(a \cdot \sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i\right) &= \widehat{\zeta}\left(\sum_i \sum_j \left[\left(a \cdot l^{i,j}\right) \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i\right)\\ &= \sum_i \sum_j a l^{i,j} S\left(h^{i,j}\right) g^i = a \cdot \left(\sum_i \sum_j l^{i,j} S\left(h^{i,j}\right) g^i\right)\\ &= a \cdot \widehat{\zeta}\left(\sum_i \sum_j \left[l^{i,j} \otimes h^{i,j}\right]_{\widehat{Q}} \otimes_E g^i\right).\end{aligned}$$

As observed at the beginning of this section, this reproduces what happens in the dual case of the [Scha4] setting where, starting from a Hopf-Galois extension, one can produce a new Hopf algebra such that the Hopf-Galois object turns into a Hopf bi-Galois object and Hopf algebras are Morita-Takeuchi equivalent. In our setting, coming from a coGalois coextension we get a coherd, which allows us to compute the monads and in particular a new monad together with the new bimodule functor. Following the theory developed in the previous sections, we could then calculate in details also the equivalence between the module categories with respects to the two monads.

9.3. **Galois comodules.** Let ${}_B\Sigma_A$ be a B-A-bimodule. Let $L = - \otimes_B \Sigma_A, R = \mathrm{Hom}_A\left({}_B\Sigma_A, -\right)$. Let $\mathcal{C}$ be an A-coring and let $\mathbb{C} = \left(- \otimes_A \mathcal{C}, - \otimes_A \Delta, r \circ \left(- \otimes_A \varepsilon\right)\right)$. Assume that (Σ, ρ_Σ) is a B-$\mathcal{C}$-*comodule* i.e. (Σ, ρ_Σ) is a $\mathcal{C}$-comodule and

$$\rho_\Sigma : \Sigma \to \Sigma \otimes_A \mathcal{C}$$

is a morphism of B-A-bimodules. In particular the map

$$\lambda : B \to \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}\left((\Sigma, \rho_\Sigma)\right) \text{ defined by setting } \lambda\left(b\right)\left(x\right) = bx$$

is well-defined and is a ring morphism. Moreover λ is a monomorphism. In this case $\beta = - \otimes_B \rho_\Sigma : - \otimes_B \Sigma_A \to - \otimes_B \Sigma_A \otimes_A \mathcal{C}$ is a left $\mathbb{C}$-comodule functor. The associated functorial morphism can $= \varphi = (C\epsilon) \circ (\beta R) : \mathbb{L}\mathbb{R} \to \mathbb{C}$,

$$\begin{gathered}\mathrm{can} : \mathrm{Hom}_A\left({}_B\Sigma_A, -\right) \otimes_B \Sigma_A \overset{\beta R}{\to} \mathrm{Hom}_A\left({}_B\Sigma_A, -\right) \otimes_B \Sigma_A \otimes_A \mathcal{C} \overset{C\epsilon}{\to} - \otimes_A \mathcal{C} \\ f \otimes_B x \mapsto f \otimes_B x_0 \otimes_A x_1 \mapsto f\left(x_0\right) \otimes_A x_1\end{gathered}$$

$$\begin{gathered}\mathrm{can}_M : \mathrm{Hom}_A\left({}_B\Sigma_A, M\right) \otimes_B \Sigma_A \to M \otimes_A \mathcal{C} \\ f \otimes_B x \mapsto f\left(x_0\right) \otimes_A x_1\end{gathered}$$

$$\begin{aligned}\mathrm{can} &= (C\epsilon) \circ (\beta R) = (\epsilon \otimes_A \mathcal{C}) \circ \mathrm{Hom}_A\left({}_B\Sigma_A, -\right) \otimes_B \rho_\Sigma \\ \mathrm{can}_M &= \varphi_M\left(f \otimes_B t\right) = (\epsilon \otimes_A \mathcal{C})\left(f \otimes_B t_0 \otimes_A t_1\right) = f\left(t_0\right) \otimes_A t_1.\end{aligned}$$

We have

$$\begin{aligned}K_\varphi &: Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A) = Comod\text{-}\mathcal{C} \\ M &\mapsto \left(M \otimes_B \Sigma, M \otimes_B \rho_\Sigma\right).\end{aligned}$$

Since Mod-B has all equalizers, K_φ has a right adjoint

$$\begin{aligned}D_\varphi\left(X, x\right) &= \mathrm{Equ}\left(\left(- \otimes_A \mathcal{C}\right) \circ \rho_\Sigma, \mathrm{Hom}_A\left({}_B\Sigma_A, x\right)\right) \\ &= \left\{f \in \mathrm{Hom}_A\left({}_B\Sigma_A, X\right) \mid x \circ f = \left(f \otimes_A \mathcal{C}\right) \circ \rho_\Sigma\right\} \\ &= \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma, \rho_\Sigma\right), \left(X, x\right)\right)\end{aligned}$$

Hence $D_\varphi = \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma, \rho_\Sigma\right), -\right) : {}^{\mathbb{C}}(Mod\text{-}A) = Comod\text{-}\mathcal{C} \to Mod\text{-}B$ has a left adjoint $K_\varphi = \left(- \otimes_B \Sigma, - \otimes_B \rho_\Sigma\right)$.

Theorem 9.6 ([GT, Theorem 3.1]). *$\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma, \rho_\Sigma\right), -\right) : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}B$ is full and faithful if and only if*

1) $-\otimes_B \Sigma_A$ *preserves the equalizer*

$$\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),\left(X,x\right)\right) \xrightarrow{i} \mathrm{Hom}_A\left(\Sigma,X\right) \underset{(-\otimes_A \mathcal{C})\circ\rho_\Sigma}{\overset{x\circ -}{\rightrightarrows}} \mathrm{Hom}_A\left(\Sigma,X\otimes_A \mathcal{C}\right).$$

2) $\mathrm{can}:\mathrm{Hom}_A\left({}_B\Sigma_A,-\right)\otimes_B \Sigma_A \to -\otimes_A \mathcal{C}$ *is a comonad isomorphism.*

Proof. Apply Theorem 4.53 to the adjunction $\left(-\otimes_B \Sigma_A, \mathrm{Hom}_A\left({}_B\Sigma_A,-\right)\right)$. □

THEOREM 9.7 ([GT, Theorem 3.2]). $K_\varphi : Mod\text{-}B \to {}^{\mathbb{C}}\left(Mod\text{-}A\right) = Comod\text{-}\mathcal{C}$ *is an equivalence of categories if and only if*

1) $-\otimes_B \Sigma_A$ *preserves the equalizer*

$$\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),\left(X,x\right)\right) \xrightarrow{i} \mathrm{Hom}_A\left(\Sigma,X\right) \underset{(-\otimes_A C)\circ\rho_\Sigma}{\overset{x\circ -}{\rightrightarrows}} \mathrm{Hom}_A\left(\Sigma,X\otimes_A \mathcal{C}\right).$$

2) $-\otimes_B \Sigma_A$ *reflects isomorphisms and*
3) $\mathrm{can}:\mathrm{Hom}_A\left({}_B\Sigma_A,-\right)\otimes_B \Sigma_A \to -\otimes_A \mathcal{C}$ *is a comonad isomorphism.*

Proof. Apply Theorem 4.55 to the adjunction $\left(-\otimes_B \Sigma_A, \mathrm{Hom}_A\left({}_B\Sigma_A,-\right)\right)$. □

Let us now consider a particular case of the previous situation.

Let $\mathcal{C}$ be an A-coring and let Σ be a right $\mathcal{C}$-comodule. Set $T=\mathrm{End}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right)\right)$. Then it is easy to check that $\left(\Sigma,\rho_\Sigma\right)$ is a T-$\mathcal{C}$-comodule. Following [Wis], we say that Σ is a Galois $\mathcal{C}$-*comodule* whenever $\mathrm{can}:\mathrm{Hom}_A\left({}_T\Sigma_A,-\right)\otimes_T \Sigma \to -\otimes_A \mathcal{C}$ is an isomorphism. The adjunction $\left({}^{\mathbb{C}}U,{}^{\mathbb{C}}F\right)$ for $\mathbb{C}=\left(-\otimes_A \mathcal{C}, -\otimes_A \Delta, r\circ\left(-\otimes_A \varepsilon\right)\right)$ gives us the following

PROPOSITION 9.8. *Let $\mathcal{C}$ be an A-coring and let Σ be a right $\mathcal{C}$-comodule. Set $T=\mathrm{End}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right)\right)$. Then the map*

$$\psi L:\mathrm{Hom}_A\left({}_T\Sigma_A,L\right)\to \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),{}^{\mathbb{C}}FL\right) \text{ defined by setting}$$
$$\psi L\left(f\right)=\left(f\otimes_A \mathcal{C}\right)\circ\rho_\Sigma$$

is an isomorphism whose inverse is defined by setting $\left(\psi L\right)^{-1}\left(h\right)=r_L\circ\left(L\otimes_A \varepsilon\right)\circ h$, for every $L\in Mod\text{-}A$. In this way we get a functorial isomorphism

$$\psi:\mathrm{Hom}_A\left({}_T\Sigma_A,-\right)\to \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),{}^{\mathbb{C}}F\right).$$

9.9. Note that, in particular, we have

$$\psi A:\mathrm{Hom}_A\left({}_T\Sigma_A,A\right)\to \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),{}^{\mathbb{C}}FA\right)$$

where

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),{}^{\mathbb{C}}FA\right) &= \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),A\otimes_A \mathcal{C}\right)\\ &\simeq \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),\mathcal{C}\right)\end{aligned}$$

so that

$$\psi A:\mathrm{Hom}_A\left({}_T\Sigma_A,A\right)\to \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\left(\Sigma,\rho_\Sigma\right),\mathcal{C}\right)$$

and is defined by setting

$$\left[\psi A\left(f\right)\right]\left(t\right)=\left[l_{\mathcal{C}}\circ\left(f\otimes_A \mathcal{C}\right)\circ\rho_\Sigma\right]\left(t\right)=f\left(t_0\right)t_1.$$

THEOREM 9.10 ([GT]). *Let $\mathcal{C}$ be an A-coring and let Σ be a right $\mathcal{C}$-comodule. Assume that ${}_A\mathcal{C}$ is flat. Set $T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma))$. Then the following are equivalent:*

(a) *The functor $\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -) : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}T$ is full and faithful where $\mathbb{C} = - \otimes_A \mathcal{C}$.*

(b) *$\epsilon : \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), - \otimes_T \Sigma) \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an isomorphism.*

(c) *(Σ, ρ_Σ) is a generator of ${}^{\mathbb{C}}(Mod\text{-}A)$.*

(d) *$\mathrm{can} : \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -) \otimes_T \Sigma \to - \otimes_A \mathcal{C}$ is an isomorphism and ${}_T\Sigma$ is flat.*

Proof. By Proposition A.12, ${}_A\mathcal{C}$ is flat if and only if $(Mod\text{-}A)^{\mathcal{C}}$ is a Grothendieck category and the forgetful functor $U : (Mod\text{-}A)^{\mathcal{C}} \to Mod\text{-}A$ is left exact. Also, by the foregoing, $D_\varphi = \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -) : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}T$ has a left adjoint $K_\varphi = (- \otimes_T \Sigma, - \otimes_T \rho_\Sigma)$.

$(a) \Leftrightarrow (b)$ It follows by Proposition 2.32.

$(a) \Leftrightarrow (c)$ It follows by Proposition A.3.

$(c) \Rightarrow (d)$ Since (Σ, ρ_Σ) is a generator of ${}^{\mathbb{C}}(Mod\text{-}A)$ and since $(- \otimes_T \Sigma, - \otimes_T \rho_\Sigma) : Mod\text{-}T \to {}^{\mathbb{C}}(Mod\text{-}A)$ is a left adjoint of $\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -) : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}T$, by Gabriel-Popescu Theorem A.9, $(- \otimes_T \Sigma, - \otimes_T \rho_\Sigma)$ is a left exact functor. Since the forgetful functor $U : (Mod\text{-}A)^{\mathcal{C}} \to Mod\text{-}A$ is also left exact, we deduce that $- \otimes_T \Sigma : Mod\text{-}T \to Mod\text{-}A$ is left exact i.e. ${}_T\Sigma$ is flat. Since $\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -)$ is full and faithful, by Theorem 9.6, can is an isomorphism.

$(d) \Rightarrow (a)$ It follows by Theorem 9.6. $\square$

THEOREM 9.11 ([GT]). *Let $\mathcal{C}$ be an A-coring, let B be a ring and assume that ${}_A\mathcal{C}$ is flat. Let (Σ, ρ_Σ) be a B-$\mathcal{C}$-comodule. Then the following are equivalent:*

(a) *The functor $- \otimes_B \Sigma_A : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories where $\mathbb{C} = - \otimes_A \mathcal{C}$.*

(b) *$\mathrm{can} : \mathrm{Hom}_A({}_B\Sigma_A, -) \to - \otimes_A \mathcal{C}$ is an isomorphism and ${}_B\Sigma$ is faithfully flat.*

(c) *(Σ, ρ_Σ) is a generator of ${}^{\mathbb{C}}(Mod\text{-}A)$ and the functor $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is full and faithful.*

(d) *(Σ, ρ_Σ) is a generator of ${}^{\mathbb{C}}(Mod\text{-}A)$, the functor $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is faithful and $\lambda : B \to T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma))$ is an isomorphism.*

Proof. $(a) \Rightarrow (b)$ By Theorem 9.7, can is an isomorphism. Since ${}_A\mathcal{C}$ is flat, by Proposition A.12, the forgetful functor $U : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}A$ is exact. Since U is also faithful, we get that the functor

$$- \otimes_B \Sigma_A : Mod\text{-}B \to Mod\text{-}A$$

is faithful and exact.

$(b) \Rightarrow (a)$ It follows by Theorem 9.7.

$(a) \Rightarrow (c)$ Since B is a generator of $Mod\text{-}B$, ${}_B\Sigma \simeq B \otimes_B \Sigma$ is a generator of ${}^{\mathbb{C}}(Mod\text{-}A)$.

$(c) \Rightarrow (d)$ Since $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is full and faithful and it is the left adjoint of the adjunction $\left(- \otimes_B \Sigma, \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -)\right)$, the unit is a functorial isomorphism. In particular we have that

$$\eta B : B \to \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left((\Sigma,\rho_\Sigma),(B\otimes_B\Sigma,\rho_\Sigma)\right) \simeq \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}\left((\Sigma,\rho_\Sigma)\right)$$

is an isomorphism. Note that ηB is exactly λ.

$(d) \Rightarrow (b)$ By Theorem 9.10, can : $\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left((\Sigma,\rho_\Sigma),-\right)\otimes_B\Sigma \to -\otimes_A\mathcal{C}$ is an isomorphism and ${}_B\Sigma$ is flat. Since $-\otimes_B\Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is faithful and $U : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}A$ is also faithful, the functor $-\otimes_B\Sigma_A = U(-\otimes_B\Sigma) : Mod\text{-}B \to (Mod\text{-}A)$ is faithful. Then ${}_B\Sigma$ is faithfully flat. □

Remark 9.12. By Theorem 9.11 we deduce that if $-\otimes_B\Sigma_A : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories then Σ is a Galois $\mathcal{C}$-comodule.

9.13. Let ${}_B\Sigma_A$ be a B-A-bimodule. In the case that Σ_A is finitely generated and projective we have a natural isomorphism

$$\Lambda : \mathrm{Hom}_A(\Sigma,-) \to -\otimes_A\Sigma^*$$
$$f \mapsto f(x_i)\otimes_A x_i^*$$

where $\Sigma^* = \mathrm{Hom}_A(\Sigma,A)$ and $(x_i,x_i^*)_{i=1,\dots,n}$ is a dual basis for Σ_A. We can consider the adjunction $(-\otimes_B\Sigma_A, -\otimes_A\Sigma^*)$ and the associated comonad $-\otimes_A\Sigma^*\otimes_B\Sigma_A : Mod\text{-}A \to Mod\text{-}A$, then the A-coring $\Sigma^*\otimes_B\Sigma_A$ is called the *comatrix coring* associated to the bimodule ${}_B\Sigma_A$. Moreover, when (Σ,ρ_Σ) is a B-$\mathcal{C}$-comodule then we have the following commutative diagram

(218)
$$\begin{tikzcd} \mathrm{Hom}_A(\Sigma,-)\otimes_B\Sigma \arrow[rr, "\Lambda\otimes_B\Sigma"] \arrow[dr, "\mathrm{can}"'] & & -\otimes_A\Sigma^*\otimes_B\Sigma \arrow[dl, "-\otimes_A\mathbf{can}"] \\ & -\otimes_A\mathcal{C} & \end{tikzcd}$$

where

$$\mathbf{can} : \mathbf{\Sigma}^*\otimes_B\Sigma \to \mathcal{C}$$

defined by setting

$$\mathbf{can}(\phi\otimes_B s) = \phi(s_0)\,s_1$$

is a morphism of A-corings where $\Sigma^*\otimes_B\Sigma$ is an A-coring via comultiplication $\Delta(\varphi\otimes_B t) = \sum\limits_i \varphi\otimes_B x_i\otimes_A x_i^*\otimes_B t$ and counit $\varepsilon(\varphi\otimes_B t) = \varphi(t)$.

Remark 9.14. Following [BrWi, pag 189] we say that Σ is a Galois $\mathcal{C}$-*comodule* when Σ_A is finitely generated and projective and $ev : \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}(\Sigma,\mathcal{C})\otimes_B\Sigma \to \mathcal{C}$ is an isomorphism.

Corollary 9.15 ([GT]). *Let $\mathcal{C}$ be an A-coring, let B be a ring and let (Σ,ρ_Σ) be a B-$\mathcal{C}$-bicomodule. Then the following are equivalent:*

(a) *${}_A\mathcal{C}$ is flat and the functor $-\otimes_B\Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories where $\mathbb{C} = -\otimes_A\mathcal{C}$*

(b) *Σ_A is finitely generated and projective, the canonical map* $\mathbf{can} : \mathbf{\Sigma}^*\otimes_B\Sigma \to \mathcal{C}$ *is an isomorphism and ${}_B\Sigma$ is faithfully flat*

(c) *${}_A\mathcal{C}$ is flat, Σ is a finitely generated projective generator of ${}^{\mathbb{C}}(Mod\text{-}A)$ and $\lambda : B \to T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((\Sigma,\rho_\Sigma))$ is an isomorphism.*

Proof. $(a) \Rightarrow (c)$ Apply Proposition A.19 to the functor $T = - \otimes_B \Sigma$. Since B is a finitely generated and projective generator of $Mod\text{-}B$, $\Sigma^{\mathcal{C}} \simeq B \otimes_B \Sigma^{\mathcal{C}}$ is a finitely generated and projective generator of ${}^{\mathbb{C}}(Mod\text{-}A)$. By the equivalence $(a) \Leftrightarrow (d)$ of Theorem 9.11 we get that $\lambda : B \to T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma))$ is an isomorphism.

$(c) \Rightarrow (b)$ Let us consider $U : {}^{\mathbb{C}}(Mod\text{-}A) \to Mod\text{-}A$ which is the left adjoint of the free functor $- \otimes_A \mathcal{C} : Mod\text{-}A \to {}^{\mathbb{C}}(Mod\text{-}A)$. We have to prove that Σ_A is finitely generated and projective. Now, by Proposition A.18, we prove that Σ_A is finite, i.e. that $\mathrm{Hom}_{Mod\text{-}A}(\Sigma_A, -)$ preserves coproducts. Let us consider a family $(A_i)_{i \in I} \in Mod\text{-}A$. We have the following

$$\coprod_{i\in I} \mathrm{Hom}_{Mod\text{-}A}(U(\Sigma), A_i) \overset{(U,-\otimes_A \mathcal{C})\mathrm{adj}}{\simeq} \coprod_{i\in I} \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}(\Sigma, A_i \otimes_A \mathcal{C})$$

$$\overset{\Sigma\mathrm{finite}}{\simeq} \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\Sigma, \coprod_{i\in I}(A_i \otimes_A \mathcal{C})\right) \overset{-\otimes_A \mathcal{C}\mathrm{right\ adj}}{\simeq} \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}\left(\Sigma, \left(\coprod_{i\in I} A_i\right) \otimes_A \mathcal{C}\right)$$

$$\overset{(U,-\otimes_A \mathcal{C})\mathrm{adj}}{\simeq} \mathrm{Hom}_{Mod\text{-}A}\left(U(\Sigma), \coprod_{i\in I} A_i\right)$$

Since $\Sigma_A = U(\Sigma)$ we deduce that $\mathrm{Hom}_{Mod\text{-}A}(\Sigma_A, -)$ preserves coproducts. Since by assumption ${}_A\mathcal{C}$ is flat, by Theorem 9.10 $(c) \Rightarrow (d)$ we get that
$\mathrm{can} : \mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -) \otimes_B \Sigma \to - \otimes_A \mathcal{C}$ is an isomorphism and ${}_B\Sigma$ is flat. By diagram 218 we obtain that **can** is also an isomorphism. Since Σ is a finitely generated projective generator of ${}^{\mathbb{C}}(Mod\text{-}A)$, by Corollary A.21 $\mathrm{Hom}_{\mathbb{C}(Mod\text{-}A)}((\Sigma, \rho_\Sigma), -)$ is an equivalence of categories, hence so is $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ so that ${}_B\Sigma$ is faithfully flat.

$(b) \Rightarrow (a)$ Since **can** is an isomorphism, we have that ${}_A\mathcal{C}$ is flat if and only if ${}_A\mathbf{\Sigma}^* \otimes_B \Sigma$ is flat. By assumption we know that ${}_B\Sigma$ is flat. Since Σ_A is finitely generated and projective, also ${}_A\Sigma^*$ is finitely generated and projective so ${}_A\Sigma^*$ is flat. Therefore the functor $- \otimes_A \Sigma^* \otimes_B \Sigma$ is left exact and, since **can** is an isomorphism, $- \otimes_A \mathcal{C}$ is also left exact. By diagram 218, since **can** is an isomorphism, can is also an isomorphism. Now, ${}_A\mathcal{C}$ is flat and ${}_B\Sigma$ is faithfully flat, then we can apply Theorem 9.11 $(b) \Rightarrow (a)$ to deduce that $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories. □

REMARK 9.16. By Corollary 9.15 we deduce that if ${}_A\mathcal{C}$ is flat and $- \otimes_B \Sigma_A : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories, then Σ is a Galois $\mathcal{C}$-comodule.

COROLLARY 9.17 ([GT, Theorem 3.10] Generalized Descent for Modules). *Let ${}_B\Sigma_A$ be a B-A-bimodule such that Σ_A is finitely generated and projective. Let* $\Sigma^* = \mathrm{Hom}_A(\Sigma, A)$. *Then the following are equivalent:*

(a) ${}_A(\Sigma^* \otimes_B \Sigma)$ *is flat and the functor* $- \otimes_B \Sigma : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A)$ *is an equivalence of categories where* $\mathbb{C} = - \otimes_A \Sigma^* \otimes_B \Sigma$
(b) ${}_B\Sigma$ *is faithfully flat.*

Proof. By (9.13) we have that $\Sigma^* \otimes_B \Sigma$ is an A-coring and thus $\mathbb{C} = - \otimes_A \Sigma^* \otimes_B \Sigma$ is a comonad on $Mod\text{-}A$. Note that Σ is a B-$\mathcal{C}$-bicomodule via a canonical right coaction $\rho_{\Sigma}^{\mathcal{C}}:\Sigma\to\Sigma\otimes_A\Sigma^*\otimes_B\Sigma$ defined by setting $\rho_{\Sigma}^{\mathcal{C}}(s)=\sum_{i=1}^{n}x_i\otimes_A x_i^*\otimes_B s$ where $(x_i,x_i^*)_{i=1,\dots,n}$ is a dual basis for Σ_A. Then we can apply Corollary 9.15 $(a)\Leftrightarrow(b)$ to the case "$\mathcal{C}$" $=\Sigma^*\otimes_B\Sigma$ so that $\mathbf{can}:\mathbf{\Sigma}^*\otimes_B\Sigma\to\mathcal{C}$ is the identity map. $\square$

9.18. Let $B\to A$ a k-algebra extension. Let "${}_B\Sigma_A$" $={}_BA_A$ in 9.13. Then the comatrix coring becomes $\mathcal{C}=A\otimes_B A$ which is an A-coring with coproduct $\Delta^{\mathcal{C}}:\mathcal{C}=A\otimes_B A\to\mathcal{C}\otimes_A\mathcal{C}=A\otimes_B A\otimes_A A\otimes_B A$ defined by setting

$$\Delta^{\mathcal{C}}(a\otimes_B a')=a\otimes_B 1_A\otimes_A 1_A\otimes_B a'$$

and counit $\varepsilon^{\mathcal{C}}:\mathcal{C}=A\otimes_B A\to A$ defined by setting

$$\varepsilon^{\mathcal{C}}(a\otimes_B a')=aa'$$

for every $a,a'\in A$. Such A-coring $\mathcal{C}=A\otimes_B A$ is called *canonical coring* or *Sweedler coring* associated to the algebra extension $B\to A$.

Definition 9.19. Let B be a k-algebra and let $B\xrightarrow{\sigma}A$ be an algebra extension. A *right descent datum from A to B* is a right A-module M together with a right A-module morphism $\delta:M\to M\otimes_B A$ such that

$$(\delta\otimes_B A)\circ\delta=(M\otimes_B\sigma\otimes_B A)\circ\left(M\otimes_B l_A^{-1}\right)\circ\delta \tag{219}$$

and

$$\mu_M\circ\delta=M \tag{220}$$

where $l_A^{-1}:B\otimes_B A\to A$ is the canonical isomorphism and $\mu_M:M\otimes_B A\to M$ is induced by the A-module structure of $M,\mu_M^A:M\otimes A\to A$. Given $(M,\delta),(M',\delta')$ two right descent data from A to B, *a morphism of right descent data from A to B* is a right A-module map $f:M\to M'$ such that

$$\delta'\circ f=(f\otimes_B A)\circ\delta.$$

We will denote by $\mathcal{D}(A\downarrow B)$ the category of right descent data. Similarly one can define *left descent data from A to B* and their category $(A\downarrow B)\mathcal{D}$.

Let $\mathbb{A}=(A,m_A,u_A)$ be a monad on a category $\mathcal{A}$. Then we can consider the adjunction $({}_{\mathbb{A}}F,{}_{\mathbb{A}}U)$, where ${}_{\mathbb{A}}F:\mathcal{A}\to{}_{\mathbb{A}}\mathcal{A}$ and ${}_{\mathbb{A}}U:{}_{\mathbb{A}}\mathcal{A}\to\mathcal{A}$, with unit u_A which is the unit of the monad and counit λ_A determined by ${}_{\mathbb{A}}U\left(\lambda_A\left(X,{}^A\mu_X\right)\right)={}^A\mu_X$ for every $\left(X,{}^A\mu_X\right)\in{}_{\mathbb{A}}\mathcal{A}$. Then ${}_{\mathbb{A}}F{}_{\mathbb{A}}U$ is a comonad on the category ${}_{\mathbb{A}}\mathcal{A}$ by Proposition 4.4. Hence we can consider the category of comodules for the comonad $\mathbb{C}={}_{\mathbb{A}}F{}_{\mathbb{A}}U,{}^{{}_{\mathbb{A}}F{}_{\mathbb{A}}U}({}_{\mathbb{A}}\mathcal{A})={}^{\mathbb{C}}({}_{\mathbb{A}}\mathcal{A})$ which is the *category of descent data with respect to the monad* $\mathbb{A}$ and it is denoted by $\mathfrak{Des}_{\mathcal{A}}(\mathbb{A})$.

Example 9.20. Let $B\xrightarrow{\sigma}A$ be a k-algebra extension. Then A is a B-ring. In fact $m_A:A\otimes A\to A$ induces $m:A\otimes_B A\to A$ as follows. We have to prove that $m_A(ab\otimes a')=m_A(a\otimes ba')$. We compute

$$\begin{aligned}m_A(ab\otimes a')&=m_A(a\sigma(b)\otimes a')=(a\sigma(b))a'\\&=a(\sigma(b)a')=m_A(a\otimes\sigma(b)a')=m_A(a\otimes ba').\end{aligned}$$

Moreover the unit is $u = \sigma : B \to A$. Then $\mathbb{A} = \left(- \otimes_B A, - \otimes_B m, (- \otimes_R u) \circ r_-^{-1}\right)$ is a monad on the category of right B-modules, $Mod\text{-}B$, as in Example 3.3. Note that we have an iso of categories $K : Mod\text{-}A \to {}_{\mathbb{A}}(Mod\text{-}B)$ given by

$$\begin{array}{ccc} Mod\text{-}A & \longrightarrow & {}_{\mathbb{A}}(Mod\text{-}B) \\ \left(X, \mu_X^A\right) & \mapsto & \left(X, \overline{\mu}_X^A\right) \end{array}$$

where $\overline{\mu}_X^A : X \otimes_B A \to X$ is well-defined starting from $\mu_X^A : X \otimes A \to X$. In fact we have

$$\begin{aligned} \mu_X^A (xb \otimes a) &= \mu_X^A \left(\mu_X^A (x \otimes \sigma (b)) \otimes a\right) \\ &\overset{X \text{is} A\text{-mod}}{=} \mu_X^A (x \otimes m_A (\sigma (b) \otimes a)) = \mu_X^A (x \otimes ba) . \end{aligned}$$

Now, since $\mathbb{A} = \left(- \otimes_B A, - \otimes_B m, (- \otimes_R u) \circ r_-^{-1}\right)$ is a monad, we can consider ${}_{\mathbb{A}}F = - \otimes_B A : Mod\text{-}B \to {}_{\mathbb{A}}(Mod\text{-}B) \simeq Mod\text{-}A$ and ${}_{\mathbb{A}}U = - \otimes_A A_B : {}_{\mathbb{A}}(Mod\text{-}B) \simeq Mod\text{-}A \to Mod\text{-}B$ so that $\mathbb{C} = {}_{\mathbb{A}}F\,{}_{\mathbb{A}}U = - \otimes_A A \otimes_B A$ is a comonad on ${}_{\mathbb{A}}(Mod\text{-}B) \simeq Mod\text{-}A$ associated to the A-coring $\mathcal{C} = A \otimes_B A$. The category of comodules for the comonad $\mathbb{C} = {}_{\mathbb{A}}F\,{}_{\mathbb{A}}U = - \otimes_A A \otimes_B A$ is then the category of right comodules for the A-coring $\mathcal{C} = A \otimes_B A$

$$\begin{aligned} {}^{{}_{\mathbb{A}}F{}_{\mathbb{A}}U}\left({}_{\mathbb{A}}(Mod\text{-}B)\right) &= {}^{\mathbb{C}}\left({}_{\mathbb{A}}(Mod\text{-}B)\right) = \left({}_{\mathbb{A}}(Mod\text{-}B)\right)^{\mathcal{C}} \simeq (Mod\text{-}A)^{\mathcal{C}} \\ &= {}^{\mathbb{C}}(Mod\text{-}A) = {}^{{}_{\mathbb{A}}F{}_{\mathbb{A}}U}(Mod\text{-}A) \end{aligned}$$

and it is the category of right descent data from A to B, usually denoted by $\mathcal{D}(A \downarrow B)$.

Corollary 9.21 ([Scha4, Theorem 4.5.2] Faithfully flat descent). *Let A be a k-algebra and let $B \subseteq A$ be a k-algebra extension. Let $\mathcal{C} = A \otimes_B A$ be the canonical A-coring. The following statements are equivalent:*

- (a) *${}_B A$ is flat and the functor $- \otimes_B A : Mod\text{-}B \to \mathcal{D}(A \downarrow B)$ is an equivalence of categories;*
- (b) *${}_B A$ is faithfully flat.*

Proof. Apply Corollary 9.17 to the case "${}_B\Sigma_A$" $= {}_B A_A$, noting that by Example 9.20 ${}^{\mathbb{C}}(Mod\text{-}A) = \mathcal{D}(A \downarrow B)$ where $\mathbb{C} = - \otimes_A \mathcal{C} = - \otimes_A A \otimes_B A$. ∎

Remark 9.22. The inverse equivalence of the induction functor $- \otimes_B A : Mod\text{-}B \to \mathcal{D}(A \downarrow B) = {}^{\mathbb{C}}(Mod\text{-}A)$ where $\mathbb{C} = - \otimes_A \mathcal{C} = - \otimes_A A \otimes_B A$, maps a descent datum (M, δ) into $M^{co\delta} = \{m \in M \mid \delta(m) = m \otimes_B 1_A\} \simeq M^{co\mathcal{C}}$. Moreover, since we have an equivalence, in particular the counit is an isomorphism, so that the map

$$M^{co\delta} \otimes_B A \xrightarrow{\epsilon_M} M$$
$$m \otimes_B a \mapsto ma$$

is an isomorphism with inverse given by

$$\begin{array}{ccc} M & \to & M^{co\delta} \otimes_B A \\ m & \mapsto & \delta(m) . \end{array}$$

In fact we have $[(\delta \otimes_B A) \circ \delta](m) = m_{0_0} \otimes_B m_{0_1} \otimes_B m_1 = m_0 \otimes_B 1_A \otimes_B m_1$ so that $\delta(m) \in M^{co\delta} \otimes_B A$.

Now, we consider a particular case of the setting investigated above.

LEMMA 9.23. *Let $\mathcal{C}$ be an A-coring. Then A can be endowed with a right $\mathcal{C}$-comodule structure $\rho_A^{\mathcal{C}}$ if and only if $\mathcal{C}$ has a grouplike element, namely $\left[\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A)\right]$.*

Proof. Assume first that A has a right $\mathcal{C}$-comodule structure given by $\rho_A^{\mathcal{C}}$. We want to prove that $g = \left[\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A)\right]$ is a grouplike element for $\mathcal{C}$. First, from

$$g = \left[\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A)\right]$$

we deduce that

$$\rho_A^{\mathcal{C}}(1_A) = \left(l_{\mathcal{C}}^A\right)^{-1}(g) = 1_A \otimes_A g \tag{221}$$

Let us compute

$$\begin{aligned}
\Delta^{\mathcal{C}}\left(\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A)\right) &= \left(\Delta^{\mathcal{C}} \circ l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A) = \left[\left(l_{\mathcal{C}}^A \otimes_A \mathcal{C}\right) \circ \left(A \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}}\right](1_A) \\
&\overset{A\mathrm{right}\mathcal{C}\text{-com}}{=} \left[\left(l_{\mathcal{C}}^A \otimes_A \mathcal{C}\right) \circ \left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \rho_A^{\mathcal{C}}\right](1_A) = \left[\left(l_{\mathcal{C}}^A \otimes_A \mathcal{C}\right) \circ \left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right)\right]\left(\rho_A^{\mathcal{C}}(1_A)\right) \\
&\overset{(221)}{=} \left[\left(l_{\mathcal{C}}^A \otimes_A \mathcal{C}\right) \circ \left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right)\right](1_A \otimes_A g) = \left[\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right) \otimes_A \mathcal{C}\right](1_A \otimes_A g) \\
&= \left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A) \otimes_A g = g \otimes_A g.
\end{aligned}$$

Moreover

$$\begin{aligned}
\varepsilon^{\mathcal{C}}\left(\left(l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A)\right) &= \left(\varepsilon^{\mathcal{C}} \circ l_{\mathcal{C}}^A \circ \rho_A^{\mathcal{C}}\right)(1_A) \\
&= \left[l_A \circ \left(A \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}}\right](1_A) \overset{A\mathrm{right}\mathcal{C}\mathrm{com}}{=} 1_A.
\end{aligned}$$

Conversely, let us assume that $g \in \mathcal{C}$ is a grouplike element and let us define $\rho_A^{\mathcal{C}} : A \to A \otimes_A \mathcal{C}$ by setting

$$\rho_A^{\mathcal{C}}(a) = 1_A \otimes_A g \cdot a.$$

We have to check that it defines a $\mathcal{C}$-comodule structure on A. We compute, for every $a \in A$,

$$\begin{aligned}
\left[\left(A \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}}\right](a) &= \left(A \otimes_A \Delta^{\mathcal{C}}\right)(1_A \otimes_A g \cdot a) \overset{\Delta^{\mathcal{C}}A\mathrm{lin}}{=} 1_A \otimes_A g \otimes_A g \cdot a \\
&= \left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right)(1_A \otimes_A g \cdot a) = \left[\left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \rho_A^{\mathcal{C}}\right](a)
\end{aligned}$$

so that

$$\left(A \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}} = \left(\rho_A^{\mathcal{C}} \otimes_A \mathcal{C}\right) \circ \rho_A^{\mathcal{C}}.$$

We also have, for every $a \in A$,

$$\begin{aligned}
\left[r_A^A \circ \left(A \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}}\right](a) &= \left[r_A^A \circ \left(A \otimes_A \varepsilon^{\mathcal{C}}\right)\right](1_A \otimes_A g \cdot a) \\
&= r_A^A\left(1_A \otimes_A \varepsilon^{\mathcal{C}}(g \cdot a)\right) \overset{\varepsilon^{\mathcal{C}}A\mathrm{lin}}{=} 1_A \varepsilon^{\mathcal{C}}(g)\, a = a
\end{aligned}$$

so that

$$r_A^A \circ \left(A \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho_A^{\mathcal{C}} = \mathrm{Id}_A.$$

$\square$

Let $\mathcal{C}$ be an A-coring and assume that $g \in \mathcal{C}$ is a grouplike element. Then we can consider the map $\rho_A : A \to A \otimes_A \mathcal{C}$ defined by setting

$$\rho_A(a) = 1_A \otimes_A (g \cdot a) \text{ for every } a \in A.$$

We denote by $\mathbb{C} = - \otimes_A \mathcal{C}$. Then, by Lemma 9.23, (A, ρ_A) is a right $\mathcal{C}$-comodule and

$$\begin{aligned} &\mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((A,\rho_A)) \simeq \{b \in A \mid 1_A \otimes_A (g \cdot b) = b \otimes_A g\} \\ &= \{b \in A \mid 1_A \otimes_A (g \cdot b) = 1_A \otimes_A bg\} = \{b \in A \mid g \cdot b = bg\} = A^{co\mathcal{C}}. \end{aligned}$$

In this case the map

$$\mathbf{can} : \mathbf{\Sigma}^* \otimes_B \Sigma = A \otimes_B A \to \mathcal{C}$$

is defined by setting

$$\mathbf{can}(a \otimes_B a') = aga'$$

and $\mathcal{C}$ is called a *Galois coring* iff **can** is an isomorphism and $B = A^{co\mathcal{C}}$.

PROPOSITION 9.24. *Let $\mathcal{C}$ be an A-coring and assume that $g \in \mathcal{C}$ is a grouplike element. Let $B \subseteq A^{co\mathcal{C}}$. Then the following statements are equivalent:*

(a) *${}_A\mathcal{C}$ is flat and the functor $- \otimes_B A : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A) = (Mod\text{-}A)^{\mathcal{C}}$ is an equivalence of categories;*
(b) *the canonical map **can** $: A \otimes_B A \to \mathcal{C}$ is an isomorphism and ${}_BA$ is faithfully flat;*
(c) *${}_A\mathcal{C}$ is flat, A is a finitely generated projective generator of ${}^{\mathbb{C}}(Mod\text{-}A)$ and $\lambda : B \to T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((A,\rho_A)) = A^{co\mathcal{C}}$ is an isomorphism.*

THEOREM 9.25. [BRZ2002, Theorem 5.6]*Let $\mathcal{C}$ be an A-coring and assume that $g \in \mathcal{C}$ is a grouplike element.*

1) *If $\mathcal{C}$ is a Galois coring and ${}_{A^{co\mathcal{C}}}A$ is faithfully flat, then the functor $- \otimes_{A^{co\mathcal{C}}} A : Mod\text{-}A^{co\mathcal{C}} \to {}^{\mathbb{C}}(Mod\text{-}A) = (Mod\text{-}A)^{\mathcal{C}}$ is an equivalence of categories and ${}_A\mathcal{C}$ is flat.*
2) *If the functor $- \otimes_{A^{co\mathcal{C}}} A : Mod\text{-}A^{co\mathcal{C}} \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories, then $\mathcal{C}$ is a Galois coring.*
3) *If ${}_A\mathcal{C}$ is flat and the functor $- \otimes_{A^{co\mathcal{C}}} A : Mod\text{-}A^{co\mathcal{C}} \to {}^{\mathbb{C}}(Mod\text{-}A)$ is an equivalence of categories, then ${}_{A^{co\mathcal{C}}}A$ is faithfully flat.*

Proof. 1) follows from Proposition 9.24 $(b) \Rightarrow (a)$.
2) follows from Theorem 9.7.
3) follows from Proposition 9.24 $(a) \Rightarrow (b)$. □

COROLLARY 9.26. *Let $\mathcal{C}$ be an A-coring and assume that $g \in \mathcal{C}$ is a grouplike element. Assume that ${}_A\mathcal{C}$ is flat. Let $B \subseteq A^{co\mathcal{C}}$. Then the following statements are equivalent:*

(a) *the functor $- \otimes_B A : Mod\text{-}B \to {}^{\mathbb{C}}(Mod\text{-}A) = (Mod\text{-}A)^{\mathcal{C}}$ is an equivalence of categories;*
(b) *the canonical map **can** $: A \otimes_B A \to \mathcal{C}$ is an isomorphism and ${}_BA$ is faithfully flat;*
(c) *A is a finitely generated projective generator of ${}^{\mathbb{C}}(Mod\text{-}A)$ and $\lambda : B \to T = \mathrm{End}_{\mathbb{C}(Mod\text{-}A)}((A,\rho_A)) = A^{co\mathcal{C}}$ is an isomorphism.*

Definition 9.27. Let k be a commutative ring. An *entwining structure* (A, C, ψ) over k consists of

- $A = (A, m, u)$ a k-algebra
- $C = (C, \Delta, \varepsilon)$ a k-coalgebra
- $\psi : C \otimes A \to A \otimes C$ satisfying the following relations

(222)
$$(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) = \psi \circ (C \otimes m) \quad \text{and} \quad \psi \circ (C \otimes u) \circ r_C^{-1} = (u \otimes C) \circ l_C^{-1}$$

and

(223)
$$(\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A) = (A \otimes \Delta) \circ \psi \quad \text{and} \quad r_A \circ (A \otimes \varepsilon) \circ \psi = l_A \circ (\varepsilon \otimes A).$$

Notation 9.28. *Let (A, C, ψ) be an entwining structure over k. We will use sigma notation*
$$\psi(c \otimes a) = \sum a_\alpha \otimes c^\alpha$$
or with summation understood
$$\psi(c \otimes a) = a_\alpha \otimes c^\alpha.$$
Using this notation we can rewrite (222) *and* (223) *as follows*

$$(ab)_\alpha \otimes c^\alpha = a_\alpha b_\beta \otimes c^{\alpha\beta}, \quad \psi(c \otimes 1_A) = (1_A)_\alpha \otimes c^\alpha = 1_A \otimes c \tag{224}$$
$$a_\alpha \otimes c_1^\alpha \otimes c_2^\alpha = a_{\alpha\beta} \otimes c_1^\beta \otimes c_2^\alpha, \quad a_\alpha \varepsilon_C(c^\alpha) = \varepsilon_C(c)\, a \tag{225}$$

Moreover we set, for every $a, b, a', b' \in A$ and $c \in C$
$$a(b \otimes c) = ab \otimes c \qquad \text{and} \qquad (b \otimes c)\, b' = b\psi(c \otimes b') = bb'_\alpha \otimes c^\alpha.$$
We also define a map $\Delta^{\mathcal{C}} : \mathcal{C} = A \otimes C \to \mathcal{C} \otimes_A \mathcal{C} = A \otimes C \otimes_A A \otimes C$, by setting
$$\Delta^{\mathcal{C}}(a \otimes c) = a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}$$
and a map $\varepsilon^{\mathcal{C}} : \mathcal{C} \to A$, as follows
$$\varepsilon^{\mathcal{C}}(a \otimes c) = a\varepsilon(c).$$

Definition 9.29. Let (A, C, ψ) be an entwining structure. An *entwined (A, C, ψ)-module* is a triple (M, μ_M^A, ρ_M^C) where (M, μ_M^A) is a right A-module, (M, ρ_M^C) is a right C-comodule such that the structures are compatible
$$(\mu_M^A \otimes C) \circ (M \otimes \psi) \circ (\rho_M^C \otimes A) = \rho_M^C \circ \mu_M^A$$
i.e. for every $m \in M$ and for every $a \in A$ we have
$$\sum (ma)_0 \otimes (ma)_1 = \sum m_0 a_\alpha \otimes m_1^\alpha. \tag{226}$$
A *morphism of entwined modules* $f : (M, \mu_M^A, \rho_M^C) \to (N, \mu_N^A, \rho_N^C)$ is a morphism of right A-modules and a morphism of right C-comodules. We denote by $\mathcal{M}_A^C(\psi)$ the category of entwined (A, C, ψ)-modules.

Proposition 9.30 ([BrWi, 32.6 pg. 325]). *Let k be a commutative ring, let $A = (A, m, u)$ be a k-algebra and let $C = (C, \Delta, \varepsilon)$ be a k-coalgebra.*

1) *If (A, C, ψ) is an entwining structure, then, using the notations introduced in* (9.28), *$(\mathcal{C} = A \otimes C, \Delta^{\mathcal{C}}, \varepsilon^{\mathcal{C}})$ is an A-coring that will be called the* A-coring associated *to the entwining (A, C, ψ).*

2) *If $A \otimes C$ is an A-coring then (A, C, ψ) is an entwining structure where $\psi(c \otimes a) = (1_A \otimes c) \cdot a$.*
3) *If $\mathcal{C} = A \otimes C$ is the A-coring associated to the entwining (A, C, ψ), then $\mathcal{M}_A^{\mathcal{C}} = (Mod\text{-}A)^{\mathcal{C}} \simeq \mathcal{M}_A^C(\psi)$.*

Proof. 1) Let us define the A-bimodule structures on $\mathcal{C} = A \otimes C$. Set, for every $a, b, a', b' \in A$ and $c \in C$

$$a(b \otimes c) = ab \otimes c \qquad \text{and} \qquad (b \otimes c)b' = b\psi(c \otimes b') = bb'_\alpha \otimes c^\alpha$$

i.e.

$$a'(b \otimes c)b' = a'b\psi(c \otimes b') = a'bb'_\alpha \otimes c^\alpha.$$

We check the right module structure. Let us compute

$$(a \otimes c)(bb') = a\psi(c \otimes bb') = a[\psi(C \otimes m)(c \otimes b \otimes b')]$$

$$\overset{\psi\text{entw}}{=} a[((m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A))(c \otimes b \otimes b')]$$

$$= a[((m \otimes C) \circ (A \otimes \psi))(b_\alpha \otimes c^\alpha \otimes b')] = a\left[(m \otimes C)\left(b_\alpha \otimes b'_\beta \otimes (c^\alpha)^\beta\right)\right]$$

$$= a\left(b_\alpha b'_\beta \otimes (c^\alpha)^\beta\right) = ab_\alpha b'_\beta \otimes (c^\alpha)^\beta = (ab_\alpha \otimes c^\alpha)b' = ((a \otimes c)b)b'.$$

Let us calculate

$$(a \otimes c)1_A = a\psi(c \otimes 1_A) = a[(\psi \circ (C \otimes u))(c \otimes 1_k)]$$

$$= a\left[\left(\psi \circ (C \otimes u) \circ r_C^{-1}\right)(c)\right] \overset{\psi\text{entw}}{=} a\left[\left((u \otimes C) \circ l_C^{-1}\right)(c)\right]$$

$$= a[(u \otimes C)(1_k \otimes c)] = a(1_A \otimes c) = a \otimes c.$$

Now, let us check that it is a bimodule

$$(a'(a \otimes c))b' = a'a\psi(c \otimes b') = a'(a\psi(c \otimes b')) = a'((a \otimes c)b').$$

We define the coproduct on $\mathcal{C} = A \otimes C$, $\Delta^{\mathcal{C}} : \mathcal{C} = A \otimes C \to \mathcal{C} \otimes_A \mathcal{C} = A \otimes C \otimes_A A \otimes C$, by setting

$$\Delta^{\mathcal{C}}(a \otimes c) = a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}$$

where we denote $\Delta(c) = c_{(1)} \otimes c_{(2)}$. It is straightforward to check that it is left A-linear. Let us check it is also right A-linear. Let us compute

$$\Delta^{\mathcal{C}}((a \otimes c)b') = \Delta^{\mathcal{C}}(a\psi(c \otimes b')) = \Delta^{\mathcal{C}}(ab'_\alpha \otimes c^\alpha)$$

$$= ab'_\alpha \otimes c^\alpha_{(1)} \otimes_A 1_A \otimes c^\alpha_{(2)} \overset{(225)}{=} a(b'_\alpha)_\beta \otimes c^\beta_{(1)} \otimes_A 1_A \otimes c^\alpha_{(2)}$$

$$= a\psi\left(c_{(1)} \otimes b'_\alpha\right) \otimes_A 1_A \otimes c^\alpha_{(2)} \overset{(224)}{=} a\psi\left(c_{(1)} \otimes b'_\alpha\right) \otimes_A \psi\left(c^\alpha_{(2)} \otimes 1_A\right)$$

$$= \left(a \otimes c_{(1)}\right)b'_\alpha \otimes_A \psi\left(c^\alpha_{(2)} \otimes 1_A\right) = \left(a \otimes c_{(1)}\right) \otimes_A b'_\alpha\psi\left(c^\alpha_{(2)} \otimes 1_A\right)$$

$$= a \otimes c_{(1)} \otimes_A \left(b'_\alpha \otimes c^\alpha_{(2)}\right)1_A = a \otimes c_{(1)} \otimes_A b'_\alpha \otimes c^\alpha_{(2)}$$

$$= a \otimes c_{(1)} \otimes_A \psi\left(c_{(2)} \otimes b'\right) = a \otimes c_{(1)} \otimes_A \psi\left(c_{(2)} \otimes b'\right)$$

$$= a \otimes c_{(1)} \otimes_A \left(1_A \otimes c_{(2)}\right)b' = \left(a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}\right)b' = \left(\Delta^{\mathcal{C}}(a \otimes c)\right)b'$$

Let us check the coassociativity

$$\left(\Delta^{\mathcal{C}} \otimes \mathcal{C}\right)\left(a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}\right) = a \otimes c_{(1)(1)} \otimes_A 1_A \otimes c_{(1)(2)} \otimes_A 1_A \otimes c_{(2)}$$

$$\overset{\Delta\text{coass}}{=} a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)(1)} \otimes_A 1_A \otimes c_{(2)(2)}$$
$$= \left(\mathcal{C} \otimes \Delta^{\mathcal{C}}\right) \left(a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}\right).$$

We define the counit of $\mathcal{C}$, $\varepsilon^{\mathcal{C}} : \mathcal{C} \to A$, as follows

$$\varepsilon^{\mathcal{C}} (a \otimes c) = a\varepsilon (c).$$

It is straightforward to check that $\varepsilon^{\mathcal{C}}$ is left A-linear. Let us check it is also right A-linear. Let us compute

$$\varepsilon^{\mathcal{C}} ((a \otimes c) b') = \varepsilon^{\mathcal{C}} (a\psi (c \otimes b')) = \varepsilon^{\mathcal{C}} (ab'_\alpha \otimes c^\alpha) = ab'_\alpha \varepsilon (c^\alpha)$$
$$\overset{(225)}{=} a\varepsilon (c) b' = \left(\varepsilon^{\mathcal{C}} (a \otimes c)\right) b'.$$

Let now check the counitality

$$\left(r_{\mathcal{C}} \circ \left(\mathcal{C} \otimes \varepsilon^{\mathcal{C}}\right) \circ \Delta^{\mathcal{C}}\right) (a \otimes c) = \left(r_{\mathcal{C}} \circ \left(\mathcal{C} \otimes \varepsilon^{\mathcal{C}}\right)\right) \left(a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}\right)$$
$$= r_{\mathcal{C}} \left(a \otimes c_{(1)} \otimes_A \varepsilon \left(c_{(2)}\right)\right) = a \otimes c$$

and similarly

$$\left(l_{\mathcal{C}} \circ \left(\varepsilon^{\mathcal{C}} \otimes \mathcal{C}\right) \circ \Delta^{\mathcal{C}}\right) (a \otimes c) = \left(l_{\mathcal{C}} \circ \left(\varepsilon^{\mathcal{C}} \otimes \mathcal{C}\right)\right) \left(a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)}\right)$$
$$= l_{\mathcal{C}} \left(a\varepsilon \left(c_{(1)}\right) \otimes_A 1_A \otimes c_{(2)}\right) = a \otimes c$$

the right counitality is proved.

2) Assume that $A \otimes C$ is an A-coring with the coproduct and counit as above, i.e.

$$\Delta^{\mathcal{C}} (a \otimes c) = a \otimes c_{(1)} \otimes_A 1_A \otimes c_{(2)} \quad \text{and} \quad \varepsilon^{\mathcal{C}} (a \otimes c) = a\varepsilon (c).$$

Let us set

$$\psi (c \otimes a) = (1_A \otimes c) \cdot a.$$

We want to prove that ψ is an entwining for A and C. Since $A \otimes C$ is an A-coring, it is in particular a right A-module, so that

$$((a \otimes c) \cdot a') \cdot b' = (a \otimes c) \cdot (a'b')$$

i.e.

$$aa'_\alpha b'_\beta \otimes c^{\alpha\beta} = a (a'b')_\alpha \otimes c^\alpha \tag{227}$$

Let us compute, for every $a, b \in A$ and $c \in C$

$$[(m \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A)] (c \otimes a \otimes b)$$
$$= [(m \otimes C) \circ (A \otimes \psi)] (\psi (c \otimes a) \otimes b)$$
$$= [(m \otimes C) \circ (A \otimes \psi)] (a_\alpha \otimes c^\alpha \otimes b) = (m \otimes C) \left(a_\alpha \otimes b_\beta \otimes (c^\alpha)^\beta\right)$$
$$= a_\alpha b_\beta \otimes (c^\alpha)^\beta \overset{(227)}{=} (ab)_\alpha \otimes c^\alpha$$
$$= \psi (c \otimes ab) = [\psi \circ (C \otimes m)] (c \otimes a \otimes b)$$

and

$$\left[\psi \circ (C \otimes u) \circ r_C^{-1}\right] (c) = [\psi \circ (C \otimes u)] (c \otimes 1_k) = \psi (c \otimes 1_A)$$
$$= (1_A \otimes c) \cdot 1_A \overset{A\otimes C\text{right}A\text{mod}}{=} 1_A \otimes c = (u \otimes C) (1_k \otimes c) = \left[(u \otimes C) \circ l_C^{-1}\right] (c).$$

On the other hand, $\Delta^{\mathcal{C}}$ and $\varepsilon^{\mathcal{C}}$ are A-bilinear maps and in particular right A-module map so that we have

$$\Delta^{\mathcal{C}}((a \otimes c) b') = \left(\Delta^{\mathcal{C}}(a \otimes c)\right) b' \quad \text{and} \quad \varepsilon^{\mathcal{C}}((a \otimes c) b') = \left(\varepsilon^{\mathcal{C}}(a \otimes c)\right) b'$$

i.e.

$$ab'_{\alpha} \otimes c^{\alpha}_{(1)} \otimes_A 1_A \otimes c^{\alpha}_{(2)} = a \otimes c_{(1)} \otimes_A b'_{\alpha} \otimes c^{\alpha}_{(2)} \tag{228}$$

and

$$ab'_{\alpha} \varepsilon(c^{\alpha}) = a\varepsilon(c) b'. \tag{229}$$

Then, for every $c \in C$ and $a \in A$, we have

$$\begin{aligned} [(A \otimes \Delta) \circ \psi](c \otimes a) &= (A \otimes \Delta)(a_{\alpha} \otimes c^{\alpha}) = a_{\alpha} \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} \\ &\overset{(228)}{=} 1_A \otimes c_{(1)} a_{\alpha} \otimes c^{\alpha}_{(2)} = \psi\left(c_{(1)} \otimes a_{\alpha}\right) \otimes c^{\alpha}_{(2)} \\ &= (\psi \otimes C)\left(c_{(1)} \otimes a_{\alpha} \otimes c^{\alpha}_{(2)}\right) = [(\psi \otimes C) \circ (C \otimes \psi)]\left(c_{(1)} \otimes c_{(2)} \otimes a\right) \\ &= [(\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A)](c \otimes a) \end{aligned}$$

and

$$\begin{aligned} [r_A \circ (A \otimes \varepsilon) \circ \psi](c \otimes a) &= [r_A \circ (A \otimes \varepsilon)](a_{\alpha} \otimes c^{\alpha}) = a_{\alpha} \varepsilon(c^{\alpha}) \\ \overset{(229)}{=} \varepsilon(c) a &= [l_A \circ (\varepsilon \otimes A)](c \otimes a). \end{aligned}$$

3) Let $M \in \mathcal{M}^C_A(\psi)$, that is ρ^C_M is a right A-module map where $A \otimes C$ has a right A-module structure given by

$$(a \otimes c) b' = a\psi(c \otimes b') = ab'_{\alpha} \otimes c^{\alpha}.$$

Since ρ^C_M is a right A-module map, then the comodule structure given by the composite

$$\rho^{\mathcal{C}}_M : M \xrightarrow{\rho^C_M} M \otimes C \simeq M \otimes_A A \otimes C = M \otimes_A \mathcal{C}$$

is a right A-module map and thus $\left(M, \rho^{\mathcal{C}}_M\right)$ is a right $\mathcal{C}$-comodule. Conversely, let $\left(M, \rho^{\mathcal{C}}_M\right)$ be a right $\mathcal{C}$-comodule, then we can consider

$$\rho^C_M : M \xrightarrow{\rho^{\mathcal{C}}_M} M \otimes_A \mathcal{C} = M \otimes_A A \otimes C \simeq M \otimes C$$

as a right A-module map and thus we can see M as a (A, C, ψ)-entwined module. In fact, (226) just means that the map ρ^C_M is a right A-module map. □

THEOREM 9.31 ([SS, Lemma 1.7]). *Let C be a k-coalgebra and let A be a k-algebra such that (A, C, ψ) is an entwining structure. Then $\mathcal{C} = A \otimes C$ is an A-coring. Assume that ${}_A\mathcal{C}$ is flat (i.e. C is k-flat) and that $\left(A, m, \rho^C_A\right) \in \mathcal{M}^C_A(\psi)$. Let $B = A^{co\mathcal{C}}$. Then the following statements are equivalent:*

(a) *the functor $- \otimes_B A : Mod\text{-}B \to \mathcal{M}^C_A(\psi)$ is an equivalence of categories;*
(b) *the canonical map* can $: A \otimes_B A \to A \otimes C$ *is an isomorphism (i.e. $B \subseteq A$ is a C-Galois extension) and ${}_BA$ is faithfully flat.*

Proof. By Proposition 9.30 we know that $\mathcal{M}^{\mathcal{C}}_A = (Mod\text{-}A)^{\mathcal{C}} \simeq \mathcal{M}^C_A(\psi)$. By hypothesis $\left(A, m, \rho^C_A\right) \in \mathcal{M}^C_A(\psi)$ and thus, by Lemma 9.23, $\mathcal{C} = A \otimes C$ has a grouplike element, that is $\rho^C_A(1_A)$. Then we can apply Corollary 9.26 to conclude. □

Definition 9.32. Let $H=\left(H,\Delta^{H},\varepsilon^{H},m_{H},u_{H}\right)$ be a k-bialgebra, let $A=\left(\left(A,m_{A},u_{A}\right),\rho_{A}^{H}\right)$ be a right H-comodule algebra, let $D=\left(\left(D,\Delta^{D},\varepsilon^{D}\right),\mu_{D}^{H}\right)$ be a right H-module coalgebra and $g\in D$ be a grouplike element. We define the category of (D,A)-*Hopf modules* (or *Doi-Koppinen Hopf modules*) denoted by $\mathcal{M}_{A}^{D}\left(H\right)$, as follows:

- $M\in Ob\left(\mathcal{M}_{A}^{D}\left(H\right)\right)$ is a right D-comodule via ρ_{M}^{D}, a right A-module via μ_{M}^{A} such that for every $m\in M$ we have

$$\left(\rho_{M}^{D}\circ\mu_{M}^{A}\right)\left(m\otimes a\right)=\sum\mu_{M}^{A}\left(m_{0}\otimes a_{0}\right)\otimes\mu_{D}^{H}\left(m_{1}\otimes a_{1}\right)\tag{230}$$

  where $\rho_{M}^{D}\left(m\right)=\sum m_{0}\otimes m_{1}\in M\otimes D$ and $\rho_{A}^{H}\left(a\right)=\sum a_{0}\otimes a_{1}\in A\otimes H$, i.e. ρ_{M}^{D} is a morphism of right A-modules or equivalently, μ_{M}^{A} is a morphism of right D-comodules
- $f\in\mathrm{Hom}_{\mathcal{M}_{A}^{D}(H)}\left(M,N\right)$ is both a morphism of right D-comodules and a morphism of right A-modules.

Lemma 9.33. *Let $H=\left(H,\Delta^{H},\varepsilon^{H},m_{H},u_{H}\right)$ be a k-bialgebra, let $A=\left(\left(A,m_{A},u_{A}\right),\rho_{A}^{H}\right)$ be a right H-comodule algebra, let $D=\left(\left(D,\Delta^{D},\varepsilon^{D}\right),\mu_{D}^{H}\right)$ be a right H-module coalgebra and $g\in D$ be a grouplike element. Then $A\in\mathcal{M}_{A}^{D}\left(H\right)$ and A is a right D-comodule algebra.*

Proof. We denote $\mu_{D}^{H}\left(d\otimes h\right)=d\cdot h$. First of all we want to prove that A is a right D-comodule. In fact we can consider

$$\rho_{A}^{D}:A\rightarrow A\otimes D$$

defined by setting

$$\rho_{A}^{D}\left(a\right)=a_{0}\otimes g\cdot a_{1}.$$

Let us compute, for every $a\in A$,

$$\begin{aligned}\left[\left(A\otimes\Delta^{D}\right)\circ\rho_{A}^{D}\right]\left(a\right)&=\left(A\otimes\Delta^{D}\right)\left(a_{0}\otimes g\cdot a_{1}\right)=a_{0}\otimes\left(g\cdot a_{1}\right)_{(1)}\otimes\left(g\cdot a_{1}\right)_{(2)}\\&\overset{D\mathrm{is}H\mathrm{modcoalg}}{=}a_{0}\otimes g_{(1)}\cdot a_{1_{(1)}}\otimes g_{(2)}\cdot a_{1_{(2)}}\overset{A\text{ is }H\mathrm{com}}{=}a_{0_{0}}\otimes g_{(1)}\cdot a_{0_{1}}\otimes g_{(2)}\cdot a_{1}\\&\overset{g\mathrm{grouplike}}{=}a_{0_{0}}\otimes g\cdot a_{0_{1}}\otimes g\cdot a_{1}=\left(\rho_{A}^{D}\otimes D\right)\left(a_{0}\otimes g\cdot a_{1}\right)=\left[\left(\rho_{A}^{D}\otimes D\right)\circ\rho_{A}^{D}\right]\left(a\right)\end{aligned}$$

so that

$$\left(A\otimes\Delta^{D}\right)\circ\rho_{A}^{D}=\left(\rho_{A}^{D}\otimes D\right)\circ\rho_{A}^{D}.$$

We compute, for every $a\in A$,

$$\begin{aligned}\left[r_{A}\circ\left(A\otimes\varepsilon^{D}\right)\circ\rho_{A}^{D}\right]\left(a\right)&=\left[r_{A}\circ\left(A\otimes\varepsilon^{D}\right)\right]\left(a_{0}\otimes g\cdot a_{1}\right)\\=r_{A}\left(a_{0}\otimes\varepsilon^{D}\left(ga_{1}\right)\right)&\overset{D\mathrm{is}H\mathrm{modcoalg}}{=}r_{A}\left(a_{0}\otimes\varepsilon^{D}\left(g\right)\varepsilon^{H}\left(a_{1}\right)\right)\\=a_{0}\varepsilon^{D}\left(g\right)\varepsilon^{H}\left(a_{1}\right)&\overset{g\mathrm{grouplike}}{=}a1_{k}=a\end{aligned}$$

so that

$$r_{A}\circ\left(A\otimes\varepsilon^{D}\right)\circ\rho_{A}^{D}=\mathrm{Id}_{A}.$$

Note that A is a right A-module via m_{A}. It remains to prove (230). Recall that

$$\rho_{A}^{D}\left(a\right)=\sum a_{0}\otimes g\cdot a_{1}\in A\otimes D$$

so that we have

$$\left(\rho_A^D \circ \mu_A^A\right)(a \otimes b) = \left(\rho_A^D \circ m_A\right)(a \otimes b) = \rho_A^D(ab)$$
$$= \sum (ab)_0 \otimes \mu_D^H\left(g \otimes (ab)_1\right) \overset{A \mathrm{is} H \mathrm{comalg}}{=} \sum a_0 b_0 \otimes g \cdot (a_1 b_1)$$
$$\overset{D \mathrm{is} H \mathrm{modcoalg}}{=} \sum a_0 b_0 \otimes (g \cdot a_1) \cdot b_1 = \sum a_0 b_0 \otimes \mu_D^H\left(\mu_D^H(g \otimes a_1) \otimes b_1\right)$$
$$= \sum m_A(a_0 \otimes b_0) \otimes \mu_D^H\left(\mu_D^H(g \otimes a_1) \otimes b_1\right)$$
$$= \sum \mu_A^A(a_0 \otimes b_0) \otimes \mu_D^H(g \cdot a_1 \otimes b_1).$$

Then we deduce that $A \in \mathcal{M}_A^D(H)$. Note that this last computation says that m_A is a morphism of right D-comodules. It remains to prove that u_A is also a morphism of right D-comodules. Let us compute

$$\left(\rho_A^D \circ u_A\right)(1_k) = \rho_A^D(1_A) = (1_A)_0 \otimes g \cdot (1_A)_1$$
$$\overset{A \mathrm{is} H \mathrm{comalg}}{=} 1_A \otimes g \cdot 1_H \overset{D \mathrm{is} H \mathrm{modcoalg}}{=} 1_A \otimes g$$
$$= (u_A \otimes D)(1_k \otimes g) = \left[(u_A \otimes D) \circ \rho_k^D\right](1_k)$$

so that we conclude that $\left((A, m_A, u_A), \rho_A^D\right)$ is a right D-comodule algebra. $\square$

Theorem 9.34 ([MeZu, Theorem 3.29 $(a) \Leftrightarrow (f)$]). *Let $H = \left(H, \Delta^H, \varepsilon^H, m_H, u_H\right)$ be a k-bialgebra, let $A = \left((A, m_A, u_A), \rho_A^H\right)$ be a right H-comodule algebra, let $D = \left(\left(D, \Delta^D, \varepsilon^D\right), \mu_D^H\right)$ be a right H-module coalgebra and let $g \in D$ be a grouplike element. Then $\left((A, m_A, u_A), \rho_A^D\right)$ is a right D-comodule algebra and $\mathcal{D} = A \otimes D$ is an A-coring. Assume that ${}_A\mathcal{D}$ is flat (i.e. D is k-flat). Let $B = A^{co\mathcal{D}}$. Then the following statements are equivalent:*

- (a) *the functor $- \otimes_B A : Mod\text{-}B \to \mathcal{M}_A^D(H)$ is an equivalence of categories;*
- (b) *the canonical map* $\mathrm{can} : A \otimes_B A \to A \otimes D$ *is an isomorphism (i.e. $B \subseteq A$ is a D-Galois extension) and ${}_BA$ is faithfully flat.*

Proof. We set $\mu_D^H(d \otimes h) = d \cdot h$. By Lemma 9.33, we know that $\left((A, m_A, u_A), \rho_A^D\right)$ is a right D-comodule algebra. First of all we want to prove that $\mathcal{D} = A \otimes D$ is an A-coring. Let us consider $\psi : D \otimes A \to A \otimes D$ defined by setting, for every $a \in A$ and $d \in D$,

$$\psi(d \otimes a) = a_0 \otimes d \cdot a_1$$

where we denote $\rho_A^H(a) = a_0 \otimes a_1$. Let us prove that (A, D, ψ) is then an entwining structure over k. We have to prove (222). Let us compute, for every $a, b \in A, d \in D$,

$$[(m_A \otimes D) \circ (A \otimes \psi) \circ (\psi \otimes A)](d \otimes a \otimes b)$$
$$= [(m_A \otimes D) \circ (A \otimes \psi)](a_0 \otimes d \cdot a_1 \otimes b)$$
$$= (m_A \otimes D)(a_0 \otimes b_0 \otimes (d \cdot a_1) \cdot b_1) = a_0 b_0 \otimes (d \cdot a_1) \cdot b_1$$
$$\overset{\left(D, \mu_D^H\right) H \mathrm{modcoalg}}{=} a_0 b_0 \otimes d \cdot (a_1 b_1) \overset{\left(A, \rho_A^H\right) H\text{-com alg}}{=} (ab)_0 \otimes d \cdot (ab)_1$$
$$= \psi(d \otimes ab) = [\psi \circ (D \otimes m_A)](d \otimes a \otimes b)$$

so that

$$(m_A \otimes D) \circ (A \otimes \psi) \circ (\psi \otimes A) = \psi \circ (D \otimes m_A).$$

Let us compute, for every $d \in D$,

$$\left[\psi \circ (D \otimes u_A) \circ r_D^{-1}\right](d) = [\psi \circ (D \otimes u_A)](d \otimes 1_k) = \psi(d \otimes 1_A)$$
$$= (1_A)_0 \otimes d \cdot (1_A)_1 \overset{(A,\rho_A^H) H\text{-com alg}}{=} 1_A \otimes d \cdot 1_H \overset{D \text{is} H \text{modcoal}}{=} 1_A \otimes d$$
$$= (u_A \otimes D)(1_k \otimes d) = \left[(u_A \otimes D) \circ l_D^{-1}\right](d)$$

so that we have

$$\psi \circ (D \otimes u_A) \circ r_D^{-1} = (u_A \otimes D) \circ l_D^{-1}.$$

Let us prove (223). Let us compute, for every $a \in A, d \in D$

$$\left[(\psi \otimes D) \circ (D \otimes \psi) \circ \left(\Delta^D \otimes A\right)\right](d \otimes a)$$
$$= [(\psi \otimes D) \circ (D \otimes \psi)]\left(d_{(1)} \otimes d_{(2)} \otimes a\right)$$
$$= (\psi \otimes D)\left(d_{(1)} \otimes a_0 \otimes d_{(2)} \cdot a_1\right) = a_{0_0} \otimes d_{(1)} \cdot a_{0_1} \otimes d_{(2)} \cdot a_1$$
$$\overset{A \text{ is} H \text{comod}}{=} a_0 \otimes d_{(1)} \cdot a_{1_{(1)}} \otimes d_{(2)} \cdot a_{1_{(2)}} \overset{D \text{is} H \text{modcoalg}}{=} a_0 \otimes (d \cdot a_1)_{(1)} \otimes (d \cdot a_1)_{(2)}$$
$$= \left(A \otimes \Delta^D\right)(a_0 \otimes d \cdot a_1) = \left[\left(A \otimes \Delta^D\right) \circ \psi\right](d \otimes a)$$

so that we get

$$(\psi \otimes D) \circ (D \otimes \psi) \circ \left(\Delta^D \otimes A\right) = \left(A \otimes \Delta^D\right) \circ \psi$$

and

$$\left[r_A \circ \left(A \otimes \varepsilon^D\right) \circ \psi\right](d \otimes a) = \left[r_A \circ \left(A \otimes \varepsilon^D\right)\right](a_0 \otimes d \cdot a_1)$$
$$= r_A\left(a_0 \otimes \varepsilon^D(d \cdot a_1)\right) \overset{D \text{is} H \text{modcoal}}{=} r_A\left(a_0 \otimes \varepsilon^D(d)\, \varepsilon^H(a_1)\right)$$
$$= a\varepsilon^D(d) = \varepsilon^D(d)\, a = l_A\left(\varepsilon^D(d) \otimes a\right) = \left[l_A \circ \left(\varepsilon^D \otimes A\right)\right](d \otimes a)$$

so that we get

$$r_A \circ \left(A \otimes \varepsilon^D\right) \circ \psi = l_A \circ \left(\varepsilon^D \otimes A\right).$$

Then by Proposition 9.30, $\left(\mathcal{D} = A \otimes D, \Delta^{\mathcal{D}}, \varepsilon^{\mathcal{D}}\right)$ is the A-coring associated to the entwining (A, D, ψ) and $\mathcal{M}_A^{\mathcal{D}} = (Mod\text{-}A)^{\mathcal{D}} \simeq \mathcal{M}_A^D(\psi)$. Note that $M \in \mathcal{M}_A^D(\psi)$, is such that $\left(M, \mu_M^A\right)$ is a right A-module, $\left(M, \rho_M^D\right)$ is a right D-comodule satisfying

$$\left(\mu_M^A \otimes D\right) \circ (M \otimes \psi) \circ \left(\rho_M^D \otimes A\right) = \rho_M^D \circ \mu_M^A$$

i.e. for every $m \in M$ and for every $a \in A$

$$\left(\rho_M^D \circ \mu_M^A\right)(m \otimes a) = \mu_M^A(m_0 \otimes a_0) \otimes \mu_D^H(m_1 \otimes a_1)$$

which is exactly the condition (230) for $M \in \mathcal{M}_A^D(H)$. Since morphisms in both categories $\mathcal{M}_A^D(\psi)$ and $\mathcal{M}_A^D(H)$ are right A-linear and right D-colinear morphisms we deduce that

$$\mathcal{M}_A^D(\psi) \simeq \mathcal{M}_A^D(H).$$

Since by Lemma 9.33 $A \in \mathcal{M}_A^D(H) \simeq \mathcal{M}_A^D(\psi)$, we can apply Theorem 9.31 to the case "C" $= D$ and "$\mathcal{M}_A^C(\psi)$" $= \mathcal{M}_A^D(\psi) \simeq \mathcal{M}_A^D(H)$. $\square$

Corollary 9.35 ([Schn1, Theorem I (2) $\Leftrightarrow$ (4)]). *Let* $H = \left(H, \Delta^H, \varepsilon^H, m_H, u_H\right)$ *be a* k*-bialgebra and let* $\left((A, m_A, u_A), \rho_A^H\right)$ *be a right* H*-comodule algebra. Then* $\mathcal{H} = A \otimes H$ *is an* A*-coring. Assume that* ${}_A\mathcal{H}$ *is flat (i.e.* H *is* k*-flat). Let* $B = A^{co\mathcal{H}}$*. Then the following statements are equivalent:*

(a) *the functor* $- \otimes_B A : Mod\text{-}B \to \mathcal{M}_A^H$ *is an equivalence of categories;*
(b) *the canonical map* $\mathrm{can} : A \otimes_B A \to A \otimes H$ *is an isomorphism (i.e.* $B \subseteq A$ *is an* H*-Galois extension) and* ${}_BA$ *is faithfully flat.*

Proof. We can apply Theorem 9.34 to the case "D" $= H$ so that $\mathcal{M}_A^D(H) = \mathcal{M}_A^H$. □

10. Bicategories

In this last part we will change some notations to be more clear and to give more evidence to a new product we introduce here.

Let C be a bicategory. For every 0-cell X in C, we denote by $1_X : X \to X$ the identity 1-cell over X. For every 1-cell A in C, we denote by $1_A : A \to A$ the identity 2-cell over A. We will use juxtaposition when we compose 2-cells vertically and we will denote by $\cdot$ the horizontal composition of 1-cells and 2-cells.

Let us assume that C is a bicategory with completeness requirement (all the categories $\mathsf{C}((X, A), (Y, B))$ have coequalizers which are preserved by composition with 1-cells).

We keep denoting by (A, m_A, u_A) a monad with its multiplication and unit.

We now want to define the 2-category $\mathbf{Mnd}(\mathsf{C})$ following the definition given in [St]. For simplicity we will always assume to work with a 2-category C even if *one* can prove similar results for an arbitrary bicategory.

Definition 10.1. Let C be a 2-category. A *monad* in C is a pair (X, A) where X is an object of C, $A : X \to X$ is a 1-cell in C together with 2-cells $m_A : 1_A \cdot 1_A \to 1_A$ and $u_A : 1_X \to 1_A$ satisfying associativity and unitality conditions, i.e.

$$m_A(1_A \cdot m_A) = m_A(m_A \cdot 1_A) \tag{231}$$

$$m_A(u_A \cdot 1_A) = 1_A = m_A(1_A \cdot u_A). \tag{232}$$

Definition 10.2. Let C be a 2-category and let $(X, A), (Y, B)$ be monads in C. A *monad functor* in C is a pair $(Q, \phi) : (X, A) \to (Y, B)$ where $Q : X \to Y$ is a 1-cell and $\phi : B \cdot Q \to Q \cdot A$ satisfying the following conditions

$$(1_Q \cdot m_A)(\phi \cdot 1_A)(1_B \cdot \phi) = \phi(m_B \cdot 1_Q) \tag{233}$$

$$\phi(u_B \cdot 1_Q) = 1_Q \cdot u_A. \tag{234}$$

Definition 10.3. Let C be a 2-category, let $(X, A), (Y, B)$ be monads in C and let $(Q, \phi), (Q', \phi') : (X, A) \to (Y, B)$ be monad functors. A *monad functor transformation* $\sigma : (Q, \phi) \to (Q', \phi')$ in C is a 2-cell $\sigma : Q \to Q'$ such that

$$\phi'(1_B \cdot \sigma) = (\sigma \cdot 1_A)\phi. \tag{235}$$

Definition 10.4. The 2-category $\mathbf{Mnd}(\mathsf{C})$ consists of

- Objects: monads in C
- 1-cells: monad functors in C

- 2-cells: monad functor transformations in C.

REMARK 10.5. We denote by $(X, 1_X)$ in C the trivial monad on the object X with trivial multiplication and unit $m_{1_X} : 1_{1_X} \cdot 1_{1_X} \to 1_{1_X}$ and $u_{1_X} : 1_X \to 1_{1_X}$.

DEFINITION 10.6. Let C, C' be two 2-categories and let $G : \mathsf{C} \to \mathsf{C}'$ be a pseudofunctor. Then the pseudofunctor $\mathbf{Mnd}(G) : \mathbf{Mnd}(\mathsf{C}) \to \mathbf{Mnd}(\mathsf{C}')$ is defined as follows:

- $\mathbf{Mnd}(G)(X, A) = (G(X), G(A))$
- $\mathbf{Mnd}(G)(Q, \phi) = (G(Q), G(\phi))$
- $\mathbf{Mnd}(G)(\sigma) = G(\sigma)$.

REMARK 10.7. Note that $\mathbf{Cmd}(\mathsf{C}) = \mathbf{Mnd}(\mathsf{C}_*)$ where C_* is the dual reversing 2-cells of C.

11. CONSTRUCTION OF $BIM(\mathsf{C})$

The idea of defining this bicategory goes back to the strict monoidal category of balanced bimodule functors that we defined in Subsection 3.2. We observed that, considering bimodule functors with respect to the same monad on both sides, we have a unit and a composition, so that they form a strict monoidal category. In the case we consider a bimodule with respect to two different monads, the unit object fails and the composition between them is no longer inside the class of objects of the category. The way to solve this problem is to look at balanced bimodule functors as 0-cells of a bicategory, changing the definition of their product.

DEFINITION 11.1. Let X be a 0-cell and let (Y, B) be a monad in C. A *left B-module* in C (or simply a *left B-module*) is a pair (Q, λ_Q) where $Q : X \to Y$ is a 1-cell and $\lambda_Q : B \cdot Q \to Q$ is a 2-cell in C, satisfying the associativity and unitality properties with respect to the monad B, i.e.

$$\lambda_Q(m_B \cdot 1_Q) = \lambda_Q(1_B \cdot \lambda_Q) \text{ and } \lambda_Q(u_B \cdot 1_Q) = 1_Q.$$

DEFINITION 11.2. Let (X, A) and $(Y, 1_Y)$ be monads in C. A *right A-module* in C (or simply a *right A-module*) is a monad functor in C_*, i.e. a 1-cell $Q : X \to Y$ and a 2-cell $\rho_Q : Q \cdot A \to 1_Y \cdot Q = Q$ in C, satisfying the associativity and unitality properties with respect to the monad A, i.e.

$$\rho_Q(1_Q \cdot m_A) = \rho_Q(\rho_Q \cdot 1_A) \text{ and } \rho_Q(1_Q \cdot u_A) = 1_Q.$$

DEFINITION 11.3. Let (X, A) and (Y, B) be monads in C. A *B-A-bimodule* in C (or simply a *B-A-bimodule*) is a triple (Q, λ_Q, ρ_Q) where

- (Q, λ_Q) is a left B-module in C
- (Q, ρ_Q) is a right A-module in C
- the compatibility condition holds

$$\lambda_Q(1_B \cdot \rho_Q) = \rho_Q(\lambda_Q \cdot 1_A).$$

LEMMA 11.4. *Let $(X, A), (Y, B)$ be monads in C and let (Q, λ_Q) be a left A-module and (Q, ρ_Q) be a right B-module. Then $(Q, \lambda_Q) = \mathrm{Coequ}_{\mathsf{C}}(m_A \cdot 1_Q, 1_A \cdot \lambda_Q)$ and $(Q, \rho_Q) = \mathrm{Coequ}_{\mathsf{C}}(1_Q \cdot m_B, \rho_Q \cdot 1_B)$.*

Proof. We will only prove the statement for the left module. Similarly can be proved the other one. Since λ_Q is associative, we deduce that

$$\lambda_Q\left(m_A \cdot 1_Q\right) = \lambda_Q\left(1_A \cdot \lambda_Q\right).$$

Now, assume that (S, σ) is such that

$$\sigma\left(m_A \cdot 1_Q\right) = \sigma\left(1_A \cdot \lambda_Q\right).$$

Then we have

$$\begin{aligned}\sigma\left(u_A \cdot 1_Q\right)\lambda_Q &\overset{u_A}{=} \sigma\left(1_A \cdot \lambda_Q\right)\left(u_A \cdot 1_A \cdot 1_Q\right)\\ &\overset{\text{prop}\sigma}{=} \sigma\left(m_A \cdot 1_Q\right)\left(u_A \cdot 1_A \cdot 1_Q\right) \overset{A\text{monad}}{=} \sigma.\end{aligned}$$

Moreover, since λ_Q is epi, we conclude that the 2-cell $\sigma\left(u_A \cdot 1_Q\right)$ is unique with respect to the property

$$\sigma\left(u_A \cdot 1_Q\right)\lambda_Q = \sigma$$

so that

$$\left(Q, \lambda_Q\right) = \mathrm{Coequ}_{\mathsf{C}}\left(m_A \cdot 1_Q, 1_A \cdot \lambda_Q\right).$$

□

Proposition 11.5. *Let $(X, A), (Y, B)$ and (W, C) be monads in C and let $Q : Y \to X$ and $Q' : W \to Y$ be respectively a A-B-bimodule with (Q, λ_Q, ρ_Q) and a B-C-bimodule in C with $(Q', \lambda_{Q'}, \rho_{Q'})$. Then $(Q \bullet_B Q', p_{Q,Q'}) = \mathrm{Coequ}_{\mathsf{C}}\left(\rho_Q \cdot 1_{Q'}, 1_Q \cdot \lambda_{Q'}\right)$ is a A-C-bimodule in C via the actions $\lambda_{Q\bullet_B Q'}$ and $\rho_{Q\bullet_B Q'}$ uniquely determined by*

$$\lambda_{Q\bullet_B Q'}\left(1_A \cdot p_{Q,Q'}\right) = p_{Q,Q'}\left(\lambda_Q \cdot 1_{Q'}\right) \tag{236}$$

and

$$\rho_{Q\bullet_B Q'}\left(p_{Q,Q'} \cdot 1_C\right) = p_{Q,Q'}\left(1_Q \cdot \rho_{Q'}\right). \tag{237}$$

Proof. Let us define the bimodule structures on $Q \bullet_B Q'$. Let us consider the following diagram

$$\begin{tikzcd}
Q \cdot B \cdot Q' \cdot C \arrow[r, shift left, "\rho_Q \cdot 1_{Q'} \cdot 1_C"] \arrow[r, shift right, "1_Q \cdot \lambda_{Q'} \cdot 1_C"'] \arrow[d, "1_Q \cdot 1_B \cdot \rho_{Q'}"'] & Q \cdot Q' \cdot C \arrow[r, "p_{Q,Q'} \cdot 1_C"] \arrow[d, "1_Q \cdot \rho_{Q'}"] & Q \bullet_B Q' \cdot C \arrow[d, dotted, "\rho_{Q\bullet_B Q'}"] \\
Q \cdot B \cdot Q' \arrow[r, shift left, "\rho_Q \cdot 1_{Q'}"] \arrow[r, shift right, "1_Q \cdot \lambda_{Q'}"'] & Q \cdot Q' \arrow[r, "p_{Q,Q'}"'] & Q \bullet_B Q'
\end{tikzcd}$$

Note that the left square serially commutes. In fact we have

$$\left(1_Q \cdot \rho_{Q'}\right)\left(\rho_Q \cdot 1_{Q'} \cdot 1_C\right) \overset{\rho_Q}{=} \left(\rho_Q \cdot 1_{Q'}\right)\left(1_Q \cdot 1_B \cdot \rho_{Q'}\right)$$

and

$$\left(1_Q \cdot \rho_{Q'}\right)\left(1_Q \cdot \lambda_{Q'} \cdot 1_C\right) \overset{Q\text{bim}}{=} \left(1_Q \cdot \lambda_{Q'}\right)\left(1_Q \cdot 1_B \cdot \rho_{Q'}\right).$$

Therefore, we get

$$p_{Q,Q'}\left(1_Q \cdot \rho_{Q'}\right)\left(\rho_Q \cdot 1_{Q'} \cdot 1_C\right) = p_{Q,Q'}\left(1_Q \cdot \rho_{Q'}\right)\left(1_Q \cdot \lambda_{Q'} \cdot 1_C\right)$$

and by the universal property of the coequalizer $(Q \bullet_B Q' \cdot C, p_{Q,Q'} \cdot 1_C) = \mathrm{Coequ}_{\mathsf{C}}\left(\rho_Q \cdot 1_{Q'} \cdot 1_C, 1_Q \cdot \lambda_{Q'} \cdot 1_C\right)$, there exists a unique 2-cell $\rho_{Q\bullet_B Q'} : Q \bullet_B Q' \cdot C \to Q \bullet_B Q'$ such that

$$\rho_{Q\bullet_B Q'}\left(p_{Q,Q'} \cdot 1_C\right) = p_{Q,Q'}\left(1_Q \cdot \rho_{Q'}\right).$$

We now want to prove that $\rho_{Q\cdot_B Q'}$ defines a structure of right C-module. Let us consider the following diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
Q\cdot B\cdot Q'\cdot C\cdot C \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}\cdot 1_C\cdot 1_C"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}\cdot 1_C\cdot 1_C"'] \arrow[d, shift right, "1_Q\cdot 1_B\cdot\rho_{Q'}\cdot 1_C"'] \arrow[d, shift left, "1_Q\cdot 1_B\cdot 1_{Q'}\cdot m_C"] & Q\cdot Q'\cdot C\cdot C \arrow[r, "p_{Q,Q'}\cdot 1_C\cdot 1_C"] \arrow[d, shift right, "1_Q\cdot\rho_{Q'}\cdot 1_C"'] \arrow[d, shift left, "1_Q\cdot 1_{Q'}\cdot m_C"] & Q\bullet_B Q'\cdot C\cdot C \arrow[d, shift right, "\rho_{Q\bullet_B Q'}\cdot 1_C"'] \arrow[d, shift left, "1_{Q\bullet_B Q'}\cdot m_C"] \\
Q\cdot B\cdot Q'\cdot C \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}\cdot 1_C"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}\cdot 1_C"'] \arrow[d, "1_Q\cdot 1_B\cdot\rho_{Q'}"'] & Q\cdot Q'\cdot C \arrow[r, "p_{Q,Q'}\cdot 1_C"] \arrow[d, "1_Q\cdot\rho_{Q'}"] & Q\bullet_B Q'\cdot C \arrow[d, "\rho_{Q\bullet_B Q'}"] \\
Q\cdot B\cdot Q' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}"'] & Q\cdot Q' \arrow[r, "p_{Q,Q'}"] & Q\bullet_B Q'
\end{tikzcd}$$

The diagram serially commutes and since the rows and the first two columns are coequalizers, also the third column is a coequalizer. In particular,

$$\rho_{Q\bullet_B Q'}\left(\rho_{Q\bullet_B Q'}\cdot 1_C\right)=\rho_{Q\bullet_B Q'}\left(1_{Q\bullet_B Q'}\cdot m_C\right)$$

i.e. $\rho_{Q\bullet_B Q'}$ is associative. Now, we also have that the following diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
Q\cdot B\cdot Q'\cdot C \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}\cdot 1_C"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}\cdot 1_C"'] \arrow[d, shift right, "1_Q\cdot 1_B\cdot\rho_{Q'}"'] & Q\cdot Q'\cdot C \arrow[r, "p_{Q,Q'}\cdot 1_C"] \arrow[d, shift right, "1_Q\cdot\rho_{Q'}"'] & Q\bullet_B Q'\cdot C \arrow[d, shift right, "\rho_{Q\bullet_B Q'}"'] \\
Q\cdot B\cdot Q' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}"'] \arrow[u, shift right, "1_Q\cdot 1_B\cdot 1_{Q'}\cdot u_C"'] & Q\cdot Q' \arrow[r, "p_{Q,Q'}"] \arrow[u, shift right, "1_Q\cdot 1_{Q'}\cdot u_C"'] & Q\bullet_B Q' \arrow[u, shift right, "1_{Q\bullet_B Q'}\cdot u_C"']
\end{tikzcd}$$

serially commutes. In particular

$$\left(1_{Q\bullet_B Q'}\cdot u_C\right)p_{Q,Q'}=\left(p_{Q,Q'}\cdot 1_C\right)\left(1_Q\cdot 1_{Q'}\cdot u_C\right)$$

so that

$$\begin{aligned}\rho_{Q\bullet_B Q'}\left(1_{Q\bullet_B Q'}\cdot u_C\right)p_{Q,Q'}&=\rho_{Q\bullet_B Q'}\left(p_{Q,Q'}\cdot 1_C\right)\left(1_Q\cdot 1_{Q'}\cdot u_C\right)\\&=p_{Q,Q'}\left(1_Q\cdot\rho_{Q'}\right)\left(1_Q\cdot 1_{Q'}\cdot u_C\right)\overset{Q'\text{mod}}{=}p_{Q,Q'}\end{aligned}$$

and since $p_{Q,Q'}$ is an epimorphism, we get that

$$\rho_{Q\bullet_B Q'}\left(1_{Q\bullet_B Q'}\cdot u_C\right)=1_{Q\bullet_B Q'}$$

so that $\rho_{Q\bullet_B Q'}$ is also unital. Therefore $(Q\bullet_B Q',\rho_{Q\bullet_B Q'})$ is a right C-module. Similarly, let us consider the following diagram

$$\begin{tikzcd}[column sep=large]
A\cdot Q\cdot B\cdot Q' \arrow[r, shift left, "1_A\cdot\rho_Q\cdot 1_{Q'}"] \arrow[r, shift right, "1_A\cdot 1_Q\cdot\lambda_{Q'}"'] \arrow[d, "\lambda_Q\cdot 1_B\cdot 1_{Q'}"'] & A\cdot Q\cdot Q' \arrow[r, "1_A\cdot p_{Q,Q'}"] \arrow[d, "\lambda_Q\cdot 1_{Q'}"] & A\cdot Q\bullet_B Q' \arrow[d, dotted, "\lambda_{Q\bullet_B Q'}"] \\
Q\cdot B\cdot Q' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}"'] & Q\cdot Q' \arrow[r, "p_{Q,Q'}"] & Q\bullet_B Q'
\end{tikzcd}$$

Since we are assuming that the coequalizers are preserved by the composition with any 1-cell, both the rows are coequalizers and the left square serially commutes. In fact

$$(\lambda_Q \cdot 1_{Q'}) (1_A \cdot \rho_Q \cdot 1_{Q'}) \overset{Q\text{bim}}{=} (\rho_Q \cdot 1_{Q'}) (\lambda_Q \cdot 1_B \cdot 1_{Q'})$$

$$(\lambda_Q \cdot 1_{Q'}) (1_A \cdot 1_Q \cdot \lambda_{Q'}) \overset{\lambda_Q}{=} (1_Q \cdot \lambda_{Q'}) (\lambda_Q \cdot 1_B \cdot 1_{Q'}) .$$

By the universal property of the coequalizer $(Q \bullet_B Q', p_{Q,Q'}) = \mathrm{Coequ}_{\mathsf{C}} (\rho_Q \cdot 1_{Q'}, 1_Q \cdot \lambda_{Q'})$, there exists a unique 2-cell $\lambda_{Q \bullet_B Q'} : A \cdot Q \bullet_B Q' \to Q \bullet_B Q'$ such that

$$\lambda_{Q \bullet_B Q'} (1_A \cdot p_{Q,Q'}) = p_{Q,Q'} (\lambda_Q \cdot 1_{Q'}) .$$

By similar computations, one can prove that $(Q \bullet_B Q', \lambda_{Q \bullet_B Q'})$ is a left A-module. Finally, we prove that the structures are compatible. In fact

$$\begin{aligned}
&\rho_{Q \bullet_B Q'} (\lambda_{Q \bullet_B Q'} \cdot 1_C) (1_A \cdot p_{Q,Q'} \cdot 1_C) \\
&\overset{(236)}{=} \rho_{Q \bullet_B Q'} (p_{Q,Q'} \cdot 1_C) (\lambda_Q \cdot 1_{Q'} \cdot 1_C) \\
&\overset{(237)}{=} p_{Q,Q'} (1_Q \cdot \rho_{Q'}) (\lambda_Q \cdot 1_{Q'} \cdot 1_C) \overset{\lambda_Q}{=} p_{Q,Q'} (\lambda_Q \cdot 1_{Q'}) (1_A \cdot 1_Q \cdot \rho_{Q'}) \\
&\overset{(236)}{=} \lambda_{Q \bullet_B Q'} (1_A \cdot p_{Q,Q'}) (1_A \cdot 1_Q \cdot \rho_{Q'}) \\
&\overset{(237)}{=} \lambda_{Q \bullet_B Q'} (1_A \cdot \rho_{Q \bullet_B Q'}) (1_A \cdot p_{Q,Q'} \cdot 1_C)
\end{aligned}$$

and since $1_A \cdot p_{Q,Q'} \cdot 1_C$ is epi, we get that

$$\rho_{Q \bullet_B Q'} (\lambda_{Q \bullet_B Q'} \cdot 1_C) = \lambda_{Q \bullet_B Q'} (1_A \cdot \rho_{Q \bullet_B Q'})$$

i.e. $(Q \bullet_B Q', \lambda_{Q \bullet_B Q'}, \rho_{Q \bullet_B Q'})$ is an A-C-bimodule. □

PROPOSITION 11.6. *Let (X, A) and (Y, B) be monads in the 2-category C, let (Q, λ_Q) be a left A-module and (Q, ρ_Q) be a right B-module. Then $A \bullet_A Q \simeq Q$ and $Q \bullet_B B \simeq Q$.*

Proof. Let us consider the trivial left A-module (A, m_A) and note that $(A \bullet_A Q, p_{A,Q}) = \mathrm{Coequ}_{\mathsf{C}} (m_A \cdot 1_Q, 1_A \cdot \lambda_Q)$. We already observed, in Lemma 11.4, that $(Q, \lambda_Q) = \mathrm{Coequ}_{\mathsf{C}} (m_A \cdot 1_Q, 1_A \cdot \lambda_Q)$. Therefore, there exists an isomorphism $l_Q : A \bullet_A Q \to Q$ such that

$$l_Q p_{A,Q} = \lambda_Q. \tag{238}$$

Similarly, if we consider the trivial right B-module (B, m_B), since $(Q \bullet_B B, p_{Q,B}) = \mathrm{Coequ}_{\mathsf{C}} (1_Q \cdot m_B, \rho_Q \cdot 1_B) = (Q, \rho_Q)$ we deduce that there exists an isomorphism $r_Q : Q \bullet_B B \to Q$ such that

$$r_Q p_{Q,B} = \rho_Q. \tag{239}$$

□

PROPOSITION 11.7. *Let (X, A), (Y, B), (Z, C), (W, D) be monads in the 2-category C and let (Q, λ_Q, ρ_Q) be an A-B-bimodule, $(Q', \lambda_{Q'}, \rho_{Q'})$ be a B-C-bimodule and $(Q'', \lambda_{Q''}, \rho_{Q''})$ be a C-D-bimodule. Then the coequalizers $(Q \bullet_B Q') \bullet_C Q'' \simeq Q \bullet_B (Q' \bullet_C Q'')$ are isomorphic as A-D-bimodules.*

Proof. Let us consider the following diagram

$$\begin{tikzcd}
Q\cdot B\cdot Q'\cdot C\cdot Q'' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}\cdot 1_C\cdot 1_{Q''}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}\cdot 1_C\cdot 1_{Q''}"'] \arrow[d, shift right, "1_Q\cdot 1_B\cdot 1_{Q'}\cdot\lambda_{Q''}"'] \arrow[d, shift left, "1_Q\cdot 1_B\cdot\rho_{Q'}\cdot 1_{Q''}1"] & Q\cdot Q'\cdot C\cdot Q'' \arrow[r, "p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}"] \arrow[d, shift right, "1_Q\cdot 1_{Q'}\cdot\lambda_{Q''}"'] \arrow[d, shift left, "1_Q\cdot\rho_{Q'}\cdot 1_{Q''}"] \arrow[dr, phantom, "2"] & Q\bullet_B Q'\cdot C\cdot Q'' \arrow[d, shift right, "1_{Q\bullet_B Q'}\cdot\lambda_{Q''}"'] \arrow[d, shift left, "\rho_{Q\bullet_B Q'}\cdot 1_{Q''}"] \\
Q\cdot B\cdot Q'\cdot Q'' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'}\cdot 1_{Q''}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'}\cdot 1_{Q''}"'] \arrow[d, "1_Q\cdot 1_B\cdot p_{Q',Q''}"'] \arrow[dr, phantom, "3"] & Q\cdot Q'\cdot Q'' \arrow[r, "p_{Q,Q'}\cdot 1_{Q''}"] \arrow[d, "1_Q\cdot p_{Q',Q''}"] \arrow[dr, phantom, "4"] & Q\bullet_B Q'\cdot Q'' \arrow[d, "p_{Q\bullet_B Q',Q''}"] \\
Q\cdot B\cdot Q'\bullet_C Q'' \arrow[r, shift left, "\rho_Q\cdot 1_{Q'\bullet_C Q''}"] \arrow[r, shift right, "1_Q\cdot\lambda_{Q'\bullet_C Q''}"'] & Q\cdot Q'\bullet_C Q'' \arrow[r, "p_{Q,Q'\bullet_C Q''}"] & Q\bullet_B (Q'\bullet_C Q'')\simeq (Q\bullet_B Q')\bullet_C Q''
\end{tikzcd}$$

Note that the left upper square serially commutes because of naturality of the 2-cells. The right upper square commutes because of naturality and of (237). The left bottom square commutes because of naturality and of (236). The rows are coequalizers and, since the 1-cells preserves coequalizers, also the columns are coequalizers. By the commutativity of the diagram, we deduce that

$$\begin{aligned}
& p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)\left(\rho_Q\cdot 1_{Q'}\cdot 1_{Q''}\right)\\
&\overset{3}{=} p_{Q,Q'\bullet_C Q''}\left(\rho_Q\cdot 1_{Q'\bullet_C Q''}\right)\left(1_Q\cdot 1_B\cdot p_{Q',Q''}\right)\\
&\overset{\text{coequ}}{=} p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot\lambda_{Q'\bullet_C Q''}\right)\left(1_Q\cdot 1_B\cdot p_{Q',Q''}\right)\\
&\overset{3}{=} p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)\left(1_Q\cdot\lambda_{Q'}\cdot 1_{Q''}\right)
\end{aligned}$$

and since $\left(Q\bullet_B Q'\cdot Q'', p_{Q,Q'}\cdot 1_{Q''}\right) = \mathrm{Coequ}_{\mathsf{C}}\left(\rho_Q\cdot 1_{Q'}\cdot 1_{Q''}, 1_Q\cdot\lambda_{Q'}\cdot 1_{Q''}\right)$, there exists a unique 2-cell $\xi : Q\bullet_B Q'\cdot Q''\to Q\bullet_B\left(Q'\bullet_C Q''\right)$ such that

$$\xi\left(p_{Q,Q'}\cdot 1_{Q''}\right) = p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right). \tag{240}$$

Moreover, we have

$$\begin{aligned}
&\xi\left(1_{Q\bullet_B Q'}\cdot\lambda_{Q''}\right)\left(p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}\right)\overset{2}{=}\xi\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(1_Q\cdot 1_{Q'}\cdot\lambda_{Q''}\right)\\
&\overset{(240)}{=} p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)\left(1_Q\cdot 1_{Q'}\cdot\lambda_{Q''}\right)\\
&\overset{p_{Q',Q''}\text{coequ}}{=} p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)\left(1_Q\cdot\rho_{Q'}\cdot 1_{Q''}\right)\\
&\overset{(240)}{=}\xi\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(1_Q\cdot\rho_{Q'}\cdot 1_{Q''}\right)\overset{2}{=}\xi\left(\rho_{Q\bullet_B Q'}\cdot 1_{Q''}\right)\left(p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}\right)
\end{aligned}$$

and since $p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}$ is epi, we get that ξ is a fork for $\left(1_{Q\bullet_B Q'}\cdot\lambda_{Q''}, \rho_{Q\bullet_B Q'}\cdot 1_{Q''}\right)$. By the universal property of the coequalizer $\left(\left(Q\bullet_B Q'\right)\bullet_C Q'', p_{Q\bullet_B Q',Q''}\right) = \mathrm{Coequ}_{\mathsf{C}}\left(1_{Q\bullet_B Q'}\cdot\lambda_{Q''}, \rho_{Q\bullet_B Q'}\cdot 1_{Q''}\right)$, there exists a unique 2-cell $\zeta : \left(Q\bullet_B Q'\right)\bullet_C Q''\to Q\bullet_B\left(Q'\bullet_C Q''\right)$ such that

$$\zeta p_{Q\bullet_B Q',Q''} = \xi$$

and thus we have

$$\zeta p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right) = \xi\left(p_{Q,Q'}\cdot 1_{Q''}\right)\overset{(240)}{=} p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)$$

i.e.

$$\zeta p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)=p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right). \tag{241}$$

Similarly, we have

$$\begin{aligned}
&p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(1_Q\cdot\rho_{Q'}\cdot 1_{Q''}\right)\\
&\overset{2}{=}p_{Q\bullet_B Q',Q''}\left(\rho_{Q\bullet_B Q'}\cdot 1_{Q''}\right)\left(p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}\right)\\
&\overset{p_{Q\bullet_B Q',Q''}\text{coequ}}{=}p_{Q\bullet_B Q',Q''}\left(1_{Q\bullet_B Q'}\cdot\lambda_{Q''}\right)\left(p_{Q,Q'}\cdot 1_C\cdot 1_{Q''}\right)\\
&\overset{2}{=}p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(1_Q\cdot 1_{Q'}\cdot\lambda_{Q''}\right)
\end{aligned}$$

and since $\left(Q\cdot Q'\bullet_C Q'',1_Q\cdot p_{Q',Q''}\right)=\mathrm{Coequ}_{\mathsf{C}}\left(\left(1_Q\cdot\rho_{Q'}\cdot 1_{Q''}\right),\left(1_Q\cdot 1_{Q'}\cdot\lambda_{Q''}\right)\right)$, there exists a unique $\xi':Q\cdot Q'\bullet_C Q''\to\left(Q\bullet_B Q'\right)\bullet_C Q''$ such that

$$\xi'\left(1_Q\cdot p_{Q',Q''}\right)=p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right). \tag{242}$$

Moreover, we have

$$\begin{aligned}
\xi'\left(\rho_Q\cdot 1_{Q'\bullet_C Q''}\right)\left(1_Q\cdot 1_B\cdot p_{Q',Q''}\right)&\overset{\rho_Q}{=}\xi'\left(1_Q\cdot p_{Q',Q''}\right)\left(\rho_Q\cdot 1_{Q'}\cdot 1_{Q''}\right)\\
&\overset{(242)}{=}p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(\rho_Q\cdot 1_{Q'}\cdot 1_{Q''}\right)\\
&\overset{p_{Q,Q'}\text{coequ}}{=}p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)\left(1_Q\cdot\lambda_{Q'}\cdot 1_{Q''}\right)\\
&\overset{(242)}{=}\xi'\left(1_Q\cdot p_{Q',Q''}\right)\left(1_Q\cdot\lambda_{Q'}\cdot 1_{Q''}\right)\overset{3}{=}\xi'\left(1_Q\cdot\lambda_{Q'\bullet_C Q''}\right)\left(1_Q\cdot 1_B\cdot p_{Q',Q''}\right)
\end{aligned}$$

and since $1_Q\cdot 1_B\cdot p_{Q',Q''}$ is epi, we deduce that ξ' is a fork for $\left(\rho_Q\cdot 1_{Q'\bullet_C Q''},1_Q\cdot\lambda_{Q'\bullet_C Q''}\right)$. Since $\left(Q\bullet_B\left(Q'\bullet_C Q''\right),p_{Q,Q'\bullet_C Q''}\right)=\mathrm{Coequ}_{\mathsf{C}}\left(\rho_Q\cdot 1_{Q'\bullet_C Q''},1_Q\cdot\lambda_{Q'\bullet_C Q''}\right)$, there exists a unique 2-cell $\zeta':Q\bullet_B\left(Q'\bullet_C Q''\right)\to\left(Q\bullet_B Q'\right)\bullet_C Q''$ such that

$$\zeta' p_{Q,Q'\bullet_C Q''}=\xi'$$

and thus

$$\zeta' p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)=\xi'\left(1_Q\cdot p_{Q',Q''}\right)\overset{(242)}{=}p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)$$

so that

$$\zeta' p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)=p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right). \tag{243}$$

We now want to prove that ζ and ζ' are two-sided inverse. We have

$$\begin{aligned}
\zeta\zeta' p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)&\overset{(243)}{=}\zeta p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)\\
&\overset{(241)}{=}p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)
\end{aligned}$$

and since $p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)$ is an epimorphism, we deduce that

$$\zeta\zeta'=1_{Q\bullet_B\left(Q'\bullet_C Q''\right)}.$$

Similarly, we have

$$\begin{aligned}
\zeta'\zeta p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)&\overset{(241)}{=}\zeta' p_{Q,Q'\bullet_C Q''}\left(1_Q\cdot p_{Q',Q''}\right)\\
&\overset{(243)}{=}p_{Q\bullet_B Q',Q''}\left(p_{Q,Q'}\cdot 1_{Q''}\right)
\end{aligned}$$

and since $p_{Q\bullet_B Q',Q''}(p_{Q,Q'}\cdot 1_{Q''})$ is an epimorphism, we get that

$$\zeta'\zeta = 1_{(Q\bullet_B Q')\bullet_C Q''}.$$

Therefore, $(Q\bullet_B Q')\bullet_C Q'' \simeq Q\bullet_B (Q'\bullet_C Q'')$ via ζ. Moreover, by Proposition 11.5, we know that $(Q\bullet_B Q')\bullet_C Q''$ and $Q\bullet_B (Q'\bullet_C Q'')$ are A-D-bimodules. We now want to prove that ζ is a morphism of left A-modules and right D-modules. Let us compute

$$\begin{aligned}
&\zeta\lambda_{(Q\bullet_B Q')\bullet_C Q''}(1_A\cdot p_{Q\bullet_B Q',Q''})(1_A\cdot p_{Q,Q'}\cdot 1_{Q''})\\
&\overset{\mathrm{def}\lambda_{(Q\bullet_B Q')\bullet_C Q''}}{=} \zeta p_{Q\bullet_B Q',Q''}\left(\lambda_{(Q\bullet_B Q')}\cdot 1_{Q''}\right)(1_A\cdot p_{Q,Q'}\cdot 1_{Q''})\\
&\overset{\mathrm{def}\lambda_{(Q\bullet_B Q')}}{=} \zeta p_{Q\bullet_B Q',Q''}(p_{Q,Q'}\cdot 1_{Q''})(\lambda_Q\cdot 1_{Q'}\cdot 1_{Q''})\\
&\overset{(241)}{=} p_{Q,Q'\bullet_C Q''}(1_Q\cdot p_{Q',Q''})(\lambda_Q\cdot 1_{Q'}\cdot 1_{Q''})\\
&\overset{\lambda_Q}{=} p_{Q,Q'\bullet_C Q''}(\lambda_Q\cdot 1_{Q'\bullet_C Q''})(1_A\cdot 1_Q\cdot p_{Q',Q''})\\
&\overset{\mathrm{def}\lambda_{(Q\bullet_B Q')}}{=} \lambda_{Q\bullet_B(Q'\bullet_C Q'')}(1_A\cdot p_{Q,Q'\bullet_C Q''})(1_A\cdot 1_Q\cdot p_{Q',Q''})\\
&\overset{(241)}{=} \lambda_{Q\bullet_B(Q'\bullet_C Q'')}(1_A\cdot\zeta)(1_A\cdot p_{Q\bullet_B Q',Q''})(1_A\cdot p_{Q,Q'}\cdot 1_{Q''})
\end{aligned}$$

and since $(1_A\cdot p_{Q\bullet_B Q',Q''})(1_A\cdot p_{Q,Q'}\cdot 1_{Q''})$ is epi, we get that

$$\zeta\lambda_{(Q\bullet_B Q')\bullet_C Q''} = \lambda_{Q\bullet_B(Q'\bullet_C Q'')}(1_A\cdot\zeta)$$

i.e. ζ is a morphism of left A-modules. Similarly, one can prove that ζ is a morphism of right D-modules. $\square$

Notation 11.8. *In the setting of Proposition 11.7, let us consider the isomorphism of bimodules*

$$\zeta : (Q\bullet_B Q')\bullet_C Q'' \to Q\bullet_B (Q'\bullet_C Q'').$$

In order to be more clear, in the following, we will denote it by

$$\zeta_{Q,Q',Q''} : (Q\bullet_B Q')\bullet_C Q'' \to Q\bullet_B (Q'\bullet_C Q'')$$

which is the unique satisfying the following

$$\zeta_{Q,Q',Q''} p_{Q\bullet_B Q',Q''}(p_{Q,Q'}\cdot 1_{Q''}) = p_{Q,Q'\bullet_C Q''}(1_Q\cdot p_{Q',Q''}). \tag{244}$$

Proposition 11.9. *Let (X,A), (Y,B), (Z,C), (W,D), (U,E) be monads in the 2-category C and let (Q,λ_Q,ρ_Q) be an A-B-bimodule, $(Q',\lambda_{Q'},\rho_{Q'})$ be a B-C-bimodule, $(Q'',\lambda_{Q''},\rho_{Q''})$ be a C-D-bimodule and $(Q''',\lambda_{Q'''},\rho_{Q'''})$ be a D-E-bimodule. Then the Pentagon Axiom holds, i.e. the following diagram is commutative*

$$\begin{tikzcd}
((Q\bullet_B Q')\bullet_C Q'')\bullet_D Q''' \arrow[rr, "\zeta_{Q,Q',Q''}\bullet_D 1_{Q'''}"] \arrow[d, "\zeta_{Q\bullet_B Q',Q'',Q'''}"'] & & (Q\bullet_B (Q'\bullet_C Q''))\bullet_D Q''' \arrow[d, "\zeta_{Q,Q'\bullet_C Q'',Q'''}"] \\
(Q\bullet_B Q')\bullet_C (Q''\bullet_D Q''') \arrow[dr, "\zeta_{Q,Q',Q''\bullet_D Q'''}"'] & & Q\bullet_B ((Q'\bullet_C Q'')\bullet_D Q''') \arrow[dl, "1_Q\bullet_B\zeta_{Q',Q'',Q'''}"] \\
& Q\bullet_B (Q'\bullet_C (Q''\bullet_D Q''')) &
\end{tikzcd}$$

Proof. We compute

$$
\begin{aligned}
&(1_Q \bullet_B \zeta_{Q',Q'',Q'''})\, \zeta_{Q,Q'\bullet_C Q'',Q'''}\, (\zeta_{Q,Q',Q''} \bullet_D 1_{Q'''})\, p_{(Q\bullet_B Q')\bullet_C Q'',Q'''} \\
&\qquad (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''}) \\
&= (1_Q \bullet_B \zeta_{Q',Q'',Q'''})\, \zeta_{Q,Q'\bullet_C Q'',Q'''} p_{Q\bullet_B (Q'\bullet_C Q''),Q'''}\, (\zeta_{Q,Q',Q''} \cdot 1_{Q'''}) \\
&\qquad (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''}) \\
&\overset{(244)}{=} (1_Q \bullet_B \zeta_{Q',Q'',Q'''})\, \zeta_{Q,Q'\bullet_C Q'',Q'''} p_{Q\bullet_B (Q'\bullet_C Q''),Q'''}\, (p_{Q,Q'\bullet_C Q''} \cdot 1_{Q'''}) \\
&\qquad (1_Q \cdot p_{Q',Q''} \cdot 1_{Q'''}) \\
&= (1_Q \bullet_B \zeta_{Q',Q'',Q'''})\, p_{Q,(Q'\bullet_C Q'')\bullet_D Q'''}\, (1_Q \cdot p_{Q'\bullet_C Q'',Q'''})\, (1_Q \cdot p_{Q',Q''} \cdot 1_{Q'''}) \\
&= p_{Q,Q'\bullet_C (Q''\bullet_D Q''')}\, (1_Q \cdot \zeta_{Q',Q'',Q'''})\, (1_Q \cdot p_{Q'\bullet_C Q'',Q'''})\, (1_Q \cdot p_{Q',Q''} \cdot 1_{Q'''}) \\
&\overset{(241)}{=} p_{Q,Q'\bullet_C (Q''\bullet_D Q''')}\, (1_Q \cdot p_{Q',Q''\bullet_D Q'''})\, (1_Q \cdot 1_{Q'} \cdot p_{Q'',Q'''})
\end{aligned}
$$

and

$$
\begin{aligned}
&\zeta_{Q,Q',Q''\bullet_D Q'''} \zeta_{Q\bullet_B Q',Q'',Q'''} p_{(Q\bullet_B Q')\bullet_C Q'',Q'''}\, (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''}) \\
&\overset{(241)}{=} \zeta_{Q,Q',Q''\bullet_D Q'''} p_{Q\bullet_B Q',Q''\bullet_D Q'''}\, (1_{Q\bullet_B Q'} \cdot p_{Q'',Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''}) \\
&\overset{p_{Q,Q'}}{=} \zeta_{Q,Q',Q''\bullet_D Q'''} p_{Q\bullet_B Q',Q''\bullet_D Q'''}\, (p_{Q,Q'} \cdot 1_{Q''\bullet_D Q'''})\, (1_Q \cdot 1_{Q'} \cdot p_{Q'',Q'''}) \\
&\overset{(241)}{=} p_{Q,Q'\bullet_C (Q''\bullet_D Q''')}\, (1_Q \cdot p_{Q',Q''\bullet_D Q'''})\, (1_Q \cdot 1_{Q'} \cdot p_{Q'',Q'''})
\end{aligned}
$$

so that we get that

$$
\begin{aligned}
&(1_Q \bullet_B \zeta_{Q',Q'',Q'''})\, \zeta_{Q,Q'\bullet_C Q'',Q'''}\, (\zeta_{Q,Q',Q''} \bullet_D 1_{Q'''})\, p_{(Q\bullet_B Q')\bullet_C Q'',Q'''}\, (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''}) \\
&\qquad (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''}) \\
&= \zeta_{Q,Q',Q''\bullet_D Q'''} \zeta_{Q\bullet_B Q',Q'',Q'''} p_{(Q\bullet_B Q')\bullet_C Q'',Q'''}\, (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''})
\end{aligned}
$$

and since $p_{(Q\bullet_B Q')\bullet_C Q'',Q'''}\, (p_{Q\bullet_B Q',Q''} \cdot 1_{Q'''})\, (p_{Q,Q'} \cdot 1_{Q''} \cdot 1_{Q'''})$ is an epimorphism, we deduce that the Pentagon Axiom holds. ☐

PROPOSITION 11.10. *Let (X,A), (Y,B), (Z,C) be monads in the 2-category C and let (Q,λ_Q,ρ_Q) be an A-B-bimodule and $(Q',\lambda_{Q'},\rho_{Q'})$ be a B-C-bimodule. Then the Triangle Axiom holds, i.e. the following diagram is commutative*

$$
\begin{tikzcd}
(Q \bullet_B B) \bullet_B Q' \arrow[rr, "\zeta_{Q,B,Q'}"] \arrow[dr, "r_Q \bullet_B 1_{Q'}"'] & & Q \bullet_B (B \bullet_B Q') \arrow[dl, "1_Q \bullet_B l_{Q'}"] \\
& Q \bullet_B Q' &
\end{tikzcd}
$$

Proof. We compute

$$
\begin{aligned}
(r_Q \bullet_B 1_{Q'})\, (p_{Q\bullet_B B,Q'})\, (p_{Q,B} \cdot 1_Q) &= p_{Q,Q'}\, (r_Q \cdot 1_{Q'})\, (p_{Q,B} \cdot 1_Q) \\
&\overset{(239)}{=} p_{Q,Q'}\, (\rho_Q \cdot 1_{Q'}) \overset{\mathrm{def}\, p}{=} p_{Q,Q'}\, (1_Q \cdot \lambda_{Q'}) \\
&\overset{(238)}{=} p_{Q,Q'}\, (1_Q \cdot l_{Q'})\, (1_Q \cdot p_{B,Q'}) \\
&= (1_Q \bullet_B l_{Q'})\, p_{Q,B\bullet_B Q'}\, (1_Q \cdot p_{B,Q'})
\end{aligned}
$$

$$\overset{(241)}{=} (1_Q \bullet_B l_{Q'}) \zeta_{Q,B,Q'} p_{Q \bullet_B B, Q'} (p_{Q,B} \cdot 1_{Q'})$$

so that, since $p_{Q \bullet_B B, Q'} (p_{Q,B} \cdot 1_{Q'})$ is an epimorphism, we get

$$r_Q \bullet_B 1_{Q'} = (1_Q \bullet_B l_{Q'}) \zeta_{Q,B,Q'}.$$

□

Proposition 11.11. *Let $(X, A), (Y, B)$ be monads in C, let $(P, \lambda_P, \rho_P), (Q, \lambda_Q, \rho_Q)$ be A-B-bimodules in C, let $(P', \lambda_{P'}, \rho_{P'}), (Q', \lambda_{Q'}, \rho_{Q'})$ be B-C-bimodules in C and let $f : P \to Q, f' : P' \to Q'$ be bimodule morphisms in C. Then there exists a unique A-C-bimodule morphism $f \bullet_B f' : P \bullet_B P' \to Q \bullet_B Q'$.*

Proof. Since f is an A-B-bimodule morphism, we have that

$$\lambda_Q (1_A \cdot f) = f \lambda_P \text{ and } \rho_Q (f \cdot 1_B) = f \rho_P. \tag{245}$$

Since f' is a B-C-bimodule morphism, we have that

$$\lambda_{Q'} (1_B \cdot f') = f' \lambda_{P'} \text{ and } \rho_{Q'} (f' \cdot 1_C) = f' \rho_{P'}. \tag{246}$$

Let us consider the following diagram

$$\begin{tikzcd}
P \cdot B \cdot P' \arrow[r, shift left, "\rho_P \cdot 1_{P'}"] \arrow[r, shift right, "1_P \cdot \lambda_{P'}"'] \arrow[d, "f \cdot 1_B \cdot f'"'] & P \cdot P' \arrow[r, "p_{P,P'}"] \arrow[d, "f \cdot f'"] & P \bullet_B P' \arrow[d, dotted, "f \bullet_B f'"] \\
Q \cdot B \cdot Q' \arrow[r, shift left, "\rho_Q \cdot 1_{Q'}"] \arrow[r, shift right, "1_Q \cdot \lambda_{Q'}"'] & Q \cdot Q' \arrow[r, "p_{Q,Q'}"] & Q \bullet_B Q'
\end{tikzcd}$$

Note that the left square serially commutes, in fact

$$\begin{aligned}
(f \cdot f') (\rho_P \cdot 1_{P'}) &= (f \cdot 1_{Q'}) (1_P \cdot f') (\rho_P \cdot 1_{P'}) \\
&\overset{\rho_P}{=} (f \cdot 1_{Q'}) (\rho_P \cdot 1_{Q'}) (1_P \cdot 1_B \cdot f') \\
&\overset{(245)}{=} (\rho_Q \cdot 1_{Q'}) (f \cdot 1_B \cdot 1_{Q'}) (1_P \cdot 1_B \cdot f') = (\rho_Q \cdot 1_{Q'}) (f \cdot 1_B \cdot f')
\end{aligned}$$

and

$$\begin{aligned}
(f \cdot f') (1_P \cdot \lambda_{P'}) &= (f \cdot 1_{Q'}) (1_P \cdot f') (1_P \cdot \lambda_{P'}) \\
&\overset{(246)}{=} (f \cdot 1_{Q'}) (1_P \cdot \lambda_{Q'}) (1_P \cdot 1_B \cdot f') \\
&\overset{f}{=} (1_Q \cdot \lambda_{Q'}) (f \cdot 1_B \cdot 1_{Q'}) (1_P \cdot 1_B \cdot f') \\
&= (1_Q \cdot \lambda_{Q'}) (f \cdot 1_B \cdot f').
\end{aligned}$$

Thus, we get that

$$p_{Q,Q'} (f \cdot f') (\rho_P \cdot 1_{P'}) = p_{Q,Q'} (f \cdot f') (1_P \cdot \lambda_{P'})$$

and since $(P \bullet_B P', p_{P,P'}) = \mathrm{Coequ}_{\mathsf{C}} (\rho_P \cdot 1_{P'}, 1_P \cdot \lambda_{P'})$ we deduce that there exists a unique 2-cell $f \bullet_B f' : P \bullet_B P' \to Q \bullet_B Q'$ such that

$$(f \bullet_B f') p_{P,P'} = p_{Q,Q'} (f \cdot f'). \tag{247}$$

We now want to prove that $f \bullet_B f'$ is a morphism of A-C-bimodules. Note that, by Proposition 11.5, $P \bullet_B P'$ and $Q \bullet_B Q'$ are A-C-bimodules. We compute

$$\begin{aligned}
\lambda_{Q \bullet_B Q'} (1_A \cdot f \bullet_B f') (1_A \cdot p_{P,P'}) &\overset{(247)}{=} \lambda_{Q \bullet_B Q'} (1_A \cdot p_{Q,Q'}) (1_A \cdot f \cdot f') \\
&\overset{(236)}{=} p_{Q,Q'} (\lambda_Q \cdot 1_{Q'}) (1_A \cdot f \cdot f') \\
&= p_{Q,Q'} (\lambda_Q \cdot 1_{Q'}) (1_A \cdot f \cdot 1_{Q'}) (1_A \cdot 1_P \cdot f') \\
&\overset{(245)}{=} p_{Q,Q'} (f \cdot 1_{Q'}) (\lambda_P \cdot 1_{Q'}) (1_A \cdot 1_P \cdot f') \\
&\overset{\lambda_P}{=} p_{Q,Q'} (f \cdot 1_{Q'}) (1_P \cdot f') (\lambda_P \cdot 1_{P'}) = p_{Q,Q'} (f \cdot f') (\lambda_P \cdot 1_{P'}) \\
&\overset{(247)}{=} (f \bullet_B f') p_{P,P'} (\lambda_P \cdot 1_{P'}) \\
&\overset{(236)}{=} (f \bullet_B f') \lambda_{P \bullet_B P'} (1_A \cdot p_{P,P'})
\end{aligned}$$

and since $1_A \cdot p_{P,P'}$ is an epimorphism, we get that

$$\lambda_{Q \bullet_B Q'} (1_A \cdot f \bullet_B f') = (f \bullet_B f') \lambda_{P \bullet_B P'}$$

i.e. $f \bullet_B f'$ is a morphism of left A-modules. Similarly, we also have

$$\begin{aligned}
\rho_{Q \bullet_B Q'} (f \bullet_B f' \cdot 1_C) (p_{P,P'} \cdot 1_C) &\overset{(247)}{=} \rho_{Q \bullet_B Q'} (p_{Q,Q'} \cdot 1_C) (f \cdot f' \cdot 1_C) \\
&\overset{(237)}{=} p_{Q,Q'} (1_Q \cdot \rho_{Q'}) (f \cdot f' \cdot 1_C) \\
&= p_{Q,Q'} (1_Q \cdot \rho_{Q'}) (1_Q \cdot f' \cdot 1_C) (f \cdot 1_{P'} \cdot 1_C) \\
&\overset{(246)}{=} p_{Q,Q'} (1_Q \cdot f') (1_Q \cdot \rho_{P'}) (f \cdot 1_{P'} \cdot 1_C) \\
&\overset{f}{=} p_{Q,Q'} (1_Q \cdot f') (f \cdot 1_{P'}) (1_P \cdot \rho_{P'}) = p_{Q,Q'} (f \cdot f') (1_P \cdot \rho_{P'}) \\
&\overset{(247)}{=} (f \bullet_B f') p_{P,P'} (1_P \cdot \rho_{P'}) \\
&\overset{(237)}{=} (f \bullet_B f') \rho_{P \bullet_B P'} (p_{P,P'} \cdot 1_C)
\end{aligned}$$

and since $p_{P,P'} \cdot 1_C$ is epi, we get that

$$\rho_{Q \bullet_B Q'} (f \bullet_B f' \cdot 1_C) = (f \bullet_B f') \rho_{P \bullet_B P'}$$

i.e. $f \bullet_B f'$ is also a morphism of right C-modules. $\square$

PROPOSITION 11.12. *For any monad (Y, B) in C, the composition denoted by $\bullet_B$ is compatible with the vertical canonical composition.*

Proof. Let $(X, A), (Y, B), (Z, C)$ be monads in C, let $(P, \lambda_P, \rho_P), (Q, \lambda_Q, \rho_Q)$, (W, λ_W, ρ_W) be A-B-bimodules in C, let $(P', \lambda_{P'}, \rho_{P'}), (Q', \lambda_{Q'}, \rho_{Q'}), (W', \lambda_{W'}, \rho_{W'})$ be B-C-bimodules in C and let $f : P \to Q, g : Q \to W$ be A-B-bimodule morphisms, $f' : P' \to Q', g' : Q' \to W'$ be B-C-bimodule morphisms in C. By Proposition 11.11 we can consider the A-C-bimodule morphisms $f \bullet_B f' : P \bullet_B P' \to Q \bullet_B Q'$ and $g \bullet_B g' : Q \bullet_B Q' \to W \bullet_B W'$ and we can compose them in order to get

$$(g \bullet_B g') (f \bullet_B f') : P \bullet_B P' \to W \bullet_B W'.$$

On the other hand, we can first consider the canonical vertical composites $gf : P \to W$ and $g'f' : P' \to W'$, which are still bimodule morphisms, and then we can compose them horizontally getting

$$(gf) \bullet_B (g'f') : P \bullet_B P' \to W \bullet_B W'.$$

We have to prove that

$$(g \bullet_B g')(f \bullet_B f') = (gf) \bullet_B (g'f').$$

Let us consider the following diagrams

$$\begin{tikzcd}
P \cdot B \cdot P' \arrow[r, shift left, "\rho_P \cdot 1_{P'}"] \arrow[r, shift right, "1_P \cdot \lambda_{P'}"'] \arrow[d, "f \cdot 1_B \cdot f'"'] & P \cdot P' \arrow[r, "p_{P,P'}"] \arrow[d, "f \cdot f'"] & P \bullet_B P' \arrow[d, "f \bullet_B f'"] \\
Q \cdot B \cdot Q' \arrow[r, shift left, "\rho_Q \cdot 1_{Q'}"] \arrow[r, shift right, "1_Q \cdot \lambda_{Q'}"'] \arrow[d, "g \cdot 1_B \cdot g'"'] & Q \cdot Q' \arrow[r, "p_{Q,Q'}"] \arrow[d, "g \cdot g'"] & Q \bullet_B Q' \arrow[d, "g \bullet_B g'"] \\
W \cdot B \cdot W' \arrow[r, shift left, "\rho_W \cdot 1_{W'}"] \arrow[r, shift right, "1_W \cdot \lambda_{W'}"'] & W \cdot W' \arrow[r, "p_{W,W'}"] & W \bullet_B W'
\end{tikzcd}$$

and

$$\begin{tikzcd}
P \cdot B \cdot P' \arrow[r, shift left, "\rho_P \cdot 1_{P'}"] \arrow[r, shift right, "1_P \cdot \lambda_{P'}"'] \arrow[d, "(gf) \cdot 1_B \cdot (g'f')"'] & P \cdot P' \arrow[r, "p_{P,P'}"] \arrow[d, "(gf) \cdot (g'f')"] & P \bullet_B P' \arrow[d, "(gf) \bullet_B (g'f')"] \\
W \cdot B \cdot W' \arrow[r, shift left, "\rho_W \cdot 1_{W'}"] \arrow[r, shift right, "1_W \cdot \lambda_{W'}"'] & W \cdot W' \arrow[r, "p_{W,W'}"'] & W \bullet_B W'
\end{tikzcd}$$

We have to prove that $(gf) \bullet_B (g'f')$ makes the external square of the first diagram commutative. Since $Bim\,(\mathsf{C})$ is a bicategory, in particular we have that $(gf) \cdot (g'f') = (g \cdot g')(f \cdot f')$ so that, by the commutativity of the first diagram, we deduce that also the left square of the second one commutes, i.e.

$$\begin{aligned}
[(gf) \cdot (g'f')](\rho_P \cdot 1_{P'}) &= (\rho_W \cdot 1_{W'})[(gf) \cdot 1_B \cdot (g'f')] \\
[(gf) \cdot (g'f')](1_P \cdot \lambda_{P'}) &= (1_W \cdot \lambda_{W'})[(gf) \cdot 1_B \cdot (g'f')].
\end{aligned}$$

Therefore, the exists the unique 2-cell $(gf) \bullet_B (g'f') : P \bullet_B P' \to W \bullet_B W'$ such that

$$[(gf) \bullet_B (g'f')]\, p_{P,P'} = p_{W,W'}\, [(gf) \cdot (g'f')].$$

Then we have

$$\begin{aligned}
[(gf) \bullet_B (g'f')]\, p_{P,P'} &= p_{W,W'}\, [(gf) \cdot (g'f')] = p_{W,W'}\, [(g \cdot g')(f \cdot f')] \\
&= p_{W,W'}\, (g \cdot g')(f \cdot f') = (g \bullet_B g')(f \bullet_B f')\, p_{P,P'}
\end{aligned}$$

and since $p_{P,P'}$ is an epimorphism, we get that

$$(gf) \bullet_B (g'f') = (g \bullet_B g')(f \bullet_B f').$$

□

DEFINITION 11.13. The bicategory $BIM\,(\mathsf{C})$ consists of

- 0-cells are monads in C

- 1-cells are bimodules in C together with their horizontal composition defined as follows. Let $(X,A),(Y,B)$ and (W,C) be monads in C and let $Q:Y\to X$ and $Q':W\to Y$ be respectively an A-B-bimodule with (Q,λ_Q,ρ_Q) and a B-C-bimodule in C with $(Q',\lambda_{Q'},\rho_{Q'})$. Then the horizontal composition of the two bimodules is given by $(Q\bullet_B Q',p_{Q,Q'})=\mathrm{Coequ}_{\mathsf{C}}(\rho_Q\cdot 1_{Q'},1_Q\cdot\lambda_{Q'})$ [Note that $Q\bullet_B Q'$ is an A-C-bimodule in C by Proposition 11.5. Moreover, such horizontal composition is weakly associative and unital by Propositions 11.7 and 11.6.]
- 2-cells are bimodule morphisms in C.

Example 11.14. Let us consider the bicategory **SetMat** as defined in [RW, 2.1]. The objects of this bicategory are sets, denoted by $A,B,\ldots$. An arrow (1-cell) $M:A\to B$ is a set valued matrix which, to fix notation ,has entries $M(a,b)$ for every $a\in A$ and $b\in B$. A 2-cell $f:M\to N:A\to B$ is a matrix of functions $f(a,b):M(a,b)\to N(a,b)$. Moreover, for $A\xrightarrow{M}B\xrightarrow{L}C$ we have $L\cdot M:A\to C$ defined by

$$(L\cdot M)(a,c)=\sum_{b\in B}L(b,c)\times M(a,b).$$

A monad in **SetMat** on an object A is thus a pair (A,M) where A is a set and $M:A\to A$ is a matrix whose entries are $M(a,b)$ for every $a,b\in A$, i.e. it is a small category with set of objects A. Hence, a monad functor is a functor $F:(A,M)\to(B,N)$ where A and B are the sets of objects of the small categories M and N. Note that, since F is a functor between categories, F is just a map $F:A\to B$ at the level of objects. This map induces a 1-cell $Q_F:A\to B$ defined as follows

$$Q_F(a,b)=\begin{Bmatrix}\varnothing & \text{if } b\neq F(a)\\ \{(a,F(a))\} & \text{if } b=F(a)\end{Bmatrix}.$$

Moreover, we can consider the following 2-cell $\phi^F:Q_FA\to BQ_F$ defined, for every $(a,b)\in A\times B$, by the map

$$\phi^F(a,b):Q_FA(a,b)\to BQ_F(a,b).$$

note that

$$Q_FA(a,b)=\sum_{a'\in A}Q_F(a',b)\times A(a,a')=\bigcup_{a'\in F^{\leftarrow}(b)}\{(a',F(a'))\}\times A(a,a')$$

where $F^{\leftarrow}(b)=\{a\in A\mid F(a)=b\}$. Similarly we have

$$BQ_F(a,b)=\sum_{b'\in B}B(b',b)\times Q_F(a,b')=B(F(a),b)\times\{(a,F(a))\}.$$

We can identify the set

$$\bigcup_{a'\in F^{\leftarrow}(b)}\{(a',F(a'))\}\times A(a,a')=Q_FA(a,b)=\bigcup_{a'\in F^{\leftarrow}(b)}A(a,a')$$

and

$$B(F(a),b)\times\{(a,F(a))\}=BQ_F(a,b)=B(F(a),b)$$

so that we define the map $\phi^F(a,b) : Q_F A(a,b) = \bigcup_{a' \in F^{\leftarrow}(b)} A(a,a') \to BQ_F(a,b) = B(F(a),b) = B(F(a),F(a'))$. Clearly, such a map is induced by the matrix map

$$\begin{array}{rcl} A(a,a') & \to & B(F(a),F(a')) \\ f & \mapsto & F(f). \end{array}$$

Since F is a functor, F preserves composition, $F(g \circ f) = F(g) \circ F(f)$, i.e. F is compatible with respect to the multiplications of the monads (A,M) and (B,N), and F preserves the identity, $F(1_a) = 1_{F(a)}$, i. e. F is compatible with respect to the units of the monads (A,M) and (B,N). Hence we get that F is a monad functor. Let now $F,G : (A,M) \to (B,N)$ be monad functors and let $\chi : (F,\phi_F) \to \left(G,\phi^G\right)$ be a functor transformation. Then we have that $\chi : Q_F \to Q_G$ is defined by setting

$$\chi(a,b) : Q_F(a,b) \to Q_G(a,b)$$
$$\left\{ \begin{array}{cc} \varnothing & \text{if } b \neq F(a) \\ \{(a,F(a))\} & \text{if } b = F(a) \end{array} \right\} \mapsto \left\{ \begin{array}{cc} \varnothing & \text{if } b \neq G(a) \\ \{(a,G(a))\} & \text{if } b = G(a) \end{array} \right\}.$$

Then, we have $Q_F(a,b) \stackrel{Q_F(f)}{\longrightarrow} Q_F(a',b')$

$$Q_F(a,b) \stackrel{Q_F(f)}{\longrightarrow} Q_F(a',b')$$
$$\left\{ \begin{array}{cc} \varnothing & \text{if } b \neq F(a) \\ \{(a,F(a))\} & \text{if } b = F(a) \end{array} \right\} \mapsto \left\{ \begin{array}{cc} \varnothing & \text{if } b' \neq F(a') \\ \{(a',F(a'))\} & \text{if } b' = F(a') \end{array} \right\}$$

and $Q_G(a,b) \stackrel{Q_G(f)}{\longrightarrow} Q_G(a',b')$ we have that

$$\chi(a',b')(Q_F(f)) = Q_G(f)(\chi(a,b))$$

i.e.χ is a monad functor transformation.

Now, let us define the following map

$$\begin{gathered} F(\mathsf{C}) : \mathbf{Mnd}(\mathsf{C}) \to BIM(\mathsf{C}) \\ (X,A) \mapsto (X,A) \\ (X,A) \stackrel{(Q,\phi)}{\to} (Y,B) \mapsto (Q \cdot A, (1_Q \cdot m_A)(\phi \cdot 1_A), 1_Q \cdot m_A) \\ (Q,\phi) \stackrel{\sigma}{\to} (P,\psi) \mapsto \sigma \cdot 1_A. \end{gathered}$$

Proposition 11.15. *The map F defined above is well-defined and it is a pseudo-functor.*

Proof. First, let us prove that $(Q \cdot A, (1_Q \cdot m_A)(\phi \cdot 1_A), 1_Q \cdot m_A)$ is a bimodule. In fact, we have

$$\begin{aligned} \lambda_{Q \cdot A}(1_B \cdot \lambda_{Q \cdot A}) &= (1_Q \cdot m_A)(\phi \cdot 1_A)(1_B \cdot 1_Q \cdot m_A)(1_B \cdot \phi \cdot 1_A) \\ &\stackrel{\phi}{=} (1_Q \cdot m_A)(1_Q \cdot 1_A \cdot m_A)(\phi \cdot 1_A \cdot 1_A)(1_B \cdot \phi \cdot 1_A) \\ &\stackrel{m_A \text{ass}}{=} (1_Q \cdot m_A)(1_Q \cdot m_A \cdot 1_A)(\phi \cdot 1_A \cdot 1_A)(1_B \cdot \phi \cdot 1_A) \\ &\stackrel{(233)}{=} (1_Q \cdot m_A)(\phi \cdot 1_A)(m_B \cdot 1_Q \cdot 1_A) = \lambda_{Q \cdot A}(m_B \cdot 1_Q \cdot 1_A) \end{aligned}$$

and

$$\lambda_{Q\cdot A}\left(u_B \cdot 1_Q \cdot 1_A\right) = \left(1_Q \cdot m_A\right)\left(\phi \cdot 1_A\right)\left(u_B \cdot 1_Q \cdot 1_A\right) \overset{(234)}{=} \left(1_Q \cdot m_A\right)\left(1_Q \cdot u_A \cdot 1_A\right) \overset{m_A \text{unit}}{=} 1_Q \cdot 1_A.$$

For the right A-module structure, we have

$$\rho_{Q\cdot A}\left(\rho_{Q\cdot A} \cdot 1_A\right) = \left(1_Q \cdot m_A\right)\left(1_Q \cdot m_A \cdot 1_A\right) \overset{m_A \text{ass}}{=} \left(1_Q \cdot m_A\right)\left(1_Q \cdot 1_A \cdot m_A\right) = \rho_{Q\cdot A}\left(1_Q \cdot 1_A \cdot m_A\right)$$

and

$$\rho_{Q\cdot A}\left(1_Q \cdot 1_A \cdot u_A\right) = \left(1_Q \cdot m_A\right)\left(1_Q \cdot 1_A \cdot u_A\right) \overset{m_A \text{unit}}{=} 1_Q \cdot 1_A.$$

Finally, we compute

$$\begin{aligned}\rho_{Q\cdot A}\left(\lambda_{Q\cdot A} \cdot 1_A\right) &= \left(1_Q \cdot m_A\right)\left(1_Q \cdot m_A \cdot 1_A\right)\left(\phi \cdot 1_A \cdot 1_A\right) \\ &\overset{m_A \text{ass}}{=} \left(1_Q \cdot m_A\right)\left(1_Q \cdot 1_A \cdot m_A\right)\left(\phi \cdot 1_A \cdot 1_A\right) \\ &\overset{\phi}{=} \left(1_Q \cdot m_A\right)\left(\phi \cdot 1_A\right)\left(1_B \cdot 1_Q \cdot m_A\right) = \lambda_{Q\cdot A}\left(1_B \cdot \rho_{Q\cdot A}\right)\end{aligned}$$

so that $\left(Q \cdot A, \left(1_Q \cdot m_A\right)\left(\phi \cdot 1_A\right), 1_Q \cdot m_A\right)$ is a B-A-bimodule. Now, let us consider the identity object $\left(X, 1_X\right) \in \mathbf{Mnd}\left(\mathsf{C}\right)$. Then $F\left(\left(X, 1_X\right)\right) = \left(X, 1_X\right)$ which is an identity object in $BIM\left(\mathsf{C}\right)$. Now, let us consider the composite of 1-cells in $\mathbf{Mnd}\left(\mathsf{C}\right)$

$$\left(X, A\right) \xrightarrow{(Q,\phi)} \left(Y, B\right) \xrightarrow{(P,\psi)} \left(Z, C\right).$$

We have to prove that

$$F\left(\left(P,\psi\right)\left(Q,\phi\right)\right) \simeq F\left(\left(P,\psi\right)\right) \bullet_B F\left(\left(Q,\phi\right)\right).$$

We have that $\left(P,\psi\right)\left(Q,\phi\right) = \left(P \cdot Q, \left(1_P \cdot \phi\right)\left(\psi \cdot 1_Q\right)\right)$ where $P \cdot Q : X \to Z$ and $\left(1_P \cdot \phi\right)\left(\psi \cdot 1_Q\right) : C \cdot P \cdot Q \to P \cdot Q \cdot A$. Then we have

$$\begin{aligned}F\left(\left(P,\psi\right)\left(Q,\phi\right)\right) &= F\left(\left(P \cdot Q, \left(1_P \cdot \phi\right)\left(\psi \cdot 1_Q\right)\right)\right) \\ &= \left(P \cdot Q \cdot A, \left(1_P \cdot 1_Q \cdot m_A\right)\left(1_P \cdot \phi \cdot 1_A\right)\left(\psi \cdot 1_Q \cdot 1_A\right), 1_P \cdot 1_Q \cdot m_A\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}F\left(\left(P,\psi\right)\right) &= \left(P \cdot B, \left(1_P \cdot m_B\right)\left(\psi \cdot 1_B\right), 1_P \cdot m_B\right) \\ F\left(\left(Q,\phi\right)\right) &= \left(Q \cdot A, \left(1_Q \cdot m_A\right)\left(\phi \cdot 1_A\right), 1_Q \cdot m_A\right)\end{aligned}$$

and thus

$$F\left(\left(P,\psi\right)\right) \bullet_B F\left(\left(Q,\phi\right)\right) = \left(P \cdot B\right) \bullet_B \left(Q \cdot A\right).$$

By definition of $\left(\left(P \cdot B\right) \bullet_B \left(Q \cdot A\right), p_{P\cdot B,Q\cdot A}\right) = \mathrm{Coequ}_{\mathsf{C}}\left(\rho_{P\cdot B} \cdot 1_Q \cdot 1_A, 1_P \cdot 1_B \cdot \lambda_{Q\cdot A}\right)$ we have the following diagram

$$P \cdot B \cdot B \cdot Q \cdot A \underset{1_P \cdot 1_B \cdot \lambda_{Q\cdot A}}{\overset{\rho_{P\cdot B} \cdot 1_Q \cdot 1_A}{\rightrightarrows}} P \cdot B \cdot Q \cdot A \xrightarrow{p_{P\cdot B,Q\cdot A}} \left(P \cdot B\right) \bullet_B \left(Q \cdot A\right)$$

Note that, $\rho_{P\cdot B} = 1_P \cdot m_B$ so that we can rewrite it in the following way

$$P \cdot B \cdot B \cdot Q \cdot A \underset{1_P \cdot 1_B \cdot \lambda_{Q\cdot A}}{\overset{1_P \cdot m_B \cdot 1_Q \cdot 1_A}{\rightrightarrows}} P \cdot B \cdot Q \cdot A \xrightarrow{p_{P\cdot B,Q\cdot A}} \left(P \cdot B\right) \bullet_B \left(Q \cdot A\right)$$

Since we have

$$\left(B \bullet_B (Q \cdot A), p_{B,Q\cdot A}\right) = \mathrm{Coequ}_{\mathsf{C}}\left(m_B \cdot 1_Q \cdot 1_A, 1_B \cdot \lambda_{Q\cdot A}\right)$$

and we are assuming that the composition with 1-cells preserves coequalizer, we also have

$$\left(P \cdot (B \bullet_B (Q \cdot A)), 1_P \cdot p_{B,Q\cdot A}\right) = \mathrm{Coequ}_{\mathsf{C}}\left(1_P \cdot m_B \cdot 1_Q \cdot 1_A, 1_P \cdot 1_B \cdot \lambda_{Q\cdot A}\right).$$

Therefore, there exists a unique isomorphism $h : (P \cdot B) \bullet_B (Q \cdot A) \to P \cdot (B \bullet_B (Q \cdot A))$ such that

$$h\left(p_{P\cdot B,Q\cdot A}\right) = 1_P \cdot p_{B,Q\cdot A}. \tag{248}$$

Moreover, by Proposition 11.6, $P \cdot (B \bullet_B (Q \cdot A)) \simeq P \cdot (Q \cdot A)$ so that we get

$$(P \cdot B) \bullet_B (Q \cdot A) \simeq P \cdot (Q \cdot A) = P \cdot Q \cdot A.$$

Now, the left C-module structure $\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}$ of $(P \cdot B) \bullet_B (Q \cdot A)$, by (236) is uniquely determined by

$$\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot p_{P\cdot B,Q\cdot A}\right) = p_{P\cdot B,Q\cdot A}\left(\lambda_{P\cdot B} \cdot 1_Q \cdot 1_A\right).$$

By (248) we get

$$p_{P\cdot B,Q\cdot A} = h^{-1}\left(1_P \cdot p_{B,Q\cdot A}\right)$$

and thus we can rewrite the above relation

$$\begin{aligned}\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot p_{P\cdot B,Q\cdot A}\right) &= \lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot \left[h^{-1}\left(1_P \cdot p_{B,Q\cdot A}\right)\right]\right) \\ &= \lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot h^{-1}\right)\left(1_C \cdot 1_P \cdot p_{B,Q\cdot A}\right)\end{aligned}$$

and

$$\begin{aligned}p_{P\cdot B,Q\cdot A}\left(\lambda_{P\cdot B} \cdot 1_Q \cdot 1_A\right) &= h^{-1}\left(1_P \cdot p_{B,Q\cdot A}\right)\left(\lambda_{P\cdot B} \cdot 1_Q \cdot 1_A\right) \\ &\overset{\mathrm{def}\lambda_{P\cdot B}}{=} h^{-1}\left(1_P \cdot p_{B,Q\cdot A}\right)\left(\lambda_P \cdot 1_B \cdot 1_Q \cdot 1_A\right) \\ &\overset{\lambda_P}{=} h^{-1}\left(\lambda_P \cdot 1_{B\bullet_B(Q\cdot A)}\right)\left(1_C \cdot 1_P \cdot p_{B,Q\cdot A}\right)\end{aligned}$$

so that

$$\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot h^{-1}\right)\left(1_C \cdot 1_P \cdot p_{B,Q\cdot A}\right) = h^{-1}\left(\lambda_P \cdot 1_{B\bullet_B(Q\cdot A)}\right)\left(1_C \cdot 1_P \cdot p_{B,Q\cdot A}\right).$$

Since $1_C \cdot 1_P \cdot p_{B,Q\cdot A}$ is epi, we get

$$\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}\left(1_C \cdot h^{-1}\right) = h^{-1}\left(\lambda_P \cdot 1_{B\bullet_B(Q\cdot A)}\right)$$

and thus

$$\lambda_{(P\cdot B)\bullet_B(Q\cdot A)} = h^{-1}\left(\lambda_P \cdot 1_{B\bullet_B(Q\cdot A)}\right)\left(1_C \cdot h\right)$$

so that we get that

$$\lambda_{(P\cdot B)\bullet_B(Q\cdot A)} \simeq \lambda_P \cdot 1_{B\bullet_B(Q\cdot A)} \simeq \lambda_P \cdot 1_{Q\cdot A} \simeq \lambda_{P\cdot Q\cdot A}.$$

Similarly, the right A-module structure $\lambda_{(P\cdot B)\bullet_B(Q\cdot A)}$ of $(P \cdot B) \bullet_B (Q \cdot A)$, by (237) is uniquely determined by

$$\rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(p_{P\cdot B,Q\cdot A} \cdot 1_A\right) = p_{P\cdot B,Q\cdot A}\left(1_P \cdot 1_B \cdot \rho_{Q\cdot A}\right).$$

By (248) we get

$$p_{P\cdot B,Q\cdot A} = h^{-1}\left(1_P \cdot p_{B,Q\cdot A}\right)$$

and thus we can rewrite the above relation

$$\begin{aligned}\rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(p_{P\cdot B,Q\cdot A}\cdot 1_A\right) &= \rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(\left[h^{-1}\left(1_P\cdot p_{B,Q\cdot A}\right)\right]\cdot 1_A\right)\\ &= \rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(h^{-1}\cdot 1_A\right)\left(1_P\cdot p_{B,Q\cdot A}\cdot 1_A\right)\end{aligned}$$

and

$$\begin{aligned}p_{P\cdot B,Q\cdot A}\left(1_P\cdot 1_B\cdot \rho_{Q\cdot A}\right) &= h^{-1}\left(1_P\cdot p_{B,Q\cdot A}\right)\left(1_P\cdot 1_B\cdot \rho_{Q\cdot A}\right)\\ &\overset{(237)}{=} h^{-1}\left(1_P\cdot \rho_{B\bullet_B(Q\cdot A)}\right)\left(1_P\cdot p_{B,Q\cdot A}\cdot 1_A\right)\end{aligned}$$

so that

$$\rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(h^{-1}\cdot 1_A\right)\left(1_P\cdot p_{B,Q\cdot A}\cdot 1_A\right) = h^{-1}\left(1_P\cdot \rho_{B\bullet_B(Q\cdot A)}\right)\left(1_P\cdot p_{B,Q\cdot A}\cdot 1_A\right).$$

Since $1_P\cdot p_{B,Q\cdot A}\cdot 1_A$ is epi, we get

$$\rho_{(P\cdot B)\bullet_B(Q\cdot A)}\left(h^{-1}\cdot 1_A\right) = h^{-1}\left(1_P\cdot \rho_{B\bullet_B(Q\cdot A)}\right)$$

and thus

$$\rho_{(P\cdot B)\bullet_B(Q\cdot A)} = h^{-1}\left(1_P\cdot \rho_{B\bullet_B(Q\cdot A)}\right)\left(h\cdot 1_A\right)$$

so that we get that

$$\rho_{(P\cdot B)\bullet_B(Q\cdot A)} \simeq 1_P\cdot \rho_{B\bullet_B(Q\cdot A)} \simeq 1_P\cdot \rho_{Q\cdot A} = \rho_{P\cdot Q\cdot A}.$$

□

12. Entwined modules and comodules

Let $(X,1_X),(Y,B)$ be monads in C and let us compute the category $\mathbf{Mnd}(\mathsf{C})((X,1_X),(Y,B))$. Note that $\mathsf{C}(X,B):\mathsf{C}(X,Y)\to\mathsf{C}(X,Y)$ is a monad over the category $\mathsf{C}(X,Y)$. In fact, we set multiplication and unit of the monad to be $\mathsf{C}(X,m_B)=m_B(-):\mathsf{C}(X,B\cdot B)\to\mathsf{C}(X,B)$ and $\mathsf{C}(X,u_B)=u_B(-):\mathsf{C}(X,1_Y)\to\mathsf{C}(X,B)$. In fact we have

$$\begin{aligned}\mathsf{C}(X,m_B)\,\mathsf{C}(X,m_B\cdot 1_B) &= m_B\left(m_B\cdot 1_B\right) = m_B\left(1_B\cdot m_B\right)\\ &= \mathsf{C}(X,m_B)\,\mathsf{C}(X,1_B\cdot m_B)\end{aligned}$$

and

$$\begin{aligned}\mathsf{C}(X,m_B)\,\mathsf{C}(X,u_B\cdot 1_B) &= m_B\left(u_B\cdot 1_B\right) = 1_B = m_B\left(1_B\cdot u_B\right)\\ &= \mathsf{C}(X,m_B)\,\mathsf{C}(X,1_B\cdot u_B)\end{aligned}$$

The objects of such category are the monad functors (Q,ϕ) from $(X,1_X)$ to (Y,B), i.e. the 1-cells $Q:X\to Y$ together with the 2-cells $\phi:B\cdot Q=\mathsf{C}(X,B)\,Q\to Q$ satisfying the following conditions

$$\begin{aligned}\phi\left(1_B\cdot\phi\right) &= \phi\left(m_B\cdot 1_Q\right)\\ \phi\left(u_B\cdot 1_Q\right) &= 1_Q\end{aligned}$$

which says that ϕ gives a structure of left $\mathsf{C}(X,B)$-module to the 1-cell $Q:X\to Y$. Therefore, we can conclude that

$$\mathbf{Mnd}(\mathsf{C})\left((X,1_X),(Y,B)\right) = {}_{\mathsf{C}(X,B)}\mathsf{C}(X,Y).$$

Now, following [St, pg. 158], we define the bicategory of comonads as follows: $\mathbf{Cmd}(\mathsf{C})=\mathbf{Mnd}(\mathsf{C}_*)_*$ where $(-)_*$ denotes the bicategory obtained by reversing

2-cells. This means that a comonad (X,C) in C is a 1-cell $C:X\to X$ together with 2-cells $\Delta^C : C \to C\cdot C$ and $\varepsilon^C : C\to 1_X$ called comultiplication and counit satisfying the reversed diagrams, i.e.

$$\left(1_C\cdot\Delta^C\right)\Delta^C=\left(\Delta^C\cdot 1_C\right)\Delta^C \tag{249}$$

$$\left(1_C\cdot\varepsilon^C\right)\Delta^C=1_C=\left(\varepsilon^C\cdot 1_C\right)\Delta^C. \tag{250}$$

A comonad functor is a pair $(P,\psi):(X,C)\to(Y,D)$ where $P:X\to Y$ is a 1-cell in C and $\psi : P\cdot C\to D\cdot P$ is a 2-cell in C satisfying

$$\left(\varepsilon^D\cdot 1_P\right)\psi=1_P\cdot\varepsilon^C \text{ and } \left(1_D\cdot\psi\right)\left(\psi\cdot 1_C\right)\left(1_P\cdot\Delta^C\right)=\left(\Delta^D\cdot 1_P\right)\psi.$$

Finally, a comonad functorial morphism $\omega:(P',\psi')\to(P,\psi)$ is $\omega:P'\to P$ is a 2-cell in C satisfying

$$\psi\left(\omega\cdot 1_C\right)=\left(1_D\cdot\omega\right)\psi'.$$

Now, we consider the category $\mathbf{Cmd}\left(\mathsf{C}\right)\left(\left(X,1_X\right),\left(Y,C\right)\right)$ where $\left(X,1_X\right)$ and $\left(Y,C\right)$ are 0-cells in $\mathbf{Cmd}\left(\mathsf{C}\right)$ respectively with trivial comultiplication and counit the former and Δ^C, ε^C the latter. Note that $\mathsf{C}\left(X,C\right):\mathsf{C}\left(X,Y\right)\to\mathsf{C}\left(X,Y\right)$ is a comonad over the category $\mathsf{C}\left(X,Y\right)$ with comultiplication and counit given by $\mathsf{C}\left(X,\Delta^C\right)=\Delta^C\left(\right):\mathsf{C}\left(X,C\right)\to\mathsf{C}\left(X,C\cdot C\right)$ and $\mathsf{C}\left(X,\varepsilon^C\right)=\varepsilon^C\left(\right):\mathsf{C}\left(X,C\right)\to\mathsf{C}\left(X,1_Y\right)$. The objects of such category are the comonad functors $\left(Q,\psi\right):\left(X,1_X\right)\to\left(Y,C\right)$ where $Q:X\to Y$ is a 1-cell and $\psi:Q\cdot 1_X\to C\cdot Q=\mathsf{C}\left(X,C\right)Q$ is a 2-cell satisfying $\left(\varepsilon^C\cdot 1_Q\right)\psi=1_Q$ and $\left(\Delta^C\cdot 1_Q\right)\psi=\left(1_C\cdot\psi\right)\psi$ so that

$$\mathbf{Cmd}\left(\mathsf{C}\right)\left(\left(X,1_X\right),\left(Y,C\right)\right)={}^{\mathsf{C}\left(X,C\right)}\mathsf{C}\left(X,Y\right).$$

Following the definition of the 2-category $\mathbf{Mnd}\left(\mathsf{C}\right)$ for any bicategory C, we can consider the 2-categories $\mathbf{Mnd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)$ and $\mathbf{Mnd}\left(BIM\left(\mathsf{C}\right)\right)$ and the functor between them

$$\mathbf{Mnd}\left(F\left(\mathsf{C}\right)\right):\mathbf{Mnd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)\to\mathbf{Mnd}\left(BIM\left(\mathsf{C}\right)\right).$$

Let us first consider $\mathbf{Mnd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)$:

- 0-cells: pairs $\left(\left(X,A\right),\left(Q,\phi\right)\right)$ where $\left(X,A\right)$ is an object in $\mathbf{Mnd}\left(\mathsf{C}\right)$ and $\left(Q,\phi\right)$ is a 1-cell in $\mathbf{Mnd}\left(\mathsf{C}\right)$ together with a pair of 2-cells in $\mathbf{Mnd}\left(\mathsf{C}\right)$ $m_{\left(Q,\phi\right)}$ and $u_{\left(Q,\phi\right)}$ satisfying associativity and unitality conditions. Therefore we have that $A:X\to X$ is a 1-cell in C together with 2-cells $m_A=m_{\left(X,A\right)}:A\cdot A\to A$ and $u_A=u_{\left(X,A\right)}:1_X\to A$ satisfying associativity and unitality conditions and we have that $Q:X\to X$ is a 1-cell in C together with the 2-cell of C $\phi:A\cdot Q\to Q\cdot A$ satisfying the following conditions

$$\phi\left(m_A\cdot 1_Q\right) = \left(1_Q\cdot m_A\right)\left(\phi\cdot 1_A\right)\left(1_A\cdot\phi\right) \tag{251}$$

$$\phi\left(u_A\cdot 1_Q\right) = 1_Q\cdot u_A \tag{252}$$

  Finally, the 2-cells of $\mathbf{Mnd}\left(\mathsf{C}\right)$ $m_{\left(Q,\phi\right)}:\left(Q,\phi\right)\cdot\left(Q,\phi\right)=\left(Q\cdot Q,\left(1_Q\cdot\phi\right)\left(\phi\cdot 1_Q\right)\right)\to\left(Q,\phi\right)$ and $u_{\left(Q,\phi\right)}:\left(1_X,1_{1_X}\right)\to\left(Q,\phi\right)$ satisfying the associativity and unitality conditions, needs to satisfy also the following

$$\phi\left(1_A\cdot m_{\left(Q,\phi\right)}\right) = \left(m_{\left(Q,\phi\right)}\cdot 1_A\right)\left(1_Q\cdot\phi\right)\left(\phi\cdot 1_Q\right) \tag{253}$$

$$\phi\left(1_A\cdot u_{\left(Q,\phi\right)}\right) = u_{\left(Q,\phi\right)}\cdot 1_A. \tag{254}$$

An object, or 0-cell, of $\mathbf{Mnd}(\mathbf{Mnd}(\mathsf{C}))$ is called *distributive law* and it gives rise to a monad structure on $Q \cdot A$. In fact, the monad functor transformation $m_{(Q,\phi)}$ induces a multiplication on $Q \cdot A$

$$m_{Q \cdot A} : Q \cdot A \cdot Q \cdot A \to Q \cdot A$$

defined by setting

$$m_{Q \cdot A} = \left(m_{(Q,\phi)} \cdot m_A\right)(1_Q \cdot \phi \cdot 1_A) = \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot \phi \cdot 1_A).$$

Using naturality and associativity of $m_{(Q,\phi)}$, naturality of ϕ, associativity of m_A, we have

$$\begin{aligned}
& m_{Q \cdot A}\left(m_{Q \cdot A} \cdot 1_Q \cdot 1_A\right) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot \phi \cdot 1_A)\left(m_{(Q,\phi)} \cdot 1_A \cdot 1_Q \cdot 1_A\right) \\
&\qquad (1_Q \cdot 1_Q \cdot m_A \cdot 1_Q \cdot 1_A)(1_Q \cdot \phi \cdot 1_A \cdot 1_Q \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)\left(m_{(Q,\phi)} \cdot 1_Q \cdot 1_A\right)(1_Q \cdot 1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_Q \cdot \phi \cdot 1_A) \\
&\qquad (1_Q \cdot 1_Q \cdot m_A \cdot 1_Q \cdot 1_A)(1_Q \cdot \phi \cdot 1_A \cdot 1_Q \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_Q \cdot \phi \cdot 1_A) \\
&\qquad (1_Q \cdot 1_Q \cdot m_A \cdot 1_Q \cdot 1_A)(1_Q \cdot \phi \cdot 1_A \cdot 1_Q \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A\right)(1_Q \cdot 1_Q \cdot \phi \cdot 1_A) \\
&\qquad (1_Q \cdot 1_Q \cdot m_A \cdot 1_Q \cdot 1_A)(1_Q \cdot \phi \cdot 1_A \cdot 1_Q \cdot 1_A) \\
&\overset{(251)}{=} \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A\right)(1_Q \cdot 1_Q \cdot 1_Q \cdot m_A \cdot 1_A) \\
&\qquad (1_Q \cdot 1_Q \cdot \phi \cdot 1_A \cdot 1_A)(1_Q \cdot 1_Q \cdot 1_A \cdot \phi \cdot 1_A)(1_Q \cdot \phi \cdot 1_A \cdot 1_Q \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_Q \cdot m_A \cdot 1_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A \cdot 1_A\right) \\
&\qquad (1_Q \cdot 1_Q \cdot \phi \cdot 1_A \cdot 1_A)(1_Q \cdot \phi \cdot 1_Q \cdot 1_A \cdot 1_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_Q \cdot 1_A \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A \cdot 1_A\right) \\
&\qquad (1_Q \cdot 1_Q \cdot \phi \cdot 1_A \cdot 1_A)(1_Q \cdot \phi \cdot 1_Q \cdot 1_A \cdot 1_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A\right)(1_Q \cdot 1_Q \cdot 1_Q \cdot 1_A \cdot m_A) \\
&\qquad (1_Q \cdot 1_Q \cdot \phi \cdot 1_A \cdot 1_A)(1_Q \cdot \phi \cdot 1_Q \cdot 1_A \cdot 1_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A\right)(1_Q \cdot 1_Q \cdot \phi \cdot 1_A) \\
&\qquad (1_Q \cdot 1_Q \cdot 1_A \cdot 1_Q \cdot m_A)(1_Q \cdot \phi \cdot 1_Q \cdot 1_A \cdot 1_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&= \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)\left(1_Q \cdot m_{(Q,\phi)} \cdot 1_A \cdot 1_A\right)(1_Q \cdot 1_Q \cdot \phi \cdot 1_A)(1_Q \cdot \phi \cdot 1_Q \cdot 1_A) \\
&\qquad (1_Q \cdot 1_A \cdot 1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&\overset{(253)}{=} \left(m_{(Q,\phi)} \cdot 1_A\right)(1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot \phi \cdot 1_A)\left(1_Q \cdot 1_A \cdot m_{(Q,\phi)} \cdot 1_A\right) \\
&\qquad (1_Q \cdot 1_A \cdot 1_Q \cdot 1_Q \cdot m_A)(1_Q \cdot 1_A \cdot 1_Q \cdot \phi \cdot 1_A) \\
&= m_{Q \cdot A}\left(1_Q \cdot 1_A \cdot m_{Q \cdot A}\right)
\end{aligned}$$

so that $m_{Q \cdot A}$ is associative. Similarly, the monad functor transformation $u_{(Q,\phi)}$ induces a unit of $Q \cdot A$

$$u_{Q \cdot A} : 1_X \to Q \cdot A$$

defined by setting

$$u_{Q\cdot A} = \left(u_{(Q,\phi)} \cdot 1_A\right) u_A.$$

Using naturality of $u_{(Q,\phi)}$, unitality of $m_{(Q,\phi)}$ and m_A we have

$$\begin{aligned} & m_{Q\cdot A}\left(u_{Q\cdot A} \cdot 1_Q \cdot 1_A\right) \\ = & \left(m_{(Q,\phi)} \cdot 1_A\right)\left(1_Q \cdot 1_Q \cdot m_A\right)\left(1_Q \cdot \phi \cdot 1_A\right)\left(u_{(Q,\phi)} \cdot 1_A \cdot 1_Q \cdot 1_A\right)\left(u_A \cdot 1_Q \cdot 1_A\right) \\ = & \left(m_{(Q,\phi)} \cdot 1_A\right)\left(u_{(Q,\phi)} \cdot 1_Q \cdot 1_A\right)\left(1_Q \cdot m_A\right)\left(\phi \cdot 1_A\right)\left(u_A \cdot 1_Q \cdot 1_A\right) \\ \overset{(252)}{=} & \left(1_Q \cdot m_A\right)\left(1_Q \cdot u_A \cdot 1_A\right) = 1_Q \cdot 1_A. \end{aligned}$$

so that we have a monad
$(Q \cdot A, m_{Q\cdot A}, u_{Q\cdot A}) = \left(Q \cdot A, \left(m_{(Q,\phi)} \cdot m_A\right)\left(1_Q \cdot \phi \cdot 1_A\right), \left(u_{(Q,\phi)} \cdot 1_A\right) u_A\right)$. We will see that such a monad is taken to an A-ring in the bimodule category.

- 1-cells: pairs $((U,\varphi),\tau) : ((X,A),(Q,\phi)) \to ((Y,B),(P,\psi))$ where $(U,\varphi) : (X,A) \to (Y,B)$ is a 1-cell in $\mathbf{Mnd}(\mathsf{C})$, i.e. a monad functor where $\varphi : B \cdot U \to U \cdot A$ satisfies $\varphi(u_B \cdot 1_U) = 1_U \cdot u_A$ and $(1_U \cdot m_A)(\varphi \cdot 1_A)(1_B \cdot \varphi) = \varphi(m_B \cdot 1_U)$, and τ is 2-cell in $\mathbf{Mnd}(\mathsf{C})$, i.e. a monad functor transformation $\tau : (P,\psi)(U,\varphi) \to (U,\varphi)(Q,\phi)$, i.e. $\tau : (P \cdot U, (1_P \cdot \varphi)(\psi \cdot 1_Q)) \to (U \cdot Q, (1_U \cdot \phi)(\varphi \cdot 1_Q))$ satisfying
$$(1_U \cdot \phi)(\varphi \cdot 1_Q)(1_B \cdot \tau) = (\tau \cdot 1_A)(1_P \cdot \varphi)(\psi \cdot 1_Q).$$
- 2-cells: $\sigma : ((U,\varphi),\tau) \to ((U',\varphi'),\tau')$ where $\sigma : (U,\varphi) \to (U',\varphi')$ is a 2-cell in $\mathbf{Mnd}(\mathsf{C})$ i.e.
$$\varphi'(1_B \cdot \sigma) = (\sigma \cdot 1_A)\varphi,$$
satisfying
$$\tau'(1_P \cdot \sigma) = (\sigma \cdot 1_Q)\tau$$

Let us now consider $\mathbf{Mnd}(BIM(\mathsf{C}))$:

- 0-cells: pairs $((X,A),(Q,\lambda_Q,\rho_Q))$ where (X,A) is an object in $BIM(\mathsf{C})$, i.e. a monad in C and $(Q,\lambda_Q,\rho_Q) : (X,A) \to (X,A)$ is a 1-cell in $BIM(\mathsf{C})$ together with 2-cells in $BIM(\mathsf{C})$, i.e. (Q,λ_Q,ρ_Q) is an A-bimodule in C together with bimodule morphisms $m_{(Q,\lambda_Q,\rho_Q)} : Q \bullet_A Q \to Q$ and $u_{(Q,\lambda_Q,\rho_Q)} : 1_X \to Q$ satisfying associativity and unitality conditions
$$\begin{aligned} m_{(Q,\lambda_Q,\rho_Q)}\left(m_{(Q,\lambda_Q,\rho_Q)} \bullet_A 1_Q\right) &= m_{(Q,\lambda_Q,\rho_Q)}\left(1_Q \bullet_A m_{(Q,\lambda_Q,\rho_Q)}\right) \\ m_{(Q,\lambda_Q,\rho_Q)}\left(u_{(Q,\lambda_Q,\rho_Q)} \bullet_A 1_Q\right) &= 1_Q = m_{(Q,\lambda_Q,\rho_Q)}\left(1_Q \bullet_A u_{(Q,\lambda_Q,\rho_Q)}\right) \end{aligned}$$
- 1-cells: pairs $((U,\lambda_U,\rho_U),\delta) : ((X,A),(Q,\lambda_Q,\rho_Q)) \to ((Y,B),(P,\lambda_P,\rho_P))$ where $(U,\lambda_U,\rho_U) : (X,A) \to (Y,B)$ is a 1-cell in $BIM(\mathsf{C})$ and $\delta : (P,\lambda_P,\rho_P)(U,\lambda_U,\rho_U) = P \bullet_B U \to (U,\lambda_U,\rho_U)(Q,\lambda_Q,\rho_Q) = U \bullet_A Q$ satisfies the following conditions
$$\delta\left(u_{(P,\lambda_P,\rho_P)} \bullet_B 1_U\right) = 1_U \bullet_A u_{(Q,\lambda_Q,\rho_Q)} \text{ and}$$
$$\left(1_U \bullet_A m_{(Q,\lambda_Q,\rho_Q)}\right)(\delta \bullet_A 1_Q)(1_P \bullet_B \delta) = \delta\left(1_U \bullet_B m_{(P,\lambda_P,\rho_P)}\right)$$

- 2-cells: $\omega : ((U,\lambda_U,\rho_U),\delta) \to ((U',\lambda_{U'},\rho_{U'}),\delta')$ where $\omega : (U,\lambda_U,\rho_U) \to (U',\lambda_{U'},\rho_{U'})$ is a 2-cell in $BIM(\mathsf{C})$, i.e. it is a B-A-bimodule morphism, satisfying
$$\delta'(1_P \bullet_B \omega) = (\omega \bullet_A 1_Q)\,\delta.$$

Now, let us apply the functor

$$\mathbf{Mnd}(F(\mathsf{C})) : \mathbf{Mnd}(\mathbf{Mnd}(\mathsf{C})) \to \mathbf{Mnd}(BIM(\mathsf{C}))$$

to the distributive law $((X,A),(Q,\phi))$. We get

$$\begin{aligned}
\mathbf{Mnd}(F(\mathsf{C}))(((X,A),(Q,\phi))) &= (F(\mathsf{C})(X,A), F(\mathsf{C})(Q,\phi)) \\
&= ((X,A),(Q\cdot A,(1_Q\cdot m_A)(\phi\cdot 1_A),1_Q\cdot m_A)) \\
\mathbf{Mnd}(F(\mathsf{C}))\left(m_{(Q,\phi)}\right) &= F(\mathsf{C})\left(m_{(Q,\phi)}\right) = m_{(Q,\phi)}\cdot 1_A \\
\mathbf{Mnd}(F(\mathsf{C}))\left(u_{(Q,\phi)}\right) &= F(\mathsf{C})\left(u_{(Q,\phi)}\right) = u_{(Q,\phi)}\cdot 1_A
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Mnd}(F(\mathsf{C}))\left(m_{(Q,\phi)}\right) &: \begin{array}{c}\mathbf{Mnd}(F(\mathsf{C}))(Q\cdot Q)\\ = Q\cdot Q\cdot A\end{array} \longrightarrow \begin{array}{c}\mathbf{Mnd}(F(\mathsf{C}))(Q)\\ = Q\cdot A\end{array} \\
\mathbf{Mnd}(F(\mathsf{C}))\left(u_{(Q,\phi)}\right) &: \mathbf{Mnd}(F(\mathsf{C}))(1_X) = A \longrightarrow \mathbf{Mnd}(F(\mathsf{C}))(Q) = Q\cdot A.
\end{aligned}$$

In particular, $(Q\cdot A,(1_Q\cdot m_A)(\phi\cdot 1_A),1_Q\cdot m_A)$ comes together with bimodule morphisms $\mathbf{Mnd}(F(\mathsf{C}))\left(m_{(Q,\phi)}\right) = m_{(Q,\phi)}\cdot 1_A$ and $\mathbf{Mnd}(F(\mathsf{C}))\left(u_{(Q,\phi)}\right) = u_{(Q,\phi)}\cdot 1_A$. Note that

$$\begin{aligned}
\mathbf{Mnd}(F(\mathsf{C}))(Q\cdot Q) &= Q\cdot Q\cdot A \simeq (Q\cdot A)\bullet_A(Q\cdot A) \\
&= \mathbf{Mnd}(F(\mathsf{C}))(Q)\bullet_A \mathbf{Mnd}(F(\mathsf{C}))(Q).
\end{aligned}$$

where the isomorphism is given by the following: by definition,

$$((Q\cdot A)\bullet_A(Q\cdot A),p_{Q\cdot A,Q\cdot A}) = \mathrm{Coequ}_{\mathsf{C}}(1_Q\cdot m_A\cdot 1_Q\cdot 1_A, 1_Q\cdot 1_A\cdot\lambda_{Q\cdot A})$$

and since we are assuming that the coequalizers are preserved, by Lemma 11.4 we also have

$$(Q\cdot Q\cdot A,1_Q\cdot\lambda_{Q\cdot A}) = \mathrm{Coequ}_{\mathsf{C}}(1_Q\cdot m_A\cdot 1_Q\cdot 1_A, 1_Q\cdot 1_A\cdot\lambda_{Q\cdot A})$$

so that there exists a unique isomorphism

$$\alpha : (Q\cdot A)\bullet_A(Q\cdot A) \to Q\cdot Q\cdot A$$

such that

$$\alpha p_{Q\cdot A,Q\cdot A} = 1_Q\cdot\lambda_{Q\cdot A} = (1_Q\cdot 1_Q\cdot m_A)(1_Q\cdot\phi\cdot 1_A). \tag{255}$$

Recall that we can consider the monad
$(Q\cdot A, m_{Q\cdot A}, u_{Q\cdot A}) = \left(Q\cdot A,\left(m_{(Q,\phi)}\cdot m_A\right)(1_Q\cdot\phi\cdot 1_A),\left(u_{(Q,\phi)}\cdot 1_A\right)u_A\right)$ and thus

$$\begin{aligned}
m_{Q\cdot A} &= \left(m_{(Q,\phi)}\cdot m_A\right)(1_Q\cdot\phi\cdot 1_A) \\
&= \left(m_{(Q,\phi)}\cdot 1_A\right)(1_Q\cdot 1_Q\cdot m_A)(1_Q\cdot\phi\cdot 1_A) \\
&= \left(m_{(Q,\phi)}\cdot 1_A\right)\alpha p_{Q\cdot A,Q\cdot A}
\end{aligned}$$

that is, $m_{Q\cdot A}$ factorizes through $(Q\cdot A)\bullet_A(Q\cdot A)$ and we denote by

$$m_{\left(Q\cdot A,\left(1_Q\cdot m_A\right)(\phi\cdot 1_A),1_Q\cdot m_A\right)} = \left(m_{(Q,\phi)}\cdot 1_A\right)\alpha \tag{256}$$

the unique A-bimodule morphism such that

$$m_{Q\cdot A}=m_{\left(Q\cdot A,\left(1_Q\cdot m_A\right)\left(\phi\cdot 1_A\right),1_Q\cdot m_A\right)}p_{Q\cdot A,Q\cdot A}.$$

Note that also $u_{(Q,\phi)}\cdot 1_A$ is an A-bimodule morphism, in fact

$$\begin{aligned}\lambda_{Q\cdot A}\left(1_A\cdot u_{(Q,\phi)}\cdot 1_A\right) &= \left(1_Q\cdot m_A\right)\left(\phi\cdot 1_A\right)\left(1_A\cdot u_{(Q,\phi)}\cdot 1_A\right)\\ &\overset{(252)}{=}\left(1_Q\cdot m_A\right)\left(u_{(Q,\phi)}\cdot 1_A\cdot 1_A\right)\overset{u_{(Q,\phi)}}{=}\left(u_{(Q,\phi)}\cdot 1_A\right)m_A\end{aligned}$$

and

$$\rho_{Q\cdot A}\left(u_{(Q,\phi)}\cdot 1_A\cdot 1_A\right)=\left(1_Q\cdot m_A\right)\left(u_{(Q,\phi)}\cdot 1_A\cdot 1_A\right)\overset{u_{(Q,\phi)}}{=}\left(u_{(Q,\phi)}\cdot 1_A\right)m_A.$$

Therefore,

$$\begin{aligned}&\left(Q\cdot A,m_{\left(Q\cdot A,\left(1_Q\cdot m_A\right)\left(\phi\cdot 1_A\right),1_Q\cdot m_A\right)},u_{\left(Q\cdot A,\left(1_Q\cdot m_A\right)\left(\phi\cdot 1_A\right),1_Q\cdot m_A\right)}\right)\\ &\qquad=\left(Q\cdot A,\left(m_{(Q,\phi)}\cdot 1_A\right)\alpha,u_{(Q,\phi)}\cdot 1_A\right)\end{aligned}$$

is an A-ring, so that the functor $\mathbf{Mnd}\left(F\left(\mathsf{C}\right)\right):\mathbf{Mnd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)\to\mathbf{Mnd}\left(BIM\left(\mathsf{C}\right)\right)$ associates distributive laws to A-rings.

Let us now consider $\mathbf{Cmd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)$:

- 0-cells: $\left(\left(X,A\right),\left(C,\gamma\right)\right)$ where $\left(X,A\right)$ is a monad, $C:X\to X$, $\gamma:A\cdot C\to C\cdot A$ together with $\Delta^C:C\to C\cdot C$ and $\varepsilon^C:C\to 1_X$ satisfying coassociativity and counitality and satisfying

$$(257)\qquad \left(1_C\cdot\gamma\right)\left(\gamma\cdot 1_C\right)\left(1_A\cdot\Delta^C\right)=\left(\Delta^C\cdot 1_A\right)\gamma$$

$$(258)\qquad 1_A\cdot\varepsilon^C=\left(\varepsilon^C\cdot 1_A\right)\gamma.$$

  Note that, if we consider $\left(C\cdot A,\Delta^{C\cdot A},\varepsilon^{C\cdot A}\right)$ where $\Delta^{C\cdot A}=\left(\gamma\cdot 1_C\cdot 1_A\right)\left(1_A\cdot\Delta^C\cdot 1_A\right)\left(u_A\cdot 1_C\cdot 1_A\right)$ and $\varepsilon^{C\cdot A}:C\cdot A\to A$ coassociativity and counitality properties are not satisfied. But, by applying the functor $\mathbf{Cmd}\left(F\left(\mathsf{C}\right)\right)$ to the comonad $\left(C,\Delta^C,\varepsilon^C\right)$ we get

$$\mathbf{Cmd}\left(F\left(\mathsf{C}\right)\right)\left(\left(C,\Delta^C,\varepsilon^C\right)\right)=\left(C\cdot A,\Delta^C\cdot 1_A,\varepsilon^C\cdot 1_A\right)$$

  where

$$\Delta^C\cdot 1_A\simeq\Delta^{C\cdot A}:C\cdot A\to C\cdot A\bullet_A C\cdot A$$

  and $\Delta^C\cdot 1_A$ and $\varepsilon^C\cdot 1_A$ are A-bimodule morphisms, clearly satisfying coassociativity and counitality conditions. Hence, $\left(C\cdot A,\Delta^C\cdot 1_A,\varepsilon^C\cdot 1_A\right)$ is an A-coring.

Since

$$\mathbf{Mnd}\left(\mathsf{C}\right)\left(\left(X,1_X\right),\left(Y,B\right)\right)={}_{\mathsf{C}(X,B)}\mathsf{C}\left(X,Y\right)$$

dually we get

$$\mathbf{Cmd}\left(\mathsf{C}\right)\left(\left(X,1_X\right),\left(Y,C\right)\right)={}^{\mathsf{C}(X,C)}\mathsf{C}\left(X,Y\right).$$

Consider the objects $\left(\left(X,1_X\right),\left(1_{(X,1_X)},1_X\right)\right),\left(\left(X,A\right),\left(C,\gamma\right)\right)\in\mathbf{Cmd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\right)$ $\left[\left(C,\gamma\right):\left(X,A\right)\to\left(X,A\right),\ \gamma:A\cdot C\to C\cdot A\right]$ and let $\left(\left(Q,\phi\right),\sigma\right)\in\mathbf{Cmd}\left(\mathbf{Mnd}\left(\mathsf{C}\right)\left(\left(X,1_X\right),\left(1_{(X,1_X)},1_X\right)\right),\left(\left(X,A\right),\left(C,\gamma\right)\right)\right)$ be a comonad functor, where $\left(Q,\phi\right):\left(X,1_X\right)\to\left(X,A\right)$ is a 1-cell in $\mathbf{Mnd}\left(\mathsf{C}\right)$, i.e. $\phi:A\cdot Q\to Q$ satisfies $\phi\left(u_A\cdot 1_Q\right)=1_Q$ and $\phi\left(1_A\cdot\phi\right)=\phi\left(m_A\cdot 1_Q\right)$ and $\sigma:\left(Q,\phi\right)\left(1_{(X,1_X)},1_X\right)=\left(Q,\phi\right)\to\left(C,\gamma\right)\left(Q,\phi\right)=\left(C\cdot Q,\left(1_C\cdot\phi\right)\left(\gamma\cdot 1_Q\right)\right)$ is a 2-cell in $\mathbf{Mnd}\left(\mathsf{C}\right)$, i.e.

$(1_C \cdot \phi)(\gamma \cdot 1_Q)(1_A \cdot \sigma) = \sigma\phi$, i.e. σ is an A-linear map. Since $((Q,\phi),\sigma)$ is a comonad functor, the 2-cell $\sigma : Q \to C \cdot Q$ satisfies $\left(\varepsilon^C \cdot 1_Q\right)\sigma = 1_Q$ and $(1_C \cdot \phi)\phi = \left(\Delta^C \cdot 1_Q\right)\phi$. This means that $((Q,\phi),\sigma) \in {}^{\mathsf{C}(X,C)}_{\mathsf{C}(X,A)}\mathsf{C}(X,X)(\gamma)$ is an entwined module. By applying the functor $\mathbf{Cmd}(F(\mathsf{C})) : \mathbf{Cmd}(\mathbf{Mnd}(\mathsf{C})) \longrightarrow \mathbf{Cmd}(BIM(\mathsf{C}))$ to the element
$((Q,\phi),\sigma) \in \mathbf{Cmd}\left(\mathbf{Mnd}(\mathsf{C})\left((X,1_X),\left(1_{(X,1_X)},1_X\right)\right),((X,A),(C,\gamma))\right)$ we get
$\mathbf{Cmd}(F(\mathsf{C}))(((Q,\phi),\sigma)) \in \mathbf{Cmd}\left(BIM(\mathsf{C})\left((X,1_X),\left(1_{(X,1_X)},1_X\right)\right),((X,A),(C,\gamma))\right)$ which is an element of ${}^{\mathsf{C}(X,C\cdot A)}BIM(X,X)$, i.e. it is a left $C \cdot A$-comodule with respect to $\bullet_A$.

Appendix A. Gabriel Popescu Theorem

Notation A.1. *Let $\mathcal{A}$ be a Grothendieck category, let U be an object of $\mathcal{A}$ and let $B = \mathrm{Hom}_{\mathcal{A}}(U,U)$. Assume that U is a generator of $\mathcal{A}$ i.e. that the functor $\mathrm{Hom}_{\mathcal{A}}(U,-) : \mathcal{A} \to Mod\text{-}B$ is faithful.*

Lemma A.2. *In the assumptions and notations of A.1, let $X \in \mathcal{A}$ and let $\lambda : U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))} \to X$ be the codiagonal morphism of the family $(f)_{f\in(\mathrm{Hom}_{\mathcal{A}}(U,X))}$. Then $\mathrm{Im}(\lambda) = X$.*

Proof. Let $J : \mathrm{Ker}(\lambda) \to U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))}$ be the canonical monomorphism and let $\lambda : U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))} \to X$ be the codiagonal morphism of the family $(f)_{f\in(\mathrm{Hom}_{\mathcal{A}}(U,X))}$ and, for every $f \in \mathrm{Hom}_{\mathcal{A}}(U,X)$ let $i_f : U \to U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))}$ the f-th canonical injection. Then we have $\lambda \circ i_f = f$. Let $\chi : X \to \mathrm{Coker}(\lambda)$ be the canonical projection and let us assume that $\chi \neq 0$. Then there exists $h : U \to X$ such that $\chi \circ h \neq 0$. Then we have

$$0 \neq \chi \circ h = \chi \circ \lambda \circ i_h = 0 \circ i_h = 0.$$

Contradiction. Thus $\mathrm{Coker}(\lambda) = 0$ and hence $X = \mathrm{KerCoker}(\lambda) = \mathrm{Im}(\lambda)$. ☐

Proposition A.3. *In the assumptions and notations of A.1, the functor $\mathrm{Hom}_{\mathcal{A}}(U,-) : \mathcal{A} \to Mod\text{-}B$ is full.*

Proof. Let $\varphi \in \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{A}}(U,X), \mathrm{Hom}_{\mathcal{A}}(U,Z))$. We have to prove that there exists a morphism $g : X \to Z$ such that $\varphi = \mathrm{Hom}_{\mathcal{A}}(U,g)$. For any subset F of $\mathrm{Hom}_{\mathcal{A}}(U,X)$ we denote by

$$i_F : U^{(F)} \to U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))}$$

the canonical injection. If $F = \{f\}$ we will write i_f instead of $i_{\{f\}}$. Let $\lambda : U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))} \to X$ be the codiagonal morphism of the family $(f)_{f\in(\mathrm{Hom}_{\mathcal{A}}(U,X))}$ and let $\mu : U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))} \to Z$ be the codiagonal morphism of the family $(\varphi(f))_{f\in(\mathrm{Hom}_{\mathcal{A}}(U,X))}$. Then, for every $f \in \mathrm{Hom}_{\mathcal{A}}(U,X)$ we have

$$\lambda \circ i_f = f \text{ and } \mu \circ i_f = \varphi(f).$$

Let F be a finite subset of $\mathrm{Hom}_{\mathcal{A}}(U, X)$ and let us consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\lambda_F) \arrow[r, "j_F"] \arrow[d, "h_F"'] & U^{(F)} \arrow[r, "\lambda_F"] \arrow[d, "i_F"'] & X \arrow[d, "\mathrm{Id}_X"'] \\
0 \arrow[r] & \mathrm{Ker}(\lambda) \arrow[r, "j"] & U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))} \arrow[r, "\lambda"] & X
\end{tikzcd}$$

where $\lambda_F : U^{(F)} \to X$ is the codiagonal morphism of the family $(f)_{f \in F}$, j and j_F are the canonical inclusions and h_F is the morphism that factorizes $i_F \circ j_F$ through $\mathrm{Ker}(\lambda)$. We have

$$\mu \circ i_F \circ j_F = \mu \circ j \circ h_F.$$

For every $f \in F$ let $\alpha_f : U \to U^{(F)}$ and $\pi_f : U^{(F)} \to U$ be respectively the canonical injections and projections. Then

$$\mathrm{Id}_{U^{(F)}} = \sum\nolimits_{f \in F} \alpha_f \circ \pi_f.$$

Let $\sigma : U \to \mathrm{Ker}(\lambda_F)$ be any morphism. We compute

$$\begin{aligned} 0 = \lambda_F \circ j_F \circ \sigma = \lambda_F \circ \mathrm{Id}_{U^{(F)}} \circ j_F \circ \sigma &= \sum\nolimits_{f \in F} \lambda_F \circ \alpha_f \circ \pi_f \circ j_F \circ \sigma \\ &= \sum\nolimits_{f \in F} f \circ \pi_f \circ j_F \circ \sigma. \end{aligned}$$

Since $\pi_f \circ j_F \circ \sigma \in B = \mathrm{Hom}_{\mathcal{A}}(U, U)$ and $\varphi \in \mathrm{Hom}_B(\mathrm{Hom}_{\mathcal{A}}(U, X), \mathrm{Hom}_{\mathcal{A}}(U, Z))$, we get that

$$0 = \varphi\left(\sum\nolimits_{f \in F} f \circ \pi_f \circ j_F \circ \sigma\right) = \left(\sum\nolimits_{f \in F} \varphi(f) \circ \pi_f \circ j_F\right) \circ \sigma$$

and hence, since U is a generator of $\mathcal{A}$, we get that

$$\sum\nolimits_{f \in F} \varphi(f) \circ \pi_f \circ j_F = 0.$$

On the other hand we have that

$$\mu \circ i_F \circ \alpha_f = \mu \circ i_f = \varphi(f)$$

and hence we obtain

$$\begin{aligned} 0 &= \sum\nolimits_{f \in F} \mu \circ i_F \circ \alpha_f \circ \pi_f \circ j_F = \mu \circ i_F \circ \left(\sum\nolimits_{f \in F} \alpha_f \circ \pi_f\right) \circ j_F \\ &= \mu \circ i_F \circ j_F = \mu \circ j \circ h_F. \end{aligned}$$

Let $u : \mathrm{Ker}(\mu) \to U^{(\mathrm{Hom}_{\mathcal{A}}(U,X))}$ be the canonical injection. Then there exists a morphism $\beta_F : \mathrm{Ker}(\lambda_F) \to \mathrm{Ker}(\mu)$ such that

$$j \circ h_F = u \circ \beta_F$$

and since both j and h_F are mono, also β_F is mono. We want to check that the family $(\beta_F)_{F \subseteq \mathrm{Hom}_{\mathcal{A}}(U,X)}$ is compatible. For every $F, G \subseteq \mathrm{Hom}_{\mathcal{A}}(U, X)$ finite subsets, let us denote $i_F^G : U^{(F)} \to U^{(G)}$. Thus we have $\lambda_G \circ i_F^G = \lambda_F$ and

$$0 = \lambda_F \circ j_F = \lambda_G \circ i_F^G \circ j_F.$$

Since $j_G : \operatorname{Ker}(\lambda_G) \to U^{(G)}$ there exists a unique morphism $\widehat{i_F^G} : \operatorname{Ker}(\lambda_F) \to \operatorname{Ker}(\lambda_G)$ such that

$$i_F^G \circ j_F = j_G \circ \widehat{i_F^G}.$$

We want to prove that $\beta_G \circ \widehat{i_F^G} = \beta_F$ for every F, G finite subsets of $\operatorname{Hom}_{\mathcal{A}}(U, X)$. Let us compute

$$\begin{aligned} u \circ \beta_G \circ \widehat{i_F^G} &= j \circ h_G \circ \widehat{i_F^G} = i_G \circ j_G \circ \widehat{i_F^G} = i_G \circ i_F^G \circ j_F = i_F \circ j_F \\ &= j \circ h_F = u \circ \beta_F. \end{aligned}$$

Since u is mono we conclude. Let us consider the exact sequence

$$0 \to \operatorname{Ker}(\lambda_F) \xrightarrow{j_F} U^{(F)} \xrightarrow{\lambda_F} X$$

Since $\mathcal{A}$ is a Grothendieck category, we have that $\lim_{\to}$ are exact and hence we get the exact sequence

$$0 \to \varinjlim \operatorname{Ker}(\lambda_F) \xrightarrow{\lim_{\to} j_F} \varinjlim U^{(F)} = U^{(\operatorname{Hom}_{\mathcal{A}}(U,X))} \xrightarrow{\lim_{\to} \lambda_F = \lambda} X.$$

It follows that $\operatorname{Ker}(\lambda) = \lim_{\to} \operatorname{Ker}(\lambda_F)$ and hence there exists a unique monomorphism $\beta = \lim_{\to} \beta_F : \lim_{\to} \operatorname{Ker}(\lambda_F) = \operatorname{Ker}(\lambda) \to \operatorname{Ker}(\mu)$ such that

$$\beta \circ h_F = \beta_F \text{ for every finite subset } F \text{ of } \operatorname{Hom}_{\mathcal{A}}(U, X).$$

Since for every finite subset F of $\operatorname{Hom}_{\mathcal{A}}(U, X)$

$$u \circ \beta \circ h_F = u \circ \beta_F = j \circ h_F$$

we get that

$$u \circ \beta = j$$

and hence

$$\mu \circ j = \mu \circ u \circ \beta = 0$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(\lambda) \arrow[r, "j"] & U^{(\operatorname{Hom}_{\mathcal{A}}(U,X))} \arrow[r, "\lambda"] \arrow[d, "p"'] & X \arrow[r, "\chi=0"] \arrow[dl, "t"'] & \operatorname{Coker}(\lambda) \\
& & \operatorname{Coker}(j) \arrow[r, "\sim", "\overline{\lambda}"'] \arrow[d, "\widetilde{\mu}"'] & \operatorname{Ker}(\chi) \arrow[u, "\sim", "k"'] & \\
& & Z & &
\end{tikzcd}$$

Therefore there exists a unique morphism $\widetilde{\mu} : \operatorname{Im}(\lambda) \simeq \operatorname{Coker}(j) \to Z$ such that $\widetilde{\mu} \circ p = \mu$ where $p : U^{(\operatorname{Hom}_{\mathcal{A}}(U,X))} \to \operatorname{Coker}(j)$ is the canonical projection. By Lemma A.2, we have that $\operatorname{Im}(\lambda) = X$ and then $X = \operatorname{Im}(\lambda) = \operatorname{Coker}(j)$. Then there exists an isomorphism $t : X \to \operatorname{Coker}(j)$ such that $t \circ \lambda = p$. Set $g = \widetilde{\mu} \circ t$ and for every $f \in \operatorname{Hom}_{\mathcal{A}}(U, X)$, we compute

$$g \circ f = \widetilde{\mu} \circ t \circ f = \widetilde{\mu} \circ t \circ \lambda \circ i_f = \widetilde{\mu} \circ p \circ i_f = \mu \circ i_f = \varphi(f).$$

This means that $\varphi = \operatorname{Hom}_{\mathcal{A}}(U, g)$. □

Lemma A.4. *Let $\mathcal{A}$ be an abelian category and let $U \in \mathcal{A}$. Then, for every exact sequence in $\mathcal{A}$*

$$0 \to K \xrightarrow{k} X \xrightarrow{f} Y,$$

the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(U,K) \overset{\mathrm{Hom}_{\mathcal{A}}(U,k)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(U,X) \overset{\mathrm{Hom}_{\mathcal{A}}(U,f)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(U,Y)$$

is an exact sequence of abelian groups.

Proof. Let $h \in \mathrm{Hom}_{\mathcal{A}}(U,K)$. Then $\mathrm{Hom}_{\mathcal{A}}(U,k)(h) = kh$. Since k is a monomorphism it follows that $kh = 0$ if and only if $h = 0$. Hence $\mathrm{Hom}_{\mathcal{A}}(U,k)$ is also a monomorphism. Also $(\mathrm{Hom}_{\mathcal{A}}(U,f) \circ \mathrm{Hom}_{\mathcal{A}}(U,k))(h) = fkh = 0$. This implies that $\mathrm{Im}(\mathrm{Hom}_{\mathcal{A}}(U,k)) \subseteq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(U,f))$. Let now $g \in \mathrm{Hom}_{\mathcal{A}}(U,X)$ and assume that $\mathrm{Hom}_{\mathcal{A}}(U,f)(g) = 0$ i.e. $fg = 0$. Since $(K,k) = \mathrm{Ker}(f)$ there exists a morphism $g' : U \to K$ such that $g = kg' = \mathrm{Hom}_{\mathcal{A}}(U,k)(g') \in \mathrm{Im}(\mathrm{Hom}_{\mathcal{A}}(U,k))$. Therefore we get that $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(U,f)) \subseteq \mathrm{Im}(\mathrm{Hom}_{\mathcal{A}}(U,k))$ and hence $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(U,f)) = \mathrm{Im}(\mathrm{Hom}_{\mathcal{A}}(U,k))$. □

Lemma A.5. *In the assumptions and notations of A.1, let (T,H) be an adjunction where $T : \mathcal{B} \to \mathcal{A}$ and $H : \mathcal{A} \to \mathcal{B}$ and let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then f is a monomorphism (resp. epimorphism) if and only if $TH(f)$ is a monomorphism (resp. epimorphism).*

Proof. First of all, for every $X \in \mathcal{A}$ let $\epsilon X : TH(X) \to X$ be the counit of the adjunction (T,H). Then, in view of Proposition A.3 and Proposition 2.32, ϵX is an isomorphism and for every morphism $f : X \to Y$ in $\mathcal{A}$ we have

$$f \circ \epsilon X = \epsilon Y \circ TH(f).$$

Thus f is mono (resp. epi) if and only if $TH(f)$ is mono (resp. epi). □

Lemma A.6. *In the assumptions and notations of A.1, let $m,n \in \mathbb{N}, m,n \geq 1$, let $\overline{f} : B^m \to B^n$ be a morphism of right B-modules, let $X = \mathrm{Coim}\left(\overline{f}\right)$ and let $j : X \to B^n$ be the canonical injection. Let $T : Mod\text{-}B \to \mathcal{A}$ be a left adjoint of the functor $\mathrm{Hom}_{\mathcal{A}}(U,-) : \mathcal{A} \to Mod\text{-}B$. Then $T(j)$ is a monomorphism.*

Proof. Let $m,n \in \mathbb{N}, m,n \geq 1$, let $f : U^m \to U^n$ be a morphism in $\mathcal{A}$. Let us consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r, "k"] & U^m \arrow[r, "f"] \arrow[d, "p"'] & U^n \\
& & \mathrm{Coim}(f) \arrow[ur, "i"'] \arrow[d] & \\
& 0 \arrow[ur] & 0 &
\end{tikzcd}$$

where p is the canonical epimorphism and i is the canonical monomorphism. By applying to it the functor $H = \mathrm{Hom}_{\mathcal{A}}(U,-)$, in view of Lemma A.4, we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(H(f)) = H(\operatorname{Ker}(f)) \arrow[r, "H(k)"] & H(U^m) \arrow[r, "H(f)"] \arrow[d, "H(p)"'] & H(U^n) \\
& & H(\operatorname{Coim}(f)) \arrow[ur, "H(i)"'] & \\
& 0 \arrow[ur] & &
\end{tikzcd}$$

Since $\operatorname{Coim}(H(f)) = \operatorname{Coker}(H(k))$ and $H(p) \circ H(k) = 0$ there exists a unique morphism $\zeta : \operatorname{Coim}(H(f)) \to H(\operatorname{Coim}(f))$ such that

(259) $$H(p) = \zeta \circ q$$

where $q : H(U^m) \to \operatorname{Coim}(H(f))$ is the canonical epimorphism. Let $j : \operatorname{Coim}(H(f)) \to H(U^n)$ be the canonical monomorphism such that $j \circ q = H(f)$. Then from $j \circ q \circ H(k) = H(f \circ k) = 0$ we get that

(260) $$q \circ H(k) = 0.$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(H(f)) = H(\operatorname{Ker}(f)) \arrow[r, "H(k)"] & H(U^m) \arrow[rr, "H(f)"] \arrow[d, "H(p)"'] \arrow[drr, "q", near start] & & H(U^n) & \\
& & H(\operatorname{Coim}(f)) \arrow[urr, "H(i)"', near end] & & \operatorname{Coim}(H(f)) = \operatorname{Coker}(H(k)) \arrow[ll, dashed, "\zeta"] \arrow[u, "j"'] \arrow[dr] & \\
& 0 \arrow[ur] & & & 0 \arrow[u] & 0
\end{tikzcd}$$

From $H(i) \circ \zeta \circ q \overset{(259)}{=} H(i) \circ H(p) = H(f) = j \circ q$, since q is an epimorphism, we get that

(261) $$H(i) \circ \zeta = j.$$

Let us apply T to it having in mind that T is right exact

$$\begin{tikzcd}
0 \arrow[r] & TH(\operatorname{Ker}(f)) \arrow[r, "TH(k)"] & TH(U^m) \arrow[r, "TH(f)"] \arrow[d, "TH(p)"'] \arrow[dr, "T(q)", near start] & TH(U^n) \\
& & TH(\operatorname{Coim}(f)) \arrow[ur, "TH(i)"', near end] \arrow[r, dashed, shift left, "\xi"] & T(\operatorname{Coim}(H(f))) \arrow[u, "T(j)"'] \arrow[l, shift left, "T(\zeta)"]
\end{tikzcd}$$

Let us prove that $T(j)$ is mono. From formula (260) we obtain that $T(q) \circ TH(k) = T(q \circ H(k)) = 0$. Since

$$\begin{aligned}
TH(\operatorname{Coim}(f)) &= \operatorname{Coim}(TH(f)) = \operatorname{Coker}(\operatorname{Ker}(TH(f))) \\
&= \operatorname{Coker}(TH(\operatorname{Ker}(f))) = \operatorname{Coker}(TH(k))
\end{aligned}$$

there exists a unique $\xi : TH(\operatorname{Coim}(f)) \to T(\operatorname{Coim}(H(f)))$ such that

(262) $$\xi \circ TH(p) = T(q).$$

We have

$$T(\zeta)\circ\xi\circ TH(p)\overset{(262)}{=}T(\zeta)\circ T(q)\overset{(259)}{=}TH(p)$$

and since $TH(p)$ is epi by Lemma A.5, we get

$$T(\zeta)\circ\xi=\mathrm{Id}_{TH(\mathrm{Coim}(f))}.$$

We compute

$$\xi\circ T(\zeta)\circ T(q)\overset{(259)}{=}\xi\circ TH(p)\overset{(262)}{=}T(q)$$

and since $T(q)$ is epi, we obtain that

$$\xi\circ T(\zeta)=\mathrm{Id}_{T(\mathrm{Coim}(H(f)))}.$$

Therefore $T(\zeta)$ is an isomorphism. From formula (261) we get that $TH(i)\circ T(\zeta)=T(j)$ and by Lemma A.5 we conclude that $T(j)$ is a monomorphism.

Let $m,n\in\mathbb{N}, m,n\geq 1$, let $\overline{f}:B^m\to B^n$ be a morphism of right B-modules and let $X=\mathrm{Coim}\left(\overline{f}\right)$. Since U is a generator, by Proposition A.3, there exists a unique morphism $f:U^m\to U^n$ such that $H(f)=\overline{f}$. Then, by the foregoing, $X=\mathrm{Coim}(H(f))$ and $T(j)$ is a monomorphism where $j:X\to B^n$ is the canonical monomorphism. $\square$

Lemma A.7. *Let B be a ring and let X be a submodule of a free module $A^{(Z)}$. Let $\mathcal{P}_0(X)$ be the set of finite subsets of X and let $j_F:X_F\to X$ be the canonical inclusion of the submodule of X spanned by $F\in\mathcal{P}_0(X)$. Then for every $F\in\mathcal{P}_0(X)$ there exists a finite subset Z_F of Z and a monomorphism $i_F:X_F\to A^{(Z_F)}$ such that the diagram*

$$\begin{tikzcd} X_F \arrow[r, "j_F"] \arrow[dr, "i_F"'] & X \arrow[r, "j"] & A^{(Z)} \\ & A^{(Z_F)} \arrow[ur, "h_F"'] & \end{tikzcd}$$

where $j:X\to A^{(Z)}$ is the canonical inclusion and $h_F:A^{(Z_F)}\to A^{(Z)}$ is the canonical section of the canonical projection $\pi_F:A^{(Z)}\to A^{(Z_F)}$, is commutative. Moreover $(i_F)_{F\in\mathcal{P}_0(X)}$ is a family of morphisms between the direct systems $(X_F)_{F\in\mathcal{P}_0(X)}$ and $\left(A^{(Z_F)}\right)_{F\in\mathcal{P}_0(X)}$ and $(h_F\circ i_F)_{F\in\mathcal{P}_0(X)}$ is a compatible family of morphisms such that

$$\varinjlim(h_F\circ i_F)=j.$$

Proof. Let $(e_z)_{z\in Z}$ be the canonical basis of $A^{(Z)}$. Then, for every $x\in X$ there exists a finite subset F_x of Z such that

$$x=\sum\nolimits_{z\in F_x}e_za_z\text{ where }a_z\in A\text{ for every }z\in F_x.$$

For every $F\in\mathcal{P}_0(X)$ let us set

$$Z_F=\bigcup_{x\in F}F_x$$

and let $\left(e_z^F\right)_{z\in Z_F}$ be the canonical basis of $A^{(Z_F)}$. Then the assignment

$$e_z^F\mapsto e_z\text{ where }z\in Z_F$$

yields the canonical section $h_F : A^{(Z_F)} \to A^{(Z)}$ of the canonical projection $\pi_F : A^{(Z)} \to A^{(Z_F)}$ since $\pi_F(e_z) = e_z^F$ for every $z \in F$. Set $i_F = \pi_F \circ j \circ j_F$. Then we have

$$\operatorname{Im}(j \circ j_F) \subseteq \sum_{z \in F} e_z A = \sum_{z \in F} (h_F \circ \pi_F)(e_z) A$$

and since

$$(h_F \circ \pi_F)(e_z) = e_z \text{ for every } z \in F$$

we obtain that

$$h_F \circ i_F = h_F \circ \pi_F \circ j \circ j_F = j \circ j_F. \tag{263}$$

Assume now that $F, G \in \mathcal{P}_0(X)$ and that $F \subseteq G$. Then $Z_F \subseteq Z_G$ so that we can consider the canonical section $h_F^G : A^{(Z_F)} \to A^{(Z_G)}$ of the canonical projection $\pi_F^G : A^{(Z_G)} \to A^{(Z_F)}$. We have

$$\pi_F^G\left(e_z^G\right) = e_z^F \text{ and } h_F^G\left(e_z^F\right) = e_z^G \text{ for every } z \in Z_F.$$

Moreover

$$h_G \circ h_F^G = h_F.$$

Let $j_F^G : X_F \to X_G$ be the canonical inclusion. Then

$$j_G \circ j_F^G = j_F \text{ and } \pi_G \circ h_F = h_F^G$$

so that we get

$$i_G \circ j_F^G = \pi_G \circ j \circ j_G \circ j_F^G = \pi_G \circ j \circ j_F =$$
$$\overset{(263)}{=} \pi_G \circ h_F \circ i_F = h_F^G \circ i_F$$

and hence

$$h_G \circ i_G \circ j_F^G = h_G \circ h_F^G \circ i_F = h_F \circ i_F$$

which proves that $(h_F \circ i_F)_{F \in \mathcal{P}_0(X)}$ is a compatible family of morphisms. Since $\lim_{\longrightarrow} (X_F)_{F \in \mathcal{P}_0(X)} = X$, to prove that

$$\lim_{\longrightarrow} (h_F \circ i_F) = j$$

it is enough to prove that

$$j \circ j_F = h_F \circ i_F$$

for every $F \in \mathcal{P}_0(X)$. This holds in view of (263). $\square$

Lemma A.8. *Let $f : X \to Y$ and $p : W \to X$ be morphisms in an abelian category $\mathcal{A}$. Assume that p is an epimorphism and that*

$$\operatorname{Ker}(f \circ p) = \operatorname{Ker}(p).$$

Then f is a monomorphism.

Proof. Since p is an epimorphism, we have that $\operatorname{Coker}(\operatorname{Ker}(p)) = (X, p)$. It follows that $\operatorname{Coker}(\operatorname{Ker}(f \circ p)) = (X, p)$. Let $\overline{f \circ p} : \operatorname{Coker}(\operatorname{Ker}(f \circ p)) \to \operatorname{Ker}(\operatorname{Coker}(f \circ p))$ be the isomorphism such that

$$f \circ p = k \circ \overline{f \circ p} \circ p \tag{264}$$

where $k : \mathrm{Ker}\left(\mathrm{Coker}\left(f \circ p\right)\right) \to Y$ is the canonical monomorphism. Since p is an epimorphism, from formula (264) we obtain that $f = k \circ \overline{f \circ p}$ and hence f is a monomorphism. $\square$

Theorem A.9 ([Po, page 112]). *Let $\mathcal{A}$ be a Grothendieck category, let U be an object of $\mathcal{A}$ and let $B = \mathrm{Hom}_{\mathcal{A}}\left(U, U\right)$. Assume that U is a generator of $\mathcal{A}$ and that there exists a left adjoint $T : Mod\text{-}B \to \mathcal{A}$ of the functor $\mathrm{Hom}_{\mathcal{A}}\left(U, -\right) : \mathcal{A} \to Mod\text{-}B$. Then T is an exact functor.*

Proof. By Proposition A.3, H is full and faithful. Since T is a left adjoint so that it preserves epimorphisms, we have only to prove that it is left exact.

Now let X be a submodule of a free right B-module $B^{(Z)}$. Let $\mathcal{P}_0\left(X\right)$ be the set of finite subset of X and let $j_F : X_F \to X$ be the canonical inclusion of the submodule of X spanned by $F \in \mathcal{P}_0\left(X\right)$. By Lemma A.7, for every $F \in \mathcal{P}_0\left(X\right)$ there exists a finite subset Z_F of Z and a monomorphism $i_F : X_F \to A^{(Z_F)}$ such that the diagram

$$\begin{tikzcd} X_F \arrow[r, "j_F"] \arrow[dr, "i_F"'] & X \arrow[r, "j"] & A^{(Z)} \\ & A^{(Z_F)} \arrow[ur, "h_F"'] & \end{tikzcd}$$

where $j : X \to A^{(Z)}$ is the canonical inclusion and $h_F : A^{(Z_F)} \to A^{(Z)}$ is the canonical section of the canonical projection $\pi_F : A^{(Z)} \to A^{(Z_F)}$, is commutative. Moreover $\left(h_F \circ i_F\right)_{F \in \mathcal{P}_0(X)}$ is a compatible family of morphisms such that

$$\varinjlim \left(h_F \circ i_F\right) = j.$$

Since T is a left adjoint functor, we have

$$T\left(j\right) = T\left(\varinjlim \left(h_F \circ i_F\right)\right) = \varinjlim T\left(h_F \circ i_F\right).$$

By Lemma A.6 we know that $T\left(i_F\right)$ is a monomorphism. On the other hand $\pi_F \circ h_F = \mathrm{Id}_{A^{(Z_F)}}$ and hence also $T\left(h_F\right)$ is a monomorphism. Since $\mathcal{A}$ is a Grothendieck category, direct limits are exact in $\mathcal{A}$ and hence $T\left(j\right)$ is a monomorphism.
Finally let

$$0 \to L \xrightarrow{f} M$$

be a monomorphism in $Mod\text{-}B$. Then we can construct the following commutative diagram with exact rows and columns

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathrm{Ker}\left(p\right) \arrow[r, "i'"] \arrow[d, "\mathrm{Id}_{\mathrm{Ker}(p)}"'] & P \arrow[r, "p'"] \arrow[d, "f'"'] & L \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\left(p\right) \arrow[r, "i"] & B^{(M)} \arrow[r, "p"] & M \arrow[r] & 0 \end{tikzcd}$$

where $p : B^{(M)} \to M$ is the usual epimorphism of right B-modules and (P, f', p') is the pullback of (p, f). Recall that

$$P = \left\{(x, y) \in B^{(M)} \times L \mid p(x) = f(y)\right\}$$

and $f' : P \to B^{(M)}$ is defined by setting $f'((x, y)) = x$ while $p' : P \to L$ is defined by setting $p'((x, y)) = y$. Moreover $i' : \mathrm{Ker}(p) \to P$ is defined by setting $i'(x) = (x, 0)$. Since f is a monomorphism we have that also f' is a monomorphism and since p is an epimorphism, also p' is an epimorphism. Then, by the foregoing, both $T(f')$ and $T(i)$ are monomorphism. Since $T(i)$ is a monomorphism we get that $T(i')$ is also a monomorphism so that $(T(\mathrm{Ker}(p)), T(i'))$ is a kernel of $T(p')$. Since $T(f')$ is a monomorphism and $T(f')T(i') = T(i)$ we get that $(T(\mathrm{Ker}(p)), T(i'))$ is also a kernel of $T(p)T(f')$. In fact $T(p)T(f')T(i') = T(p)T(i) = 0$ and if $\zeta : Z \to T(P)$ is a morphism such that $T(p)T(f')\zeta = 0$ there exists a unique morphism $\zeta' : Z \to T(\mathrm{Ker}(p))$ such that $T(f')\zeta = T(i)\zeta'$ so that $T(f')\zeta = T(i)\zeta' = T(f')T(i')\zeta'$ and since $T(f')$ is mono we get that $\zeta = T(i')\zeta'$. Since $T(p)T(f') = T(f)T(p')$ we deduce that $(T(\mathrm{Ker}(p)), T(i'))$ is a kernel of $T(f)T(p')$. Therefore we obtain that

$$\mathrm{Ker}(T(f)T(p')) = \mathrm{Ker}(T(p')).$$

Since T is right exact we know that $T(p')$ is an epimorphism and hence, in view of Lemma A.8, we deduce that $T(f)$ is a monomorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & T(\mathrm{Ker}(p)) \arrow[r, "T(i')"] \arrow[d, "\mathrm{Id}_{T(\mathrm{Ker}(p))}"'] & T(P) \arrow[r, "T(p')"] \arrow[d, "T(f')"'] & T(L) \arrow[r] \arrow[d, "T(f)"'] & 0 \\
0 \arrow[r] & T(\mathrm{Ker}(p)) \arrow[r, "T(i)"] & T\left(B^{(M)}\right) \arrow[r, "T(p)"] & T(M) \arrow[r] & 0
\end{tikzcd}$$

□

Lemma A.10. *Let $\mathcal{C}$ be an A-coring. Then the category $(Mod\text{-}A)^{\mathbb{C}}$ has coproducts and cokernels so that it is cocomplete. Moreover if $U : (Mod\text{-}A)^{\mathbb{C}} \to (Mod\text{-}A)$ is the forgetful functor we have*

$$U\left(\left(\coprod (M_i, \rho_i)_{i\in I}, \rho^M\right)\right) = \bigoplus_{i\in I} M_i \text{ and } U\left(\left(\mathrm{Coker}(f), \rho^{\mathrm{Coker}(f)}\right)\right) = \mathrm{Coker}(f).$$

Proof. Let $(M_\iota, \rho_i)_{i\in I}$ be a family of right $\mathcal{C}$-comodules and let $\varepsilon_i : M_i \to M = \bigoplus_{i\in I} M_i$ be the canonical injection and $\pi_i : M \to M_i$ the canonical projection. Since $\pi_i\varepsilon_i = \mathrm{Id}_{M_i}$ the map

$$\varepsilon_i \otimes_A \mathcal{C} : M_i \otimes_A \mathcal{C} \to M \otimes_A \mathcal{C}$$

is injective and hence also the map

$$\overline{\rho_i} = (\varepsilon_i \otimes_A \mathcal{C}) \circ \rho_i : M_i \to M \otimes_A \mathcal{C}$$

is injective. Let

$$\rho^M : M \to M \otimes_A \mathcal{C}$$

be the codiagonal map of the $\overline{\rho_i}$. Then ρ^M is uniquely defined by

$$\rho^M \circ \varepsilon_i = \overline{\rho_i}.$$

Then $\left(M, \rho^M\right) \in (Mod\text{-}A)^{\mathbb{C}}$. In fact, for every $i \in I$ we have

$$\begin{aligned}
\left(\rho^M \otimes_A \mathcal{C}\right) \circ \rho^M \circ \varepsilon_i &= \left(\rho^M \otimes_A \mathcal{C}\right) \circ \overline{\rho_i} = \left(\rho^M \otimes_A \mathcal{C}\right) \circ (\varepsilon_i \otimes_A \mathcal{C}) \circ \rho_i \\
= (\overline{\rho_i} \otimes_A \mathcal{C}) \circ \rho_i &= (\varepsilon_i \otimes_A \mathcal{C}) \circ (\rho_i \otimes_A \mathcal{C}) \circ \rho_i = (\varepsilon_i \otimes_A \mathcal{C}) \circ \left(M_i \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho_i \\
= \left(M \otimes_A \Delta^{\mathcal{C}}\right) \circ (\varepsilon_i \otimes_A \mathcal{C}) \circ \rho_i &= \left(M \otimes_A \Delta^{\mathcal{C}}\right) \circ \overline{\rho_i} = \left(M \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho^M \circ \varepsilon_i
\end{aligned}$$

and

$$\begin{aligned}
r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^M \circ \varepsilon_i &= r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \overline{\rho_i} = r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right) \circ (\varepsilon_i \otimes_A \mathcal{C}) \circ \rho_i \\
&= r_M \circ (\varepsilon_i \otimes_A A) \circ \left(M_i \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho_i = \varepsilon_i \circ r_{Mi} \circ \left(M_i \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho_i = \varepsilon_i.
\end{aligned}$$

Let $f : \left(M, \rho^M\right) \to \left(N, \rho^N\right)$ be a morphism in $(Mod\text{-}A)^{\mathbb{C}}$ so that $f : M \to N$ is a morphism in $Mod\text{-}A$ and let us consider

$$M \xrightarrow{f} N \xrightarrow{p} \mathrm{Coker}\,(f) \to 0$$

the cokernel of f in $Mod\text{-}A$. Then we have the following diagram in $Mod\text{-}A$

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\rho^M"] & N \arrow[r, "p"] \arrow[d, "\rho^N"] & \mathrm{Coker}\,(f) \arrow[r] \arrow[d, "\rho^{\mathrm{Coker}(f)}"] & 0 \\
M \otimes_A \mathcal{C} \arrow[r, "f \otimes_A \mathcal{C}"] & N \otimes_A \mathcal{C} \arrow[r, "p \otimes_A \mathcal{C}"] & \mathrm{Coker}\,(f) \otimes_A \mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

We compute

$$(p \otimes_A \mathcal{C}) \circ \rho^N \circ f = (p \otimes_A \mathcal{C}) \circ (f \otimes_A \mathcal{C}) \circ \rho^M = (pf \otimes_A \mathcal{C}) \circ \rho^M = 0$$

by the universal property of the cokernel there exists a unique morphism $\rho^{\mathrm{Coker}(f)} : \mathrm{Coker}\,(f) \to \mathrm{Coker}\,(f) \otimes_A \mathcal{C}$ such that

$$\rho^{\mathrm{Coker}(f)} \circ p = (p \otimes_A \mathcal{C}) \circ \rho^N.$$

Let us check that $\left(\mathrm{Coker}\,(f), \rho^{\mathrm{Coker}(f)}\right) \in (Mod\text{-}A)^{\mathbb{C}}$. Let us compute

$$\begin{aligned}
\left(\rho^{\mathrm{Coker}(f)} \otimes_A \mathcal{C}\right) \circ \rho^{\mathrm{Coker}(f)} \circ p &= \left(\rho^{\mathrm{Coker}(f)} \otimes_A \mathcal{C}\right) \circ (p \otimes_A \mathcal{C}) \circ \rho^N \\
= (p \otimes_A \mathcal{C} \otimes_A \mathcal{C}) \circ \left(\rho^N \otimes_A \mathcal{C}\right) \circ \rho^N &= (p \otimes_A \mathcal{C} \otimes_A \mathcal{C}) \circ \left(N \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho^N \\
= \left(\mathrm{Coker}\,(f) \otimes_A \Delta^{\mathcal{C}}\right) \circ (p \otimes_A \mathcal{C}) \circ \rho^N &= \left(\mathrm{Coker}\,(f) \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho^{\mathrm{Coker}(f)} \circ p
\end{aligned}$$

and

$$\begin{aligned}
& r_{\mathrm{Coker}(f)} \circ \left(\mathrm{Coker}\,(f) \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^{\mathrm{Coker}(f)} \circ p \\
&= r_{\mathrm{Coker}(f)} \circ \left(\mathrm{Coker}\,(f) \otimes_A \varepsilon^{\mathcal{C}}\right) \circ (p \otimes_A \mathcal{C}) \circ \rho^N \\
&= r_{\mathrm{Coker}(f)} \circ (p \otimes_A A) \circ \left(N \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^N = p \circ r_N \circ \left(N \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^N = p
\end{aligned}$$

and since p is epi we conclude. Now, let $\zeta : \left(N, \rho^N\right) \to \left(Z, \rho^Z\right)$ be a morphism in $(Mod\text{-}A)^{\mathbb{C}}$ such that $\zeta \circ f = 0$. Then, there exists a unique morphism $\zeta' : \mathrm{Coker}\,(f) \to Z$ in $Mod\text{-}A$ such that $\zeta' \circ p = \zeta$. We want to prove that $\zeta' \in (Mod\text{-}A)^{\mathbb{C}}$. We compute

$$\rho^Z \circ \zeta' \circ p = \rho^Z \circ \zeta = (\zeta \otimes_A \mathcal{C}) \circ \rho^N$$

$$= (\zeta' \otimes_A \mathcal{C}) \circ (p \otimes_A \mathcal{C}) \circ \rho^N = (\zeta' \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Coker}(f)} \circ p$$

and since p is an epimorphism we get that $\zeta' \in (Mod\text{-}A)^{\mathbb{C}}$. □

Lemma A.11. *Let $\mathcal{C}$ be an A-coring and assume that ${}_A\mathcal{C}$ is flat. Then the category $(Mod\text{-}A)^{\mathbb{C}}$ has kernels. Moreover if $U : (Mod\text{-}A)^{\mathbb{C}} \to (Mod\text{-}A)$ is the forgetful functor we have*

$$U\left(\left(\mathrm{Ker}\,(f)\,, \rho^{\mathrm{Ker}(f)}\right)\right) = \mathrm{Ker}\,(f)\,.$$

Proof. Since by Lemma A.10 the preadditive category $(Mod\text{-}A)^{\mathbb{C}}$ has coproducts, it also has products. Now, let $f : \left(M, \rho^M\right) \to \left(N, \rho^N\right)$ a morphism in $(Mod\text{-}A)^{\mathbb{C}}$. Then in $Mod\text{-}A$ we can consider the exact sequence

$$0 \to \mathrm{Ker}\,(f) \xrightarrow{k} M \xrightarrow{f} N$$

and, since ${}_A\mathcal{C}$ is flat, we get the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,(f) \arrow[r, "k"] \arrow[d, "\rho^{\mathrm{Ker}(f)}"] & M \arrow[r, "f"] \arrow[d, "\rho^M"] & N \arrow[d, "\rho^N"] \\
0 \arrow[r] & \mathrm{Ker}\,(f) \otimes_A \mathcal{C} \arrow[r, "k\otimes_A\mathcal{C}"] & M \otimes_A \mathcal{C} \arrow[r, "f\otimes_A\mathcal{C}"] & N \otimes_A \mathcal{C}
\end{tikzcd}$$

We have

$$0 = \rho^N \circ f \circ k = (f \otimes_A \mathcal{C}) \circ \rho^M \circ k.$$

By the properties of the kernel of f there exists a unique morphism $\rho^{\mathrm{Ker}(f)} : \mathrm{Ker}\,(f) \to \mathrm{Ker}\,(f) \otimes_A \mathcal{C}$ such that

$$\rho^M \circ k = (k \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Ker}(f)}.$$

We have to prove that $\left(\mathrm{Ker}\,(f)\,, \rho^{\mathrm{Ker}(f)}\right) \in (Mod\text{-}A)^{\mathbb{C}}$. Let us compute

$$(k \otimes_A \mathcal{C} \otimes_A \mathcal{C}) \circ \left(\rho^{\mathrm{Ker}(f)} \otimes_A \mathcal{C}\right) \circ \rho^{\mathrm{Ker}(f)} = \left(\rho^M \otimes_A \mathcal{C}\right) \circ (k \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Ker}(f)}$$

$$= \left(\rho^M \otimes_A \mathcal{C}\right) \circ \rho^M \circ k = \left(M \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho^M \circ k = \left(M \otimes_A \Delta^{\mathcal{C}}\right) \circ (k \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Ker}(f)}$$

$$= (k \otimes_A \mathcal{C} \otimes_A \mathcal{C}) \circ \left(\mathrm{Ker}\,(f) \otimes_A \Delta^{\mathcal{C}}\right) \circ \rho^{\mathrm{Ker}(f)}$$

and

$$k \circ r_{\mathrm{Ker}(f)} \circ \left(\mathrm{Ker}\,(f) \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^{\mathrm{Ker}(f)} = r_M \circ (k \otimes_A \mathcal{C}) \circ \left(\mathrm{Ker}\,(f) \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^{\mathrm{Ker}(f)}$$

$$= r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right) \circ (k \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Ker}(f)} = r_M \circ \left(M \otimes_A \varepsilon^{\mathcal{C}}\right) \circ \rho^M \circ k = k.$$

Since k is mono we conclude. Let now $\zeta : \left(Z, \rho^Z\right) \to \left(M, \rho^M\right)$ be a morphism in $(Mod\text{-}A)^{\mathbb{C}}$ such that $f \circ \zeta = 0$. Then by the universal property of the kernel of f in $Mod\text{-}A$ there exists a unique morphism $\zeta' : Z \to \mathrm{Ker}\,(f)$ such that

$$k \circ \zeta' = \zeta.$$

We want to prove that $\zeta' \in (Mod\text{-}A)^{\mathbb{C}}$. Let us compute

$$(k \otimes_A \mathcal{C}) \circ \rho^{\mathrm{Ker}(f)} \circ \zeta' = \rho^M \circ k \circ \zeta' = \rho^M \circ \zeta = (\zeta \otimes_A \mathcal{C}) \circ \rho^Z$$

$$= (k \otimes_A \mathcal{C}) \circ (\zeta' \otimes_A \mathcal{C}) \circ \rho^Z$$

and since $k \otimes_A \mathcal{C}$ is mono we conclude that $\zeta' \in (Mod\text{-}A)^{\mathbb{C}}$ and $U\left(\left(\mathrm{Ker}\,(f)\,, \rho^{\mathrm{Ker}(f)}\right)\right) = \mathrm{Ker}\,(f)\,.$ □

Proposition A.12 ([ELGO2, Proposition 1.2]). *Let $\mathcal{C}$ be an A-coring. Then the following are equivalent*

(a) ${}_A\mathcal{C}$ *is flat.*

(b) $(Mod\text{-}A)^{\mathbb{C}}$ *is an abelian category and the forgetful functor* $U : (Mod\text{-}A)^{\mathbb{C}} \to Mod\text{-}A$ *is left exact (and hence exact).*

(c) $(Mod\text{-}A)^{\mathbb{C}}$ *is a Grothendieck category and the forgetful functor* $U : (Mod\text{-}A)^{\mathbb{C}} \to Mod\text{-}A$ *is left exact (and hence exact).*

Proof. By Lemma A.10, the category $(Mod\text{-}A)^{\mathbb{C}}$ has coproducts and cokernels.

$(a) \Rightarrow (c)$ By Lemma A.11 has kernels. Consider the following diagram

$$\begin{tikzcd}
\left(\mathrm{Ker}\,(f), \rho^{\mathrm{Ker}(f)}\right) \arrow[r, "k"] & \left(M, \rho^M\right) \arrow[r, "f"] \arrow[d, "\chi'"'] \arrow[dr, dashed, "\rho"] & \left(N, \rho^N\right) \arrow[r, "\chi"] & \left(\mathrm{Coker}\,(f), \rho^{\mathrm{Coker}(f)}\right) \\
 & \left(\mathrm{Coker}\,(k), \rho^{\mathrm{Coker}(k)}\right) \arrow[r, dotted, "\bar{f}"'] & \left(\mathrm{Ker}\,(\chi), \rho^{\mathrm{Ker}(\chi)}\right) . \arrow[u, "k'"'] &
\end{tikzcd}$$

in $(Mod\text{-}A)^{\mathbb{C}}$. Then we get the diagram

$$\begin{tikzcd}
\mathrm{Ker}\,(f) \arrow[r, "k"] & M \arrow[r, "f"] \arrow[d, "\chi'"'] \arrow[dr, dashed, "\rho"] & N \arrow[r, "\chi"] & \mathrm{Coker}\,(f) \\
 & \mathrm{Coker}\,(k) \arrow[r, dotted, "\bar{f}"'] & \mathrm{Ker}\,(\chi) . \arrow[u, "k'"'] &
\end{tikzcd}$$

in $Mod\text{-}A$. Since $Mod\text{-}A$ is preabelian, $\overline{f}$ is an isomorphism in $Mod\text{-}A$ and hence also in $(Mod\text{-}A)^{\mathbb{C}}$. Thus also the category $(Mod\text{-}A)^{\mathbb{C}}$ is preabelian and moreover abelian (there exist products of every finite family of objects in the category). Moreover, by Lemma A.10 and Lemma A.11 U is left exact. Further, the direct limits are exacts for module categories and thus also for $(Mod\text{-}A)^{\mathbb{C}}$. We now have to find a generator for $(Mod\text{-}A)^{\mathbb{C}}$. Let $\left(M, \rho^M\right) \in (Mod\text{-}A)^{\mathbb{C}}$ and let $p : A^{(M)} \to M$ is the usual epimorphism of right A-modules. Let us consider the epimorphism l given by the following composite

$$l : \mathcal{C}^{(M)} \longrightarrow A^{(M)} \otimes_A \mathcal{C} \stackrel{p \otimes_A \mathcal{C}}{\longrightarrow} M \otimes_A \mathcal{C} \longrightarrow 0$$

where the first arrow is the canonical isomorphism and the second one is the usual epimorphism so that $l\left((c_m)_{m \in M}\right) = \sum_{m \in M} m \otimes_A c_m$ where c_m are almost all zero. Then we have the following diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P \arrow[r, "g"] \arrow[d, "\rho"'] & M \arrow[r] \arrow[d, "\rho^M"'] & 0 \\
C^{(M)} \arrow[r, "l"] & M \otimes_A \mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

where (P,ρ,g) is the pullback of $\left(l,\rho^M\right)$. Recall that P is the submodule of

$$\mathcal{C}^{(M)}\times M$$

defined by setting

$$P=\left\{(x,m)\in\mathcal{C}^{(M)}\times M\mid l\left(x\right)=\rho^M\left(m\right)\right\}$$

and $\rho:P\to\mathcal{C}^{(M)}$ is defined by setting $\rho\left((x,m)\right)=x$ while $g:P\to M$ by setting $g\left((x,m)\right)=m$. Since ρ^M is mono, ρ is also mono (thus $\rho\left(P\right)=H$ is a subcomodule of $\mathcal{C}^{(M)}$) and since l is epi, g is epi. Denote by $h_F^M:\mathcal{C}^{(F)}\to\mathcal{C}^{(M)}$ the canonical inclusion for any $F\subseteq M$. Let $m\in M$, then there exist $n\in\mathbb{N}$ and $F_m=\{y_1,y_2,\ldots,y_n\}\subseteq M$ such that $\rho^M\left(m\right)=\sum_{y\in F_m}y\otimes_A c_y=l\left(h_{F_m}^M\left(\left(c_y\right)_{y\in F_m}\right)\right)$. Then, for every $m\in M$, there exists $z\in P$ such that $m=g\left(z\right)=g\left(\left(\rho\left(z\right),m\right)\right)$ where $\rho\left(z\right)=h_{F_m}^M\left(\left(c_y\right)_{y\in F_m}\right)\in h_{F_m}^M\left(\mathcal{C}^{(F_m)}\right)\subseteq\mathcal{C}^{(M)}$. Thus, for every $m\in M$, there exists $x_m=\left(c_y\right)_{y\in F_m}\in$ such that $m=g\left(\left(h_{F_m}^M\left(x_m\right),m\right)\right)$. Then we have defined the following homomorphism

$$\begin{array}{cccc}\nu_{x_m}: & x_mA & \longrightarrow & M\\ & x_m & \mapsto & g\left(\left(h_{F_m}^M\left(x_m\right),m\right)\right).\end{array}$$

such that

$$m=\nu_{x_m}\left(x_m\right).$$

Since $x_mA\subseteq\mathcal{C}^{(F_m)}$, we deduce that the subcomodules of $\mathcal{C}^{(\mathbb{N})}$ form a set of generators for $(Mod\text{-}A)^{\mathbb{C}}$ i.e. $\bigoplus\limits_{H\subseteq\mathcal{C}^{(\mathbb{N})}}H$ is a generator for $(Mod\text{-}A)^{\mathbb{C}}$.

$(c)\Rightarrow(b)$ Obvious.

$(b)\Rightarrow(a)$ By Example 4.3 and Definition 4.10 $F:Mod\text{-}A\to(Mod\text{-}A)^{\mathbb{C}}$ is a right adjoint of U and then F is left exact. Then using the hypothesis that U is left exact, we deduce that $U\circ F:Mod\text{-}A\to Mod\text{-}A$ is also left exact, i.e. ${}_A\mathcal{C}$ is flat. □

Definition A.13. Let $\mathcal{A}$ be a Grothendieck category. An object $A\in\mathcal{A}$ is called *finitely generated* if, for every direct family of subobjects $\{A_i\}_{i\in I}$ of A such that $A=\sum\limits_{i\in I}A_i$, there exists an index $i_0\in I$ such that $A=A_{i_0}$.

Proposition A.14. *Let $\mathcal{A}$ be a Grothendieck category. An object $A\in\mathcal{A}$ is finitely generated if and only if, for every family of subobjects $\{A_i\}_{i\in I}$ of A such that $\sum\limits_{i\in I}A_i=A$, there exists a finite number of subobjects $A_1,\ldots,A_n$ such that $A=\sum\limits_{i\in I}^{n}A_i$.*

Proof. $(\Leftarrow)$ Let $\{A_i\}_{i\in I}$ be a direct family of subobjects of A closed under sums and such that $A=\sum\limits_{i\in I}A_i$. By hypothesis there exists a finite number of subobjects $A_1,\ldots,A_n$ such that $A=\sum\limits_{i\in I}^{n}A_i$. Since the family is direct and closed under sums, there exists an index $i_0\in I$ such that $A=\sum\limits_{i\in I}^{n}A_i\subseteq A_{i_0}$. Then A is finitely generated.

($\Rightarrow$) Let $\{A_i\}_{i\in I}$ be a family of subobjects of A closed under sums and which contains A itself. Assume that $A = \sum_{i\in I} A_i$. Since A is finitely generated, there exists an index $i_0 \in I$ such that $A_{i_0} = A$. $\square$

Lemma A.15. *Let $\mathcal{A}$ be a Grothendieck category and let $0 \to A' \longrightarrow A \xrightarrow{p} A'' \to 0$ be an exact sequence in $\mathcal{A}$. Then if A is finitely generated A'' is also finitely generated.*

Proof. Let $(A''_i)_{i\in I}$ be a direct family of subobjects of A'' such that $A'' = \sum_{i\in I} A''_i$. Then we have, for every $i \in I$, $A''_i = A_i / A'$ for A_i subobject of A such that $A' \subseteq A_i$. Hence $(A_i)_{i\in I}$ is a direct family of subobjects of A such that $A = \sum_{i\in I} A_i$ and since A is finitely generated there exists an index $i_0 \in I$ such that $A = A_{i_0}$. Then we have $A'' = A_{i_0} / A' = A''_{i_0}$, i.e. A'' is also finitely generated. $\square$

Lemma A.16. *Let $\mathcal{A}$ be a Grothendieck category and let $A \in \mathcal{A}$ be a finitely generated object. Let $f : A \to \coprod_{i\in I} B_i$ be a morphism in $\mathcal{A}$. Then there exist a finite subset $F \subseteq I$ such that $\operatorname{Im}(f) \subseteq \sum_{i\in F} \varepsilon_i(B_i)$.*

Proof. Let $\varepsilon_i : B_i \to \coprod_{j\in I} B_j$ the canonical inclusions and consider $\nabla(\varepsilon_i)_{i\in I} : \coprod_{i\in I} B_i \to \coprod_{j\in I} B_j$ defined by setting $\nabla(\varepsilon_i)_{i\in I}\left(\coprod_{j\in I} B_j\right) = \sum_{i\in I} \varepsilon_i(B_i)$. We prove $\nabla(\varepsilon_i)_{i\in I} = \mathrm{Id}_{\coprod_{i\in I} B_i}$. In fact we have that $\nabla(\varepsilon_j)_{j\in I} \circ \varepsilon_i = \varepsilon_i = \mathrm{Id}_{\coprod_{j\in I} B_j} \circ \varepsilon_i$. Thus, $\coprod_{i\in I} B_i = \operatorname{Im}\left(\mathrm{Id}_{\coprod_{i\in I} B_i}\right) = \operatorname{Im}\left(\nabla(\varepsilon_i)_{i\in I}\right) = \sum_{i\in I} \varepsilon_i(B_i)$ where $(\varepsilon_i(B_i))_{i\in I}$ define a family of subobjects of $\coprod_{i\in I} B_i$. Let $f : A \to \coprod_{j\in I} B_j$. By Lemma A.15, since A is finitely generated also $\operatorname{Coim}(f)$ is finitely generated and, since $\operatorname{Coim}(f) \simeq \operatorname{Im}(f) \subseteq \sum_{i\in I} \varepsilon_i(B_i)$, there exists a finite subset $F \subseteq I$ such that $\operatorname{Im}(f) \subseteq \sum_{i\in F} \varepsilon_i(B_i)$. $\square$

Definition A.17. Let $\mathcal{A}$ be an abelian category. A projective object $P \in \mathcal{A}$ is called *finite* if the functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ preserves coproducts.

Proposition A.18. *Let $\mathcal{A}$ be a Grothendieck category and let P be a projective object. Then P is finite if and only if P is finitely generated.*

Proof. Assume first that $P \in \mathcal{A}$ is finite. Let $\{P_i\}_{i\in I}$ be a family of subobjects of P such that $\sum_{i\in I} P_i = P$. Let $p_i : P_i \to P$ be the canonical inclusion for every $i \in I$ and consider $p = \nabla(p_i)_{i\in I} : \coprod_{i\in I} P_i \to P$. Then we have

$$\left(\nabla(p_i)_{i\in I}\right)\left(\coprod_{i\in I} P_i\right) = \sum_{i\in I} p_i(P_i) = \sum_{i\in I} P_i = P$$

and thus p is an epimorphism. Since P is projective there exists $i : P \to \coprod_{i\in I} P_i$ such that $p \circ i = \mathrm{Id}_P$. Note that $i \in \mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i\in I} P_i\right)$ and since P is finite we have that $\nabla\left(\mathrm{Hom}\left(P, \varepsilon_i\right)\right)_{i\in I} : \coprod_{i\in I} \mathrm{Hom}_{\mathcal{A}}\left(P, B_i\right) \to \mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i\in I} B_i\right)$ is an isomorphism. Thus there exist $n \in \mathbb{N}$ and $i_1 \in \mathrm{Hom}_{\mathcal{A}}\left(P, P_1\right), \ldots, i_n \in \mathrm{Hom}_{\mathcal{A}}\left(P, P_n\right)$ such that $i = \varepsilon_1 i_1 + \cdots + \varepsilon_n i_n$. Hence $\mathrm{Id}_P = p \circ i = p \circ \left(\varepsilon_1 i_1 + \cdots + \varepsilon_n i_n\right) : P \to \coprod_{i\in 1}^{n} P_i \to P$ and then $P = \coprod_{i\in 1}^{n} P_i$ so that P is finitely generated.

Assume now that P is finitely generated. Let us denote by $\varepsilon_j : B_j \to \coprod_{i\in I} B_i$, $\varepsilon_i^F : B_i \to \coprod_{i\in F} B_i$ the canonical injections for every $F \subseteq I$ finite subset, and let us prove that

$$\nabla\left(\mathrm{Hom}\left(P, \varepsilon_i\right)\right)_{i\in I} : \coprod_{i\in I} \mathrm{Hom}_{\mathcal{A}}\left(P, B_i\right) \longrightarrow \mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i\in I} B_i\right)$$
$$\left(f_i\right)_{i\in I} \mapsto \sum_{i\in I} \varepsilon_i \circ f_i$$

is an isomorphism. First of all we prove that it is epi. Since P is finitely generated, if $f : P \to \coprod_{i\in I} B_i$ is a morphism in $\mathcal{A}$, by Lemma $A.16$, there exists a $F \subseteq I$ finite subset such that $\mathrm{Im}\left(f\right) \subseteq \sum_{i\in F} \varepsilon_i\left(B_i\right)$. Let us denote by $\overline{f} : P \to \mathrm{Im}\left(f\right)$ and $s : \mathrm{Im}\left(f\right) \hookrightarrow \sum_{i\in F} \varepsilon_i\left(B_i\right)$ the canonical inclusion. Let us consider $h = \nabla\left(\varepsilon_i\right)_{i\in F} : \coprod_{i\in F} B_i \to \coprod_{i\in I} B_i$ satisfying

$$h \circ \varepsilon_i^F = \nabla\left(\varepsilon_i\right)_{i\in F} \circ \varepsilon_i^F = \varepsilon_i \text{ for every } i \in F. \tag{265}$$

Then, by definition of the codiagonal morphism, we have that $\mathrm{Im}\left(h\right) = \sum_{i\in F} \varepsilon_i\left(B_i\right)$ and thus, if we call $\widehat{h} : \coprod_{i\in F} B_i \to \mathrm{Im}\left(h\right) = \sum_{i\in F} \varepsilon_i\left(B_i\right)$ the canonical projection, we can write

$$h = t \circ \overline{h} \tag{266}$$

where $t : \sum_{i\in F} \varepsilon_i\left(B_i\right) \to \coprod_{i\in I} B_i$ is the inclusion. Thus also f can be factorized through $\overline{f}$ by

$$f = t \circ s \circ \overline{f}. \tag{267}$$

With these notations we can rewrite (265) as follows

$$\varepsilon_i = h \circ \varepsilon_i^F = t \circ \overline{h} \circ \varepsilon_i^F. \tag{268}$$

We will prove that $\overline{h}:\coprod_{i\in F}B_i\to\sum_{i\in F}\varepsilon_i(B_i)$ is in fact an isomorphism. We define the family $(\eta_j)_{j\in I}$ by setting

$$\eta_j=0 \text{ for every } j\in I\backslash F \text{ and } \eta_j=\varepsilon_j^F \text{ for every } j\in F.$$

Then we can take $\nabla(\eta_i)_{i\in I}:\coprod_{i\in I}B_i\to\coprod_{i\in F}B_i$. Let us compute for every $j\in F$

$$\begin{aligned}\nabla(\eta_i)_{i\in I}\circ h\circ\varepsilon_j^F &\overset{\text{def}h}{=}\nabla(\eta_i)_{i\in I}\circ\nabla(\varepsilon_i)_{i\in F}\circ\varepsilon_j^F\overset{(265)}{=}\nabla(\eta_i)_{i\in I}\circ\varepsilon_j\\&=\varepsilon_j^F=\mathrm{Id}_{\coprod_{i\in F}B_i}\circ\varepsilon_j^F\end{aligned}$$

and thus $\nabla(\eta_i)_{i\in I}\circ h=\mathrm{Id}_{\coprod_{i\in F}B_i}$. Therefore we deduce that h is mono and then $\overline{h}$ is an isomorphism. Let us consider $(\delta_{ij}:B_i\to B_j)_{j\in I}$ the family defined by setting

$$\delta_{ii}=\mathrm{Id}_{B_i} \text{ and } \delta_{ij}=0 \text{ for every } j\neq i.$$

Since $\coprod_{i\in F}B_i$ is a finite coproduct, we can view it as a product and call $\pi_j:\coprod_{i\in F}B_i=\prod_{i\in F}B_i=\bigtimes_{i\in F}\to B_j$ the projections for every $j\in F$ satisfying

$$\pi_j\circ\varepsilon_i^F=\delta_{ij}\quad\text{and}\quad\sum_{i\in F}\varepsilon_i^F\pi_i=\mathrm{Id}_{\bigtimes_{i\in F}}. \tag{269}$$

Note that, by the universal property of the coproduct, there exist $q_j:\coprod_{i\in I}B_i\to B_j$ such that

$$q_j\circ\varepsilon_i=\delta_{ij}. \tag{270}$$

Let us compute, for every $i\in F$ and for every $j\in I$,

$$q_j\circ h\circ\varepsilon_i^F=q_j\circ\varepsilon_i\overset{(270)}{=}\delta_{ij}\overset{(269)}{=}\pi_j\circ\varepsilon_i^F$$

and thus

$$q_j\circ h=\pi_j. \tag{271}$$

We define the family $(f_i)_{i\in I}\in\coprod_{i\in I}\mathrm{Hom}_{\mathcal{A}}(P,B_i)$ by setting

$$f_i=\pi_i\circ\overline{h}^{-1}\circ s\circ\overline{f} \text{ for every } i\in F \text{ and } f_i=0 \text{ for every } i\in I\backslash F.$$

Note that

$$\begin{aligned}f_i=\pi_i\circ\overline{h}^{-1}\circ s\circ\overline{f}&\overset{(271)}{=}q_i\circ h\circ\overline{h}^{-1}\circ s\circ\overline{f}\\&\overset{(266)}{=}q_i\circ t\circ\overline{h}\circ\overline{h}^{-1}\circ s\circ\overline{f}=q_i\circ t\circ s\circ\overline{f}\overset{(267)}{=}q_i\circ f\end{aligned}$$

i.e.

$$f_i=q_i\circ f. \tag{272}$$

For such a family $(f_i)_{i\in I}$, we have to prove that

$$f=\sum_{i\in I}\varepsilon_i\circ f_i.$$

Since $f_i = 0$ for every $i \in I \backslash F$ we have

$$\sum_{i \in I} \varepsilon_i \circ f_i = \sum_{i \in F} \varepsilon_i \circ f_i$$

so that it is sufficient to prove that

$$f = \sum_{i \in F} \varepsilon_i \circ f_i$$

and by (267) and (268)

$$t \circ s \circ \overline{f} = \sum_{i \in F} t \circ \overline{h} \circ \varepsilon_i^F \circ f_i.$$

Since t is mono we only need to prove that

$$s \circ \overline{f} = \sum_{i \in F} \overline{h} \circ \varepsilon_i^F \circ f_i$$

and thus, for every $j \in F$, that

$$\pi_j \circ \overline{h}^{-1} \circ s \circ \overline{f} = \pi_j \circ \overline{h}^{-1} \circ \sum_{i \in F} \overline{h} \circ \varepsilon_i^F \circ f_i.$$

Let us compute

$$\begin{aligned} \pi_j \circ \overline{h}^{-1} \circ s \circ \overline{f} &\overset{(271)}{=} q_j \circ h \circ \overline{h}^{-1} \circ s \circ \overline{f} \\ &\overset{(266)}{=} q_j \circ t \circ \overline{h} \circ \overline{h}^{-1} \circ s \circ \overline{f} \\ &= q_j \circ t \circ s \circ \overline{f} \overset{(267)}{=} q_j \circ f \overset{(272)}{=} f_j. \end{aligned}$$

On the other hand

$$\begin{aligned} \pi_j \circ \overline{h}^{-1} \circ \sum_{i \in F} \overline{h} \circ \varepsilon_i^F \circ f_i &= \sum_{i \in F} \pi_j \circ \overline{h}^{-1} \circ \overline{h} \circ \varepsilon_i^F \circ f_i = \sum_{i \in F} \pi_j \circ \varepsilon_i^F \circ f_i \\ &\overset{(269)}{=} \sum_{i \in F} \delta_{ij} \circ f_i = f_j \end{aligned}$$

so that we conclude that $\coprod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(P, B_i) \overset{\nabla(\mathrm{Hom}(P,\varepsilon_i))_{i \in I}}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i \in I} B_i\right)$ is an epimorphism. Let now $(f_i)_{i \in I} \in \coprod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(P, B_i)$ where f_i's are almost all zero, be such that $\nabla(\mathrm{Hom}(P, \varepsilon_i))_{i \in I}\left((f_i)_{i \in I}\right) = \sum_{i \in I} \varepsilon_i \circ f_i = 0$. Since f_i's are almost all zero, let $F \subseteq I$ be a finite subset such that $f_i \neq 0$ for every $i \in F$ and $f_i = 0$ for every $i \in I \backslash F$. Then $0 = \sum_{i \in I} \varepsilon_i \circ f_i \overset{(265)}{=} \sum_{i \in F} h \circ \varepsilon_i^F \circ f_i$. Since h is mono, we also have $0 = \sum_{i \in F} \varepsilon_i^F \circ f_i \in \mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i \in F} B_i\right)$ so that, for every $j \in F$,

$$0 = \pi_j \circ \sum_{i \in I} \varepsilon_i^F \circ f_i = \sum_{i \in I} \pi_j \circ \varepsilon_i^F \circ f_i \overset{(269)}{=} f_j$$

and thus $f_i = 0$ for every $i \in I$. $\square$

Proposition A.19. *Let (T,H) be an adjunction where $T:\mathcal{A}\to\mathcal{B}$, $H:\mathcal{B}\to\mathcal{A}$ and $\mathcal{A}$, $\mathcal{B}$ are Grothendieck categories. If T is an equivalence of categories, then*

1) *if P is a generator of $\mathcal{A}$ then TP is a generator of $\mathcal{B}$*
2) *if P is projective in $\mathcal{A}$ then TP is projective $\mathcal{B}$*
3) *if P is finite in $\mathcal{A}$ then TP is finite $\mathcal{B}$*
4) *if P is finitely generated in $\mathcal{A}$ then TP is finitely generated in $\mathcal{B}$*
5) *if P is finitely generated and projective in $\mathcal{A}$ then TP is finitely generated and projective $\mathcal{B}$*
6) *if P is finite projective in $\mathcal{A}$ then TP is finite projective in $\mathcal{B}$.*

Proof. 1) Let $f:Y\to Y'$ be a non zero morphism in $\mathcal{B}$. Since T is an equivalence, there exists a non zero morphism $g:X\to X'$ in $\mathcal{A}$ such that $f=T(g)$. Since P is a generator of $\mathcal{A}$ there exists a morphism $p:P\to X$ such that $g\circ p\neq 0$. Then $0\neq T(g\circ p)=T(g)\circ T(p)=f\circ T(p)$ and $T(p):TP\to TX=Y$ so that TP is a generator of $\mathcal{B}$.

2) Let $f:Y\to Y'$ be a morphism in $\mathcal{B}$. Since T is an equivalence, there exists a morphism $g:X\to X'$ in $\mathcal{A}$ such that $f=T(g)$, $Y=TX$ and $Y'=TX'$. Let $l:TP\to Y$ a morphism in $\mathcal{B}$, then $l:TP\to TX$ then there exists $h:P\to X$ such that $l=T(h)$. Since P is projective $\mathcal{A}$, there exists $k:P\to X'$ such that $g\circ h=k$. By applying the functor T we get $T(g\circ h)=T(g)\circ T(h)=f\circ l=T(k)$ then TP is projective in $\mathcal{B}$.

3) Let $(N_i)_{i\in I}$ be a family of objects in $\mathcal{B}$. Since T is an equivalence, there exists a family $(M_i)_{i\in I}$ of objects in $\mathcal{A}$ such that $(N_i)_{i\in I}=(TM_i)_{i\in I}$. Denote by $\varepsilon_i: M_i\to\coprod_{i\in I}M_i$ and by $\mathrm{Hom}_{\mathcal{A}}(P,\varepsilon_i):\mathrm{Hom}_{\mathcal{A}}(P,M_i)\to\mathrm{Hom}_{\mathcal{A}}\left(P,\coprod_{i\in I}M_i\right)$. Then we can consider the codiagonal morphism $\nabla\left(\mathrm{Hom}_{\mathcal{A}}(P,\varepsilon_i)\right)_{i\in I}:\mathrm{Hom}_{\mathcal{A}}\left(P,\coprod_{i\in I}M_i\right)\to\coprod_{i\in I}\mathrm{Hom}_{\mathcal{A}}(P,M_i)$. Since P is finite in $\mathcal{A}$ we have that P preserves coproducts, i.e. $\nabla\left(\mathrm{Hom}_{\mathcal{A}}(P,\varepsilon_i)\right)_{i\in I}$ is an isomorphism. Let

$$\begin{aligned}\phi_{X,X'}:\mathrm{Hom}_{\mathcal{A}}(X,X')&\longrightarrow\mathrm{Hom}_{\mathcal{B}}(TX,TX')\\ f&\mapsto T(f)\end{aligned}$$

Since T is an equivalence $\phi_{X,X'}$ is bijective for every $X,X'\in\mathcal{A}$ and $\coprod_{i\in I}(TM_i)=T\left(\coprod_{i\in I}M_i\right)$. Then we can consider

$\mathrm{Hom}_{\mathcal{B}}(TP,T(\varepsilon_i)):\mathrm{Hom}_{\mathcal{B}}(TP,T(M_i))\to\mathrm{Hom}_{\mathcal{B}}\left(TP,T\left(\coprod_{i\in I}M_i\right)\right)=\mathrm{Hom}_{\mathcal{B}}\left(TP,\coprod_{i\in I}(TM_i)\right)$

and their codiagonal morphism $\nabla\left(\mathrm{Hom}_{\mathcal{B}}(TP,T(\varepsilon_i))\right)_{i\in I}:\mathrm{Hom}_{\mathcal{B}}\left(TP,\coprod_{i\in I}(TM_i)\right)\to$

$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{B}}(TP, T(M_i))$. Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}\left(P, \coprod_{i\in I} M_i\right) & \xrightarrow{\nabla(\mathrm{Hom}_{\mathcal{A}}(P,\varepsilon_i))_{i\in I}} & \coprod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P, M_i) \\
\downarrow{\scriptstyle \phi_{P,\coprod_{i\in I} M_i}} & & \downarrow{\scriptstyle \coprod_{i\in I}\phi_{P,M_i}} \\
\mathrm{Hom}_{\mathcal{B}}\left(TP, \coprod_{i\in I}(TM_i)\right) & \xrightarrow{\nabla(\mathrm{Hom}_{\mathcal{B}}(TP,T(\varepsilon_i)))_{i\in I}} & \coprod_{i\in I} \mathrm{Hom}_{\mathcal{B}}(TP, T(M_i))
\end{array}$$

where we observed that the first row is an isomorphism and also the ϕ's are isomorphism. Then we deduce that $\nabla(\mathrm{Hom}_{\mathcal{B}}(TP, T(\varepsilon_i)))_{i\in I}$ is an isomorphism, so that TP preserves coproducts, i.e. TP is finite.

4) Let $\{Q_i\}_{i\in I}$ be a direct family of subobjects of TP such that $TP = \sum_{i\in I} Q_i$. Then, since T is an equivalence, there exists a direct family $\{P_i\}_{i\in I}$ of subobjects of P such that $TP_i = Q_i$ for every $i \in I$ and $P = \sum_{i\in I} P_i$. Since P is finitely generated, there exists an index $i_0 \in I$ such that $P = P_{i_0}$ and then $TP = TP_{i_0} = Q_{i_0}$, i.e. TP is finitely generated.

5) By Proposition A.18, P is finite and thus by 3) we deduce that TP is also finite. Since TP is moreover projective, by Proposition A.18 we conclude that TP is finitely generated.

6) By Proposition A.18, since P is finite projective, P is finitely generated and projective. Then we conclude by 5). □

Definition A.20. Let $\mathcal{A}$ be an abelian category. A finite projective generator P in $\mathcal{A}$ is called *progenerator*.

Corollary A.21. *Let $\mathcal{A}$ be a Grothendieck category. There exists an equivalence $\mathcal{F} : \mathcal{A} \to Mod\text{-}B$, where B is a ring, if and only if $\mathcal{A}$ contains a progenerator P. Moreover*

- *If P is a progenerator of $\mathcal{A}$, then* $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to Mod\text{-}T$ *where* $T = \mathrm{Hom}_{\mathcal{A}}(P, P)$.
- *If $\mathcal{F}$ is an equivalence, there exists a progenerator P in $\mathcal{A}$ such that* $\mathrm{Hom}_{\mathcal{A}}(P, P) \simeq B$ *and* $\mathcal{F} \simeq \mathrm{Hom}_{\mathcal{A}}(P, -)$.

Proof. Assume first that $\mathcal{A}$ contains a progenerator P. Let $B = \mathrm{Hom}_{\mathcal{A}}(P, P)$ and consider the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to Mod\text{-}B$. Since P is a generator and $\mathcal{A}$ is a Grothendieck category, by Proposition $A.3$ we deduce that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is full and faithful. Hence we only have to prove that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is surjective on the objects. Let $M \in Mod\text{-}B$. Then we have the following exact sequence in $Mod\text{-}B$

$$B^{(X)} \longrightarrow B^{(M)} \longrightarrow M \to 0.$$

Since $B = \mathrm{Hom}_{\mathcal{A}}(P, P)$ we can rewrite is as

$$\mathrm{Hom}_{\mathcal{A}}(P, P)^{(X)} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, P)^{(M)} \longrightarrow M \to 0.$$

Since P is finite $\mathrm{Hom}_{\mathcal{A}}(P,P)^{(X)} \simeq \mathrm{Hom}_{\mathcal{A}}\left(P,P^{(X)}\right)$ and $\mathrm{Hom}_{\mathcal{A}}(P,P)^{(M)} \simeq \mathrm{Hom}_{\mathcal{A}}\left(P,P^{(M)}\right)$ and then we have an exact sequence in $Mod\text{-}B$

$$\mathrm{Hom}_{\mathcal{A}}\left(P,P^{(X)}\right) \xrightarrow{f} \mathrm{Hom}_{\mathcal{A}}\left(P,P^{(M)}\right) \longrightarrow Q \to 0 \tag{273}$$

where $Q = \mathrm{Coker}(f)$. Then $Q \simeq M$. Since $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is full (and faithful) we have

$$\mathrm{Hom}_{\mathcal{A}}(A,A') \simeq \mathrm{Hom}_{Mod\text{-}B}(\mathrm{Hom}_{\mathcal{A}}(P,A), \mathrm{Hom}_{\mathcal{A}}(P,A')),$$

hence there exists a unique morphism $g : P^{(X)} \to P^{(M)}$ in $\mathcal{A}$ such that $f = \mathrm{Hom}_{\mathcal{A}}(P,-)(g)$. Let us consider in $\mathcal{A}$

$$P^{(X)} \xrightarrow{g} P^{(M)} \longrightarrow X \to 0 \tag{274}$$

where $X = \mathrm{Coker}(g)$. Since P is projective, $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is exact, and applying it to (274) we get the exact sequence

$$\mathrm{Hom}_{\mathcal{A}}\left(P,P^{(X)}\right) \overset{f=\mathrm{Hom}_{\mathcal{A}}(P,g)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}\left(P,P^{(M)}\right) \to \mathrm{Hom}_{\mathcal{A}}(P,X) \to 0.$$

From this sequence and (273) we deduce that $Q \simeq \mathrm{Hom}_{\mathcal{A}}(P,X)$ where $X = \mathrm{Coker}(g) \in \mathcal{A}$.
Conversely, let us assume that $\mathcal{F} : \mathcal{A} \to Mod\text{-}B$ is an equivalence of categories. Let $\mathcal{G} : Mod\text{-}B \to \mathcal{A}$ be its inverse equivalence. Since B is a progenerator and $\mathcal{G}$ is an equivalence of categories, by Proposition A.19 1) and 6), we deduce that $\mathcal{G}(B)$ is a progenerator in $\mathcal{A}$. Moreover we have

$$B \simeq \mathrm{Hom}_{Mod\text{-}B}(B,B) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), \mathcal{G}(B)).$$

Observe that $\mathcal{G}$ is a left adjoint to $\mathcal{F}$ and thus we have

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), -) \simeq \mathrm{Hom}_{Mod\text{-}B}(B, \mathcal{F}-).$$

Since $\mathrm{Hom}_{Mod\text{-}B}(B, \mathcal{F}-) \simeq \mathcal{F}$ as functors, we deduce that

$$\mathcal{F} \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), -)$$

where $\mathcal{G}(B)$ is a progenerator in $\mathcal{A}$. □

THEOREM A.22. *Let $\mathcal{A}$ be an abelian category. There exists an equivalence $\mathcal{F} : \mathcal{A} \to Mod\text{-}B$, where B is a ring, if and only if $\mathcal{A}$ contains a progenerator P and arbitrary coproducts of copies of P. If $\mathcal{F}$ is an equivalence, there exists a progenerator P in $\mathcal{A}$ such that* $\mathrm{Hom}_{\mathcal{A}}(P,P) \simeq B$ *and* $\mathcal{F} \simeq \mathrm{Hom}_{\mathcal{A}}(P,-)$.

Proof. Assume first that $\mathcal{A}$ contains a progenerator P and arbitrary coproducts of copies of P. Let $B = \mathrm{Hom}_{\mathcal{A}}(P,P)$ and consider the functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to Ab$. Let us endow any abelian group $\mathrm{Hom}_{\mathcal{A}}(P,A)$, for every $A \in \mathcal{A}$, with a right B-module structure given by the composition with morphisms of $\mathrm{Hom}_{\mathcal{A}}(P,P) = B$. This means we have the following map

$$\begin{gathered}\mathrm{Hom}_{\mathcal{A}}(P,A) \times \mathrm{Hom}_{\mathcal{A}}(P,P) \to \mathrm{Hom}_{\mathcal{A}}(P,A) \\ (h,\xi) \mapsto h \circ \xi.\end{gathered}$$

For every morphism $f : A \to B$ in $\mathcal{A}$ we define a morphism in $Mod\text{-}B$ as follows

$$\mathrm{Hom}_{\mathcal{A}}(P,f) : \mathrm{Hom}_{\mathcal{A}}(P,A) \to \mathrm{Hom}_{\mathcal{A}}(P,B)$$

$$\xi \mapsto f \circ \xi.$$

Then it is well-defined a functor $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to Mod\text{-}B$. We want to prove that $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is an equivalence of category. To be full and faithful for $\mathrm{Hom}_{\mathcal{A}}(P,-)$ means that the map

$$\begin{array}{rcl} \phi_{A,A'} & : & \mathrm{Hom}_{\mathcal{A}}(A,A') \longrightarrow \mathrm{Hom}_{Mod\text{-}B}(\mathrm{Hom}_{\mathcal{A}}(P,A), \mathrm{Hom}_{\mathcal{A}}(P,A')) \\ f & \mapsto & \left(\begin{array}{c} \mathrm{Hom}_{\mathcal{A}}(P,f) : \mathrm{Hom}_{\mathcal{A}}(P,A) \to \mathrm{Hom}_{\mathcal{A}}(P,A') \\ \xi \mapsto f \circ \xi \end{array} \right) \end{array}$$

is bijective for every $A, A' \in \mathcal{A}$. Note that $\mathrm{Hom}_{\mathcal{A}}(P,-)$ induces an isomorphism

$$\phi_{P,P} : \mathrm{Hom}_{\mathcal{A}}(P,P) \longrightarrow \mathrm{Hom}_{Mod\text{-}B}(\mathrm{Hom}_{\mathcal{A}}(P,P), \mathrm{Hom}_{\mathcal{A}}(P,P)).$$

In fact, for every $\zeta \in \mathrm{Hom}_{\mathcal{A}}(P,P)$ such that $\mathrm{Hom}_{\mathcal{A}}(P,\zeta) = 0$ we have that, for every $\xi \in \mathrm{Hom}_{\mathcal{A}}(P,P)$, $0 = \mathrm{Hom}_{\mathcal{A}}(P,\zeta)(\xi) = \zeta \circ \xi$. Since P is a generator, we deduce that $\zeta = 0$. Now, let $f : \mathrm{Hom}_{\mathcal{A}}(P,P) \to \mathrm{Hom}_{\mathcal{A}}(P,P)$ be a morphism in $Mod\text{-}B$ and set $f(\mathrm{Id}_P) = \chi$. Then, for every $\xi \in \mathrm{Hom}_{\mathcal{A}}(P,P)$, we have

$$f(\xi) = f(\mathrm{Id}_P \circ \xi) \overset{f \in Mod\text{-}B}{=} f(\mathrm{Id}_P) \circ \xi = \chi \circ \xi = \mathrm{Hom}_{\mathcal{A}}(P,\chi)(\xi)$$

and thus

$$f = \mathrm{Hom}_{\mathcal{A}}(P,\chi) = \mathrm{Hom}_{\mathcal{A}}(P,-)(\chi)$$

so that $\phi_{P,P}$ is an epimorphism. Let us consider families $(P_i)_{i \in I}$ and $(P_j)_{j \in J}$ where $P_i \simeq P \simeq P_j$ for every $i \in I$ and $j \in J$. Set $B_i = \mathrm{Hom}_{\mathcal{A}}(P,P_i)$ and $B_j = \mathrm{Hom}_{\mathcal{A}}(P,P_j)$. Then $B_i = \mathrm{Hom}_{\mathcal{A}}(P,P_i) \simeq \mathrm{Hom}_{\mathcal{A}}(P,P) = B$ and similarly $B_j \simeq B$. Let us compute

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}\left(\coprod_{j \in J} P_j, \coprod_{i \in I} P_i\right) &\overset{\text{coprod}}{\simeq} \prod_{j \in J} \mathrm{Hom}_{\mathcal{A}}\left(P_j, \coprod_{i \in I} P_i\right) \overset{P\text{finite}}{\simeq} \prod_{j \in J} \coprod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(P_j, P_i) \\ &\overset{\phi_{P,P}}{\simeq} \prod_{j \in J} \coprod_{i \in I} \mathrm{Hom}_{Mod\text{-}B}(\mathrm{Hom}_{\mathcal{A}}(P,P_j), \mathrm{Hom}_{\mathcal{A}}(P,P_i)) = \prod_{j \in J} \coprod_{i \in I} \mathrm{Hom}_{Mod\text{-}B}(B_j, B_i) \\ &\overset{B_i \simeq B\text{finite}}{\simeq} \prod_{j \in J} \mathrm{Hom}_{Mod\text{-}B}\left(B_j, \coprod_{i \in I} B_i\right) \overset{\text{coprod}}{\simeq} \mathrm{Hom}_{Mod\text{-}B}\left(\coprod_{j \in J} B_j, \coprod_{i \in I} B_i\right) \end{aligned}$$

hence

$$\mathrm{Hom}_{\mathcal{A}}\left(\coprod_{j \in J} P_j, \coprod_{i \in I} P_i\right) \simeq \mathrm{Hom}_{Mod\text{-}B}\left(\coprod_{j \in J} B_j, \coprod_{i \in I} B_i\right) \tag{275}$$

which says that $\mathrm{Hom}_{\mathcal{A}}(P,-)$ induces a bijection between the full subcategory of the coproducts of copies of P in $\mathcal{A}$ and the full subcategory of coproducts of copies of B in $Mod\text{-}B$, i.e. $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is full and faithful on the full subcategory of the coproducts of copies of P in $\mathcal{A}$. Let us denote by $\varepsilon_i^P : P \to P^{(I)}$, $p_i^P : P^{(I)} \to P$ and $\varepsilon_j : B = \mathrm{Hom}_{\mathcal{A}}(P,P) \to B^{(I)} = \mathrm{Hom}_{\mathcal{A}}(P,P)^{(I)}$, $p_j : B^{(I)} = \mathrm{Hom}_{\mathcal{A}}(P,P)^{(I)} \to B = \mathrm{Hom}_{\mathcal{A}}(P,P)$ the canonical maps. Now, let $A, A' \in \mathcal{A}$. Since P is a generator, we have

$$P^{(J)} \xrightarrow{\widetilde{f}} P^{(I)} \xrightarrow{h} A = \mathrm{Coker}(h) \to 0$$

and

$$P^{(J')} \xrightarrow{\widetilde{f}'} P^{(I')} \xrightarrow{h'} A' = \mathrm{Coker}\,(h') \to 0.$$

Let $z : \mathrm{Hom}_{\mathcal{A}}(P, A) \to \mathrm{Hom}_{\mathcal{A}}(P, A')$ be a morphism in Mod-B. We have to prove that there exists a morphism $a : A \to A'$ such that $z = \mathrm{Hom}_{\mathcal{A}}(P, a)$. Since P is projective, $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact so that we get the exact sequences

$$\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \overset{\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}\right)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \overset{\mathrm{Hom}_{\mathcal{A}}(P,h)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(P, A) \to 0$$

and

$$\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right) \overset{\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}'\right)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right) \overset{\mathrm{Hom}_{\mathcal{A}}(P,h')}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(P, A') \to 0$$

Then we can consider

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}\right)"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h)"] & \mathrm{Hom}_{\mathcal{A}}(P, A) \arrow[r] \arrow[d, "z"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}'\right)"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h')"] & \mathrm{Hom}_{\mathcal{A}}(P, A') \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \simeq B^{(I)}$ it is projective, so that there exists a morphism $y : \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \to \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right)$ and a morphism $x : \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \to \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right)$. Thus we have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}\right)"] \arrow[d, "x"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h)"] \arrow[d, "y"] & \mathrm{Hom}_{\mathcal{A}}(P, A) \arrow[r] \arrow[d, "z"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}'\right)"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h')"] & \mathrm{Hom}_{\mathcal{A}}(P, A') \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is full and faithful on coproducts of copies of P, every morphism $x : \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \to \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right)$ is of the form $x = \mathrm{Hom}_{\mathcal{A}}(P, \widetilde{x})$ for $\widetilde{x} : P^{(J)} \to P^{(J')}$, so that we can rewrite the diagram as follows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}\right)"] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(P,\widetilde{x})"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h)"] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(P,\widetilde{y})"] & \mathrm{Hom}_{\mathcal{A}}(P, A) \arrow[r] \arrow[d, "z"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\left(P,\widetilde{f}'\right)"] & \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(P,h')"] & \mathrm{Hom}_{\mathcal{A}}(P, A') \arrow[r] & 0
\end{tikzcd}$$

Thus we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}\left(P, \widetilde{f}' \circ \widetilde{x}\right) &= \mathrm{Hom}_{\mathcal{A}}\left(P, \widetilde{f}'\right) \circ \mathrm{Hom}_{\mathcal{A}}(P, \widetilde{x}) \\
&= \mathrm{Hom}_{\mathcal{A}}(P, \widetilde{y}) \circ \mathrm{Hom}_{\mathcal{A}}\left(P, \widetilde{f}\right) = \mathrm{Hom}_{\mathcal{A}}\left(P, \widetilde{y} \circ \widetilde{f}\right).
\end{aligned}$$

Since P is a generator we already now that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is faithful, so that we deduce that

$$\widetilde{f}' \circ \widetilde{x} = \widetilde{y} \circ \widetilde{f}$$

i.e.

$$\begin{tikzcd}
P^{(J)} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{x}"] & P^{(I)} \arrow[r, "h"] \arrow[d, "\widetilde{y}"] & A \arrow[r] & 0 \\
P^{(J')} \arrow[r, "\widetilde{f'}"] & P^{(I')} \arrow[r, "h'"] & A' \arrow[r] & 0
\end{tikzcd}$$

Since $A = \operatorname{Coker}\left(\widetilde{f}\right)$ and by the commutative of the diagram, we deduce that there exists a unique morphism $a : A \to A'$ in $\mathcal{A}$ such that the diagram

$$\begin{tikzcd}
P^{(J)} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{x}"] & P^{(I)} \arrow[r, "h"] \arrow[d, "\widetilde{y}"] & A \arrow[r] \arrow[d, "a"] & 0 \\
P^{(J')} \arrow[r, "\widetilde{f'}"] & P^{(I')} \arrow[r, "h'"] & A' \arrow[r] & 0
\end{tikzcd}$$

is commutative. By applying the exact functor $\operatorname{Hom}_{\mathcal{A}}(P, -)$ thus we get

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}\left(P, P^{(J)}\right) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}\left(P,\widetilde{f}\right)"] \arrow[d, "\operatorname{Hom}_{\mathcal{A}}(P,\widetilde{x})"] & \operatorname{Hom}_{\mathcal{A}}\left(P, P^{(I)}\right) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}(P,h)"] \arrow[d, "\operatorname{Hom}_{\mathcal{A}}(P,\widetilde{y})"] & \operatorname{Hom}_{\mathcal{A}}(P, A) \arrow[r] \arrow[d, shift right, "z"'] \arrow[d, shift left, "\operatorname{Hom}_{\mathcal{A}}(P,a)"] & 0 \\
\operatorname{Hom}_{\mathcal{A}}\left(P, P^{(J')}\right) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}\left(P,\widetilde{f'}\right)"] & \operatorname{Hom}_{\mathcal{A}}\left(P, P^{(I')}\right) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}(P,h')"] & \operatorname{Hom}_{\mathcal{A}}(P, A') \arrow[r] & 0
\end{tikzcd}$$

which says

$$\begin{aligned}
z \circ \operatorname{Hom}_{\mathcal{A}}(P, h) &= \operatorname{Hom}_{\mathcal{A}}(P, h') \circ \operatorname{Hom}_{\mathcal{A}}(P, \widetilde{y}) \\
&= \operatorname{Hom}_{\mathcal{A}}(P, a) \circ \operatorname{Hom}_{\mathcal{A}}(P, h)
\end{aligned}$$

and since $\operatorname{Hom}_{\mathcal{A}}(P, h)$ is epi we deduce that

$$z = \operatorname{Hom}_{\mathcal{A}}(P, a).$$

This proves that $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is full. Since P is a generator, $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is faithful. Then we only need to prove that $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is surjective on objects. Let $M \in$ $Mod\text{-}B$. Then we have the following exact sequence in $Mod\text{-}B$

$$B^{(X)} \to B^{(M)} \to M \to 0.$$

Since $B = \operatorname{Hom}_{\mathcal{A}}(P, P)$ we can rewrite is as

$$\operatorname{Hom}_{\mathcal{A}}(P, P)^{(X)} \to \operatorname{Hom}_{\mathcal{A}}(P, P)^{(M)} \to M \to 0.$$

Since P is finite $\operatorname{Hom}_{\mathcal{A}}(P, P)^{(X)} \simeq \operatorname{Hom}_{\mathcal{A}}\left(P, P^{(X)}\right)$ and
$\operatorname{Hom}_{\mathcal{A}}(P, P)^{(M)} \simeq \operatorname{Hom}_{\mathcal{A}}\left(P, P^{(M)}\right)$ and then we have an exact sequence in $Mod\text{-}B$

$$\operatorname{Hom}_{\mathcal{A}}\left(P, P^{(X)}\right) \xrightarrow{f} \operatorname{Hom}_{\mathcal{A}}\left(P, P^{(M)}\right) \to Q \to 0 \tag{276}$$

where $Q = \operatorname{Coker}(f)$. Then $Q \simeq M$. Since $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is full (and faithful) we have

$$\operatorname{Hom}_{\mathcal{A}}(A, A') \simeq \operatorname{Hom}_{Mod\text{-}B}\left(\operatorname{Hom}_{\mathcal{A}}(P, A), \operatorname{Hom}_{\mathcal{A}}(P, A')\right),$$

hence there exists a unique morphism $g : P^{(X)} \to P^{(M)}$ in $\mathcal{A}$ such that $f = \mathrm{Hom}_{\mathcal{A}}(P, -)(g)$. Let us consider in $\mathcal{A}$

$$P^{(X)} \xrightarrow{g} P^{(M)} \to X \to 0$$

where $X = \mathrm{Coker}(g)$. Since P is projective, $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact, and applying it we get the exact sequence

$$\mathrm{Hom}_{\mathcal{A}}\left(P, P^{(X)}\right) \overset{f=\mathrm{Hom}_{\mathcal{A}}(P,g)}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}\left(P, P^{(M)}\right) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X) \to 0.$$

From this sequence and (276) we deduce that $Q \simeq \mathrm{Hom}_{\mathcal{A}}(P, X)$ where $X = \mathrm{Coker}(g) \in \mathcal{A}$.

Conversely, let us assume that $\mathcal{F} : \mathcal{A} \to Mod\text{-}B$ is an equivalence of categories. Let $\mathcal{G} : Mod\text{-}B \to \mathcal{A}$ be its inverse equivalence. Since B is a progenerator and $\mathcal{G}$ is an equivalence of categories, by Proposition A.19 1) and 6), we deduce that $\mathcal{G}(B)$ is a progenerator in $\mathcal{A}$. Moreover we have

$$B \simeq \mathrm{Hom}_{Mod\text{-}B}(B, B) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), \mathcal{G}(B)).$$

Observe that $\mathcal{G}$ is a left adjoint to $\mathcal{F}$ and thus we have

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), -) \simeq \mathrm{Hom}_{Mod\text{-}B}(B, \mathcal{F}-).$$

Since $\mathrm{Hom}_{Mod\text{-}B}(B, \mathcal{F}-) \simeq \mathcal{F}$ as functors, we deduce that

$$\mathcal{F} \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(B), -)$$

where $\mathcal{G}(B)$ is a progenerator in $\mathcal{A}$. Moreover, since $\mathcal{G}$ is an equivalence, $\mathcal{G}\left(B^{(X)}\right) \simeq \mathcal{G}(B)^{(X)}$. $\square$

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