Four stress analysis strategies to use the Modified Wöhler Curve Method to perform the fatigue assessment of weldments subjected to constant and variable amplitude multiaxial fatigue loading

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Abstract
The present paper investigates the different ways of using the Modified Wöhler Curve Method (MWCM) to perform the fatigue assessment of steel and aluminium welded joints subjected to in-service variable amplitude (VA) multiaxial load histories. Thanks to its specific features, the above critical plane approach can efficiently be applied in terms of both nominal, hot-spot, and local quantities, that is, by using any of the stress analysis strategies suggested by the Design Recommendations of the International Institute of Welding (IIW). The MWCM can efficiently be used also along with the so-called Theory of Critical Distances applied in the form of the Point Method (PM). The accuracy of the different formalisations of the MWCM investigated in the present paper was systematically checked against a large number of experimental results taken from the literature and generated by testing, under VA biaxial nominal loading, welded samples having different geometries. Such a systematic validation exercise allowed us to prove that our multiaxial fatigue criterion is successful in designing welded joints against VA multiaxial fatigue, this holding true independently from both definition adopted to calculate the necessary stress quantities and complexity of the assessed load history.

1. Introduction
To design un-cracked welded components of both steel and aluminium against fatigue, the available Design Codes and Recommendations [1–3] allow structural engineers to perform the stress analysis according to three different strategies, i.e., in terms of either nominal, hot-spot, or local stresses. In more detail, nominal stresses are suggested as being calculated according to classical continuum mechanics, without explicitly modelling the stress concentration phenomena at the weld toe. All the calculations are done by assuming that the designed material obeys a linear-elastic constitutive law and the effects of macro-geometrical features as well as the presence of concentrated loadings must always be taken into account [3,4].

Hot-spot stresses are instead used either when a reference design curve is not available for the specific welded detail being assessed, or when a nominal section cannot unambiguously be defined due to the complexity of the geometry of the component being designed. Hot-spot stress quantities can be determined from conventional linear-elastic finite element (FE) models as well as by directly measuring the local strains through strain gauges attached, in the vicinity of the assumed crack initiation locations, to the surface of the welded component being assessed [3–5]; as soon as the linear-elastic stress is known either at two or at three superficial reference points, the structural stress is directly extrapolated to the weld toe at the hot spot. Finally, according to the IIW [3] the stress analysis can be performed also in terms of local quantities, where the critical stress states are suggested as being determined by rounding either the weld toe or the weld root with a reference radius having length equal to either 1 mm or to 0.05 mm [4].

Other than those approaches suggested by the available Standard Codes and Recommendations as being adopted in situations of practical interest, examination of the state of the art shows that in recent years many different attempts have been made to devise alternative design techniques taking full advantage of local quantities. Amongst the different approaches which have been proposed so far, and somehow validated through experimental results, certainly the N-SIF approach [6,7], the Strain Energy Density parameter [8,9], and the Theory of Critical Distances [10,11] deserve to be mentioned explicitly.

As far as multiaxial fatigue loadings are concerned, apart from those simplified methods recommended by some design codes [3,12], examination of the state of the art shows that the most accurate estimates are obtained by using those critical plane approaches specifically devised to post-process either structural or local quantities [12–17].
Turning back to uniaxial nominal situations, the three types of stress analysis mentioned above can be used also to perform the fatigue assessment of weldments subjected to VA fatigue loading [4], in such circumstances [3,4] cumulative fatigue damage being usually calculated according to the linear rule due to Palmgren [18] and Miner [19]. The most tricky aspect behind the use of the above classical rule to design real weldments against VA fatigue is that the critical value of the damage sum is seen to vary as the degree of multiaxiality and non-proportionality of the investigated load history varies [14,21–23]. The most tricky one being that based on the use of the maximum chord concept [25]. Even if the classical definitions [24] can successfully be adopted to calculate the maximum shear stress amplitude, \( \tau_m \), one of the most tricky aspects in using them to design real components against CA fatigue is that, in theory, the shear stress amplitude relative to any plane experiencing the maximum shear stress amplitude itself reaches its maximum value. Since this \textit{modus operandi} can become extremely time-consuming when complex and long load histories are involved [26,27], in recent years we have made a big effort in order to devise an alternative definition [24,28] based on the maximum variance concept [29]. The Maximum Variance Method (MVM) [28] postulates that the critical plane can be defined as that plane containing the direction (passing through the assumed critical point) that experiences the maximum variance of the resolved shear stress, \( \tau_{Mv} \) – see Fig. 1a and b. From a practical point of view, the most remarkable peculiarity of the MVM is that, as soon as the variance and co-variance terms of the stress components at the critical location are known, the computational time required to determine the orientation of the critical plane does not depend on the length of the input load

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>( a, b, x, \beta )</td>
<td>constants in the MWCM's calibration equations</td>
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<tr>
<td>( f_\Sigma, f_T )</td>
<td>frequency of the axial stress component, frequency of the torsional stress component</td>
</tr>
<tr>
<td>( f(RCP) )</td>
<td>enhancement factor</td>
</tr>
<tr>
<td>( k, k_0, k_r )</td>
<td>negative inverse slope of the uniaxial fatigue curve, negative inverse slope of the torsional fatigue curve, modified Wöhler curve's negative inverse slope</td>
</tr>
<tr>
<td>( m_r )</td>
<td>modified Wöhler curve's negative inverse slope in the high-cycle fatigue regime</td>
</tr>
<tr>
<td>( n )</td>
<td>number of cycles</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( D_{cr} )</td>
<td>critical value of the damage sum</td>
</tr>
<tr>
<td>( D_{tot} )</td>
<td>damage sum</td>
</tr>
<tr>
<td>( F )</td>
<td>ratio between ( f_\Sigma ) and ( f_T )</td>
</tr>
<tr>
<td>( M-D_{\nu} )</td>
<td>multiaxial critical distance</td>
</tr>
<tr>
<td>( N_f )</td>
<td>number of cycles to failure</td>
</tr>
<tr>
<td>( N_{f,p} )</td>
<td>estimated number of cycles to failure</td>
</tr>
<tr>
<td>( N_{ip} )</td>
<td>number of cycles to failure defining the position of the knee point</td>
</tr>
<tr>
<td>( N_{k,p} )</td>
<td>reference number of cycles to failure</td>
</tr>
<tr>
<td>( Oxyz )</td>
<td>system of coordinates</td>
</tr>
<tr>
<td>( P_s )</td>
<td>probability of survival</td>
</tr>
<tr>
<td>( R )</td>
<td>load ratio</td>
</tr>
<tr>
<td>( RCP )</td>
<td>critical plane load ratio (( R_{CP} = \sigma_{n,\text{min}}/\sigma_{n,\text{max}} ))</td>
</tr>
<tr>
<td>( T )</td>
<td>time interval</td>
</tr>
<tr>
<td>( T_{\nu}, T_r )</td>
<td>scatter ratio of the stress amplitude for 90% and 10% probabilities of survival</td>
</tr>
<tr>
<td>( \delta )</td>
<td>out-of-phase angle</td>
</tr>
<tr>
<td>( \rho_\nu )</td>
<td>critical plane stress ratio</td>
</tr>
<tr>
<td>( \rho_{\nu,\text{lim}} )</td>
<td>limit value for the critical plane stress ratio</td>
</tr>
<tr>
<td>( \sigma_i(t) )</td>
<td>instantaneous value of the normal stress (( i = x, y, z ))</td>
</tr>
<tr>
<td>( \sigma_s(t) )</td>
<td>instantaneous value of the stress perpendicular to the critical plane</td>
</tr>
<tr>
<td>( \sigma_{n,\nu} )</td>
<td>amplitude of the stress perpendicular to the critical plane</td>
</tr>
<tr>
<td>( \sigma_{n,m} )</td>
<td>maximum stress perpendicular to the critical plane</td>
</tr>
<tr>
<td>( \sigma_{n,\text{max}} )</td>
<td>maximum stress perpendicular to the critical plane</td>
</tr>
<tr>
<td>( \tau_{n,\text{min}} )</td>
<td>minimum stress perpendicular to the critical plane</td>
</tr>
<tr>
<td>( \tau_s )</td>
<td>maximum shear stress amplitude</td>
</tr>
<tr>
<td>( \tau_{Mv}(t) )</td>
<td>instantaneous value of the shear stress (( i, j = x, y, z ))</td>
</tr>
<tr>
<td>( \tau_m )</td>
<td>mean shear stress</td>
</tr>
<tr>
<td>( \tau_{Mv,\text{max}} )</td>
<td>maximum shear stress resolved along the maximum variance direction</td>
</tr>
<tr>
<td>( \tau_{Mv,\text{min}} )</td>
<td>minimum shear stress resolved along the maximum variance direction</td>
</tr>
<tr>
<td>( \Delta \sigma_A )</td>
<td>range of the reference normal stress at ( N_A ) cycles to failure</td>
</tr>
<tr>
<td>( \Delta \tau )</td>
<td>shear stress range relative to the critical plane</td>
</tr>
<tr>
<td>( \Delta \sigma_{R_{Ref}} )</td>
<td>range of the reference shear stress at ( N_s ) cycles to failure</td>
</tr>
<tr>
<td>( \Delta \Sigma_i )</td>
<td>range of the uniaxial nominal stress during the ( i )-th loading cycle</td>
</tr>
<tr>
<td>( \Delta \tau_i )</td>
<td>range of the torsional nominal stress during the ( i )-th loading cycle</td>
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#### 2. Stress components relative to the critical plane and cycle counting under variable amplitude multiaxial fatigue loading

The MWCM is a bi-parametrical critical plane approach whose formalisation takes as a starting point the assumption that fatigue damage, under both variable and constant amplitude (CA) loading, reaches its maximum value on that plane (i.e., the so-called critical plane) experiencing the maximum shear stress amplitude.

By initially focussing attention solely on the CA problem, examination of the state of the art shows that different definitions [24] can successfully be adopted to calculate the maximum shear stress amplitude, \( \tau_m \), one of the most tricky aspects in using them to design real components against CA fatigue is that, in theory, the shear stress amplitude relative to any plane passing through the assumed critical point should be calculated, by determining \textit{a posteriori} that plane on which the shear stress amplitude itself reaches its maximum value. Since this \textit{modus operandi} can become extremely time-consuming when complex and long load histories are involved [26,27], in recent years we have made a big effort in order to devise an alternative definition [24,28] based on the maximum variance concept [29]. The Maximum Variance Method (MVM) [28] postulates that the critical plane can be defined as that plane containing the direction (passing through the assumed critical point) that experiences the maximum variance of the resolved shear stress, \( \tau_{Mv} \) – see Fig. 1a and b. From a practical point of view, the most remarkable peculiarity of the MVM is that, as soon as the variance and co-variance terms of the stress components at the critical location are known, the computational time required to determine the orientation of the critical plane does not depend on the length of the input load.
history being assessed [30]. Since the main features of the MVM have already been discussed in Refs [24,28,30] in great detail, by also investigating the different issues behind the numerical solution of the associated multi-variable optimisation problem [30], only the different definitions suitable for calculating the stress quantities relative to the critical plane under both constant and variable amplitude loading will briefly be reviewed in what follows.

Consider then a body subjected to a complex system of time-variable forces resulting in a time-variable multiaxial stress state at the assumed critical location (point O in Fig. 1a). According to the MVM [24,28,30], the critical plane is then the one containing that direction, MV, experiencing the maximum variance of the resolved shear stress, \( \tau_{MV}(t) \) (Fig. 1b). It is worth recalling here that, by definition, the variance of a time-variable signal is the expected value of the square of the deviation of that signal from its mean value. According to the above definition, the variance of a time-variable signal gives a measure of the amount of variation of the signal itself within the two extremes defining the maximum range, the variance being obviously independent from the mean value of the considered signal.

This remark should make it evident that the above definition can be used to determine the orientation of the critical plane not only under CA, but also under VA multiaxial fatigue loading [31]. Turning back to the CA problem, if the body sketched in Fig. 1a is

\[ \tau_n = \frac{1}{2} (\tau_{MV,\max} + \tau_{MV,\min}) \]
\[ \tau_s = \frac{1}{2} (\tau_{MV,\max} - \tau_{MV,\min}) \]

\[ \sigma_n = \frac{1}{2} (\sigma_{n,\max} + \sigma_{n,\min}) \]
\[ \sigma_s = \frac{1}{2} (\sigma_{n,\max} - \sigma_{n,\min}) \]

\[ \tau_n = \frac{1}{T} \int_0^T \tau_{MV}(t) \cdot dt \]
\[ \text{Var}[\tau_{MV}(t)] = \frac{1}{T} \left[ \tau_{MV}(t) - \tau_n \right]^2 \cdot dt \]
\[ \tau_s = \sqrt{2 \cdot \text{Var}[\tau_{MV}(t)]} \]

\[ \sigma_n = \frac{1}{T} \int_0^T \sigma_n(t) \cdot dt \]
\[ \text{Var}[\sigma_n(t)] = \frac{1}{T} \left[ \sigma_n(t) - \sigma_{n,\min} \right]^2 \cdot dt \]
\[ \sigma_s = \sqrt{2 \cdot \text{Var}[\sigma_n(t)]} \]
being the maximum and minimum value of $\tau_{MV}$, are the maximum and minimum value of $\tau_{MV}(t)$, respectively. As to the polarity of the shear stress, it is worth recalling here that the sign of the shear stress components have to be unambiguously and consistently assigned according to the convention being adopted. In a similar way, the amplitude, $\sigma_n$, and the mean value, $\sigma_{n,m}$, of the stress perpendicular to the critical plane, $\sigma_n(t)$, turn out to be (Fig. 1c):

$$\sigma_{n,a} = \frac{1}{2} (\sigma_{n,max} - \sigma_{n,min})$$  \hspace{1cm} (3)

$$\sigma_{n,m} = \frac{1}{2} (\sigma_{n,max} + \sigma_{n,min})$$  \hspace{1cm} (4)

$\sigma_{n,max}$ and $\sigma_{n,min}$ being the maximum and minimum value of $\sigma_n(t)$ during the loading cycle.

Assume now that the body of Fig. 1a is subjected to a complex system of time-variable forces resulting in a stress state at point O whose components vary randomly over the time interval $[0,T]$. According to the MVM [24,28,30], the critical plane can be determined also in such circumstances by directly locating that plane containing the direction, $MV$, experiencing the maximum variance of resolved shear stress $\tau_{MV}(t)$. As soon as the orientation of the critical plane is known, the mean value of the shear stress relative to the critical plane takes on the following value:

$$\tau_m = \frac{1}{T} \int_0^T \tau_{MV}(t) \cdot dt,$$  \hspace{1cm} (5)

whereas the equivalent stress amplitude of the resolved shear stress is equal to [30]:

$$\tau_a = \sqrt{2 \cdot \text{Var} [\tau_{MV}(t)]},$$  \hspace{1cm} (6)

the variance of stress signal $\tau_{MV}(t)$ being:

$$\text{Var} [\tau_{MV}(t)] = \frac{1}{T} \int_0^T [\tau_{MV}(t) - \tau_m]^2 \cdot dt$$  \hspace{1cm} (7)

By following the same strategy as above, the equivalent amplitude and the mean value of normal stress $\sigma_n(t)$ take on the following values [30]:

$$\sigma_{n,m} = \frac{1}{T} \int_0^T \sigma_n(t) \cdot dt$$  \hspace{1cm} (8)

$$\sigma_{n,a} = \sqrt{2 \cdot \text{Var} [\sigma_n(t)]},$$  \hspace{1cm} (9)

$\text{Var} [\sigma_n(t)]$ being the variance of stress component $\sigma_n(t)$, i.e.:

$$\text{Var} [\sigma_n(t)] = \frac{1}{T} \int_0^T [\sigma_n(t) - \sigma_{n,m}]^2 \cdot dt$$  \hspace{1cm} (10)

For the sake of clarity, Fig. 1d graphically shows the meaning of the above definitions: the schematic charts of Fig. 1d should make it evident that the equivalent amplitudes of both resolved shear stress $\tau_{MV}(t)$ and normal stress $\sigma_n(t)$ are proportional to the amount of variation of the stress signals themselves. Another problem which can briefly be addressed here is the way of performing the cycle counting under VA uniaxial/multiaxial fatigue loading when the fatigue assessment is performed through the MWCM.

To conclude, it is worth observing that the available Standard Codes and Recommendation usually address the problem of designing weldments against fatigue in terms of ranges. Accordingly, the ranges of the stress quantities relative to the critical plane can be calculated as follows:

$$\Delta \tau = 2 \cdot \tau_a$$  \hspace{1cm} (11)

$$\Delta \sigma_n = 2 \cdot \sigma_{n,a}$$  \hspace{1cm} (12)

where the amplitudes of the two relevant stress components have to be calculated according to the definitions reviewed above, that is, by distinguishing between constant and variable amplitude situations.

### 3. The Modified Wöhler Curve Method to design welded connections against fatigue

In the present section the main features of the MWCM are briefly reviewed by specifically describing the way our multiaxial fatigue criterion is suggested as being used to perform the fatigue assessment of welded connections. The in-field procedures to be followed to estimate lifetime under both CA and VA multiaxial fatigue loading will be summarised by addressing the problem in its most general form, that is, independently of the strategy adopted to determine the relevant stress state at the assumed critical point. In the next Sections instead, the way of applying the MWCM along with both nominal stresses, hot-spot quantities, the reference radius concept, and the Theory of Critical Distances will be investigated in depth, by also showing the existing links, and the consequent implications, between the MWCM’s *modus operandi* and the adopted definition to determine the stress state at the assumed critical location.

Turning back to the in-field usage of the MWCM, our criterion estimates the fatigue damage extent associated with the assessed load history through the ranges of the stress components relative to the critical plane, the combined effect of the shear and normal stress being taken into account by means of the following stress ratio [24,33]:

$$\rho_w = \frac{\Delta \sigma_n}{\Delta \tau}$$  \hspace{1cm} (13)

The most relevant peculiarity of the above stress quantity is that, thanks to the way it is defined, $\rho_w$ is seen to be sensitive to the degree of multiaxiality and non-proportionality of the stress state at the assessed critical point: for instance, $\rho_w$ is equal to unity under uniaxial fatigue loading, whereas it is invariably equal to zero under torsion [24]. Intentionally, the critical plane stress ratio is instead insensitive to the presence of non-zero mean stresses: this suggests that $\rho_w$ as defined above can be used solely to perform the fatigue assessment of weldments working in the as-welded condition, that is, in those circumstances in which, due to the residual stresses acting on the material in the vicinity of the weld bead, the presence of...
superimposed static stresses can be neglected with little loss of accuracy [3,4,24]. With regard to the non-damaging effect of non-zero mean stresses in as-welded joints, it is worth recalling here that such an experimental evidence is a consequence of the fact that the residual stresses resulting from the welding process change the local value of the load ratio, \( R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \), so that, under high tensile residual stresses, the local value of \( R \) can become larger than zero also under nominal load ratios approaching \(-1\). Accordingly, welded joints in the as-welded condition can accurately be designed against fatigue by simply using reference fatigue curves generated under \( R \) ratios larger than zero (usually, by setting \( R = 0.5 \) [3]), this holding true independently from the magnitude of the superimposed static stresses applied to the connection being assessed [3,4]. On the contrary, in stress relieved welded joints, the effect of non-zero mean stresses cannot be disregarded and the presence of superimposed static stresses is usually taken into account through appropriate enhancement factors [3,4,34], their in-field use being explained in the next Sections in great detail.

Consider then an as-welded joint subjected to a system of external forces resulting, at the critical location, in a CA multiaxial stress state (Fig. 2a and b). By taking full advantage of the MVM [30], the direction, MV, experiencing the maximum variance of the resolved shear stress can directly be determined (Fig. 2c). As soon as the orientation of the critical plane is known, the corresponding ranges of the shear, \( \Delta \tau \), and normal, \( \Delta \sigma_r \), stress (Fig. 2d and e) allow stress ratio \( \rho_w \) to directly be estimated through definition (13) – Fig. 2f. The way the MWCM estimates fatigue lifetime under CA fatigue loading is summarised through the log–log modified Wöhler diagram sketched in Fig. 2g. This chart plots the shear stress range relative to the critical plane, \( \Delta \tau \), against the number of cycles to failure, \( N_f \). In more detail, through a systematic reanalysis of numerous data sets, it was proven [24,35,36] that, given the material, different fatigue curves are obtained in the above Modified Wöhler diagram as ratio \( \rho_w \) varies, such curves being characterised by different values not only of the negative inverse slope \( k_s(\rho_w) \), but also of the reference shear stress range, \( \Delta \tau_{\text{Ref}}(\rho_w) \), extrapolated at \( N_A \) cycles to failure (Fig. 2g). The schematic modified Wöhler diagram of Fig. 2g should make it evident that, as soon as shear stress range \( \Delta \tau \) is known, lifetime can directly be estimated provided that the design curve for the investigated value of ratio \( \rho_w \) is defined unambiguously. Since the experimental fatigue curves which are usually available are those generated under both uniaxial (\( \rho_w = 1 \)) and

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**Fig. 2.** Design against CA multiaxial fatigue loading and modified Wöhler diagram (g).
torsional ($\rho_w = 0$) fatigue loading, the position of any other modified Wöhler curve has to be estimated. By performing a systematic investigation based on a large number of experimental results, it was seen that accurate predictions can be made by defining the $\Delta \tau$ vs. $\rho_w$ and $k$ vs. $\rho_w$ relationships through simple linear laws [24,36,37], i.e.:

$$k_c(\rho_w) = a \cdot \rho_w + b$$ (14)

$$\Delta \tau_{ref}(\rho_w) = a \cdot \rho_w + b$$ (15)

where $a$, $b$, $x$, and $\beta$ are material fatigue constants which can be determined experimentally. In particular, if the above constants are calibrated through the fatigue curves generated under uniaxial ($\rho_w = 1$) and torsional ($\rho_w = 0$) fatigue loading, respectively, they take on the following values [24]:

$$a = k - k_0; \quad \beta = k_0$$ (16)

$$a = \frac{\Delta \sigma_A}{z} - \Delta \tau_A; \quad b = \Delta \tau_A,$$ (17)

where $k$ and $k_0$ are the negative inverse slopes of the uniaxial and torsional fatigue curve, respectively, whereas $\Delta \sigma_A$ and $\Delta \tau_A$ are the ranges of the corresponding reference stresses extrapolated at $N_f$ cycles to failure (Fig. 2g).

It is worth observing here also that the range of the reference shear stress defining those modified Wöhler curves to be used to estimate fatigue lifetime under a $\rho_w$ ratio larger than $\rho_{w,lim}$ is assumed to be constant and equal to $\Delta \tau_{ref}(\rho_{w,lim})$, such a threshold value being determined as [24]:

$$\rho_{w,lim} = \frac{\Delta \tau_A}{\Delta \tau_{ref}(\rho_{w,lim}) - \Delta \sigma_A}$$ (18)

This assumption was made to take into account the experimental evidence that, under high values of $\rho_w$, the use of the MWCM as formulated above resulted in estimates that were seen to be characterised by an excessive level of conservatism [24,38,39]. Such an high level of conservatism was ascribed to the fact, that when critical planes experience large values of ratio $\rho_w$, fatigue damage is non longer solely shear stress governed. Accordingly, the use of the conventional critical plane approach as it stands is no longer justified, so that correction (18) was introduced in order to better take into account the contribution to the overall fatigue damage of the stress perpendicular to the critical plane [39].

Another important aspect which deserves to be discussed in great detail here is the way the $k$ vs. $\rho_w$ relationship, Eq. (14), is suggested as being defined to specifically design weldments against fatigue. In more detail, the negative inverse slope of modified Wöhler curves is recommended to be estimated as follows (Fig. 2g):

$$k_c(\rho_w) = [k - k_0] \cdot \rho_w + k_0 \quad \text{for} \quad \rho_w \leq 1 \quad \text{and} \quad N_f \leq N_{kp}$$ (19)

$$k_c(\rho_w) \equiv k \quad \text{for} \quad \rho_w > 1 \quad \text{and} \quad N_f \leq N_{kp}$$ (20)

$N_{kp}$ being the number of cycles to failure that defines the position of the knee point. For $N_f > N_{kp}$ instead, the slope of the Modified Wöhler Curves under CA loading is suggested, as recommended by Soncino [34], to be taken invariably equal to 22, this holding true independently from the actual value of the $\rho_w$ ratio relative to the critical plane under investigation. With regard to the above hypothesis, it is important to recall here that, under axial loading, the IIW [3] recommends to take the knee point at $N_{kp} = 10^7$ cycles to failure, whereas under torsional loading at $N_{kp} = 10^8$ cycles to failure. Owing to the fact that the MWCM is a shear stress based criterion, to simplify the formalisation of the method itself, the knee point is suggested as being taken, independently from the degree of multiaxiality and non-proportionality of the assessed stress state, always at $10^8$ cycles to failure [40].

The last relevant aspect which deserves to be commented here is the hypothesis that the negative inverse slope is constant and invariably equal to $k$ under $\rho_w$ values larger than unity – see Eq. (20). In particular, such an hypothesis is derived from the experimental evidence that, independently of the complexity of the load history damaging the welded detail being assessed, in general, the slope of the corresponding Modified Wöhler curve is never seen to be lower than the one of the uniaxial fatigue curve (having $\rho_w$ equal to unity) [40].

Turning back to the way of performing the fatigue assessment according to the MWCM, after estimating, through Eqs (14) and (15), the modified Wöhler curve for the specific value of the $\rho_w$ ratio characterising the critical plane being investigated, the number of cycles to failure can directly be calculated by using the following trivial relationship:

$$N_{f,c} = N_{a} \cdot \left( \frac{\Delta \tau_{ref}(\rho_{w,lim})}{\Delta \tau} \right)^{k_c(\rho_w)}$$ (21)

Focussing attention now on VA situations, assume that the welded detail, in the as-welded condition, sketched in Fig. 3a is subjected to a system of external forces resulting, at the critical location, in a VA multiaxial stress state (Fig. 3b). As discussed in the previous section, under VA fatigue loading as well the orientation of the critical plane can directly be determined by locating that plane containing the direction, $\text{MV}$, experiencing the maximum variance of the resolved shear stress [30] (Fig. 3c). By using the appropriate definitions, the range of the shear, $\Delta \tau$, and normal, $\Delta \sigma_{n}$, stress relative to the critical plane can then be calculated directly (Fig. 3d and e), by subsequently determining the value of stress ratio $\rho_w$ (Fig. 3f). As soon as $\rho_w$ is known, by taking full advantage of Eqs (14) and (15) the position of the pertinent modified Wöhler curve has to be estimated by correcting the curve itself in order to properly take into account the damaging effect of those stress cycles of low stress amplitude (Fig. 3g). In more detail, according to Hailbach [41], the negative inverse slope in the long-life regime has to be corrected as follows:

$$m_c(\rho_w) = 2 \cdot k_c(\rho_{w,lim}) - 1 \quad \text{for} \quad \rho_w \leq 1$$ (22)

$$m_c(\rho_w) = 2 \cdot k - 1 = \text{const} \quad \text{for} \quad \rho_w > 1$$ (23)

where, for the same reason as above, also under VA fatigue loading the knee point is recommended to be always taken at $N_{kp} = 10^8$ cycles to failure. By taking full advantage of the classical Rain-Flow method [32], the resolved shear stress cycles can now be counted (Fig. 3h) to build the corresponding load spectrum (Fig. 3i) [31]. Finally, the calculated load spectrum can directly be used, along with the adopted modified Wöhler curve, to evaluate the damage content associated with any counted shear stress cycles (Fig. 3i and g), the estimated number of cycles to failure being equal to (Fig. 3k and l):

$$D_{tot} = \sum_{i=1}^{j} N_{f,i} = D_{f} \cdot D_{tot} \cdot \sum_{i=1}^{j} N_{f,i}.$$ (24)

In the above equations, $D_{tot}$ is the total value of the damage sum, whereas $D_{f}$ is the critical value of the damage sum, i.e., the value of $D_{tot}$ resulting in the fatigue breakage of the welded component being designed. Finally, it is worth observing that, as recommended by the IIW [3], the critical value of the damage sum, $D_{f}$, is suggested as being taken invariably equal to 0.5.

To conclude, it has to be pointed out that, compared to our initial simplified proposals [28,42], the approach proposed in the present paper allows the VA problem to efficiently and accurately be addressed independently from the complexity of the assessed load history. On the contrary, since in Ref. [28,42] fatigue damage
was suggested as being estimated by following a cycle-by-cycle procedure, such a simplified method could unambiguously be applied solely to those situations involving VA multiaxial loading paths characterised by stress components having the same frequency.

4. Summary of the investigated data sets

In order to check the accuracy of the MWCM in estimating fatigue lifetime under multiaxial fatigue loading when our critical plane approach is applied in terms of either nominal stresses, hot-spot quantities, the reference radius concept, or the Theory of Critical Distances, a number of data sets were selected from the technical literature. In more detail, for comparison purposes, our approach was initially employed to estimate fatigue results generated under CA multiaxial fatigue loading. The selected CA series are listed in Table 1, whereas the investigated welded geometries are sketched in Fig. 4a–f. In order to concisely summarise the fatigue behaviour of the considered welded samples, Table 1 reports, for any investigated data sets, the uniaxial as well as the torsional fatigue curves, where such fatigue curves, estimated for a probability of survival, $P_s$, equal to 97.7%, are summarised in terms of negative inverse slope, $k$ and $k_0$, reference nominal stress range, $\Delta \sigma_N$ and $\Delta \tau_N$, and scatter ratio of the stress amplitude, $T_N$ and $T_T$, at $N_s = 2 \times 10^6$ cycles to failure for 90% and 10% probabilities of survival. With regard to the performed statistical reanalysis, the fatigue curves listed in Table 1 were calculated under the hypothesis of a log-normal distribution of the number of cycles to failure for each stress level and assuming a confidence level equal to 95% [43]. Finally, it is worth observing here also that, in light of their intrinsic VA nature [56], also the results generated both by Archer [50] and by Yousefi et al. [44] under CA bending and shear stress at different frequencies were considered in the present investigation. The main features of the results generated under VA multiaxial fatigue loading are instead summarised in Table 2, where $\Delta \sigma_{r}$ and $\Delta \tau_{r}$ are the ranges of the uniaxial and torsional nominal stress during the loading cycles, respectively, $\delta$ is the out of phase angle and, finally, $F$ is the ratio between the frequency of the axial stress component, $f_R$, and the frequency of the torsional stress component, $f_T$. It is important to highlight here that, for any data sets listed in Table 2, the corresponding uniaxial and torsional fatigue curves generated under CA fatigue loading are reported in Table 1.

To conclude, it has to be pointed out that all the fatigue results summarised in Table 2 were generated by adopting the classical
The most simple way of using the MWCM to perform the fatigue assessment of welded joints subjected to in-service CA and VA multiaxial fatigue loading is by calculating the relevant stress states at the assumed critical locations in terms of nominal stresses \[24,33\]. As soon as the nominal stress components damaging the welded detail being assessed are known, the MWCM can directly be applied as discussed in the previous Sections solely to perform the fatigue assessment of weldments working in the as-welded condition. On the contrary, to reduce the estimates’ level of conservatism, stress relieved welded joints are recommended to be designed against fatigue by multiplying the reference shear stress range of the adopted modified Wöhler curve by a suitable enhancement factor, \( f_{RCP} \), that is \[3\]:

\[
\Delta \sigma_{ref}(\rho_u) \cdot f_{RCP} = (a \cdot \rho_u + b) \cdot f_{RCP}
\]

where factor \( f_{RCP} \) is assumed \[40\] to depend on the load ratio, \( R_{CP} \), calculated through the stress perpendicular to the critical plane, i.e. \[39\]:

\[
R_{CP} = \frac{\sigma_{n.m} - \sigma_{n.a}}{\sigma_{n.m} + \sigma_{n.a}}
\]

\[25\]

**Table 1**

Summary of the uniaxial and torsional experimental fatigue curves calculated in terms of nominal stresses.

<table>
<thead>
<tr>
<th>Code</th>
<th>Material</th>
<th>Reference</th>
<th>( t ) (mm)</th>
<th>( z ) (mm)</th>
<th>( R )</th>
<th>( \Delta \sigma_{ref} ) (MPa)</th>
<th>( T_f )</th>
<th>( k_0 )</th>
<th>( \Delta \sigma_{ref} ) (MPa)</th>
<th>( T_f )</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>WSP1</td>
<td>Ste 460'</td>
<td>[14]</td>
<td>10</td>
<td>9</td>
<td>-1</td>
<td>4.4</td>
<td>152.0</td>
<td>1.703</td>
<td>-1</td>
<td>4.8</td>
<td>122.4</td>
</tr>
<tr>
<td>WSP2</td>
<td>Ste 460'</td>
<td>[44]</td>
<td>8</td>
<td>10</td>
<td>-1,0</td>
<td>3.9</td>
<td>125.7</td>
<td>1.759</td>
<td>-1</td>
<td>7.4</td>
<td>151.6</td>
</tr>
<tr>
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<td>Ste 460</td>
<td>[45]</td>
<td>7.7</td>
<td>9</td>
<td>-1,0</td>
<td>5.4</td>
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<td>-1,0</td>
<td>6.2</td>
<td>104.9</td>
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<tr>
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<td>AS19</td>
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<td>7.95</td>
<td>-8</td>
<td>-1</td>
<td>5.4</td>
<td>90.7</td>
<td>1.528</td>
<td>-1</td>
<td>3.7</td>
<td>65.3</td>
</tr>
<tr>
<td>WSP5</td>
<td>A519 – A36'</td>
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<td>-1,0</td>
<td>3.8</td>
<td>123.9</td>
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<td>-1</td>
<td>5.5</td>
<td>103.9</td>
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<td>Fe S2 steel</td>
<td>[48,49]</td>
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<td>6</td>
<td>0.1-0.7</td>
<td>2.2</td>
<td>46.7</td>
<td>7.511</td>
<td>-1</td>
<td>3.5</td>
<td>85.2</td>
</tr>
<tr>
<td>WSP7</td>
<td>BS4360</td>
<td>[50]</td>
<td>6</td>
<td>10</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1.319</td>
<td>4.5</td>
</tr>
<tr>
<td>WSP8</td>
<td>BS4360 Gr. 50E</td>
<td>[51,52]</td>
<td>3.2</td>
<td>11</td>
<td>0</td>
<td>3.0</td>
<td>56.1</td>
<td>1.319</td>
<td>0</td>
<td>4.5</td>
<td>82.7</td>
</tr>
<tr>
<td>WSP9</td>
<td>42CrMo4'</td>
<td>[53]</td>
<td>5.5</td>
<td>n.a.</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>14.6</td>
<td>167.6</td>
</tr>
<tr>
<td>WSP10</td>
<td>6082-T6</td>
<td>[54]</td>
<td>10</td>
<td>16</td>
<td>-1</td>
<td>6.8</td>
<td>60.3</td>
<td>1.845</td>
<td>-1</td>
<td>5.6</td>
<td>52.0</td>
</tr>
<tr>
<td>WSP11</td>
<td>6060-T6'</td>
<td>[55]</td>
<td>3</td>
<td>3</td>
<td>-1</td>
<td>5.4</td>
<td>68.3</td>
<td>1.222</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

* Stress relieved.

\( ^{a} \) Nominal reference stress ranges are calculated for \( P_2 = 97.7\% \) and extrapolated at \( N_f = 2 \times 10^6 \) cycles to failure.

5. Estimating lifetime in terms of nominal stresses

Fig. 4. Schematic geometry of the investigated welded samples (a–f) and LBS Gaussian Spectrum (g) [14].

LBF Gaussian spectrum sketched in Fig. 4g and having sequence length equal to \( 5 \times 10^4 \) cycles [14].
\( \sigma_{a0} \) and \( \sigma_{asm} \) being the amplitude and the mean value of normal stress \( \sigma_{a}(t) \), respectively (obviously both stress quantities have to be determined according to either the CA or the VA definitions reviewed above). The idea of using solely the stress perpendicular to the critical plane to estimate in stress relieved welded joints the damaging effect of superimposed static stresses is based on the experimental evidence that in non-welded metallic materials the presence of non-zero mean shear stresses can be neglected as long as the maximum shear stress (during the loading cycle) is lower than the material yield shear stress \([57,58]\). Therefore, according the above considerations, the rules recommended in Ref. \([34]\) can directly be extended to CA and VA multiaxial fatigue situations as follows \([40]\):

\[
\begin{align*}
    f(R_{CP}) &= 1.32 \text{ for } R_{CP} < -1 \\
    f(R_{CP}) &= -0.22 \times R_{CP} + 1.1 \text{ for } -1 \leq R_{CP} \leq 0 \quad (27) \\
    f(R_{CP}) &= -0.2 \times R_{CP} + 1.1 \text{ for } 0 < R_{CP} \leq 0.5 \\
    f(R_{CP}) &= 1 \text{ for } R_{CP} > 0.5 \\
    f(R_{CP}) &= 1.88 \text{ for } R_{CP} < -1 \\
    f(R_{CP}) &= -0.55 \times R_{CP} + 1.33 \text{ for } -1 \leq R_{CP} \leq 0 \quad (28) \\
    f(R_{CP}) &= -0.66 \times R_{CP} + 1.33 \text{ for } 0 < R_{CP} \leq 0.5 \\
    f(R_{CP}) &= 1 \text{ for } R_{CP} > 0.5
\end{align*}
\]

for aluminium welded joints.

The experimental, \( N_f \), vs. estimated, \( N_{f,e} \), fatigue lifetime diagrams reported in Fig. 5 summarise the overall accuracy of the MWCM, applied in terms of nominal stresses, in estimating fatigue strength of both as-welded and stress relieved joints subjected to CA as well as to VA fatigue loading, the adopted axial and torsional calibration fatigue curves (see Table 3) being those recommended by the IIW \([3]\). With regard to these error diagrams, it is worth observing here also that the plotted scatter bands were recalculated from the reference value of 1.5 suggested by Haibach and determined by considering fatigue curves characterised by a probability of survival equal to 10% and 90%, respectively \([59]\); consistently, the scatter ratio of the stress range at \( 2 \times 10^6 \) cycles to failure for 97.7% and 2.3% probabilities of survival is equal to 1.85. Further, both the uniaxial and torsional scatter bands were shifted parallel to the abscissa by forcing the straight lines characterised by a probability of survival equal to 2.3% to coincide with the \( N_f = N_{f,e} \) straight diagonal line. The error diagrams of Fig. 5 clearly prove that the MWCM applied in terms of nominal stresses is highly accurate in estimating fatigue lifetime of welded joints subjected to both CA (Fig. 5a and b) and VA in-service fatigue loading (Fig. 5c), resulting in estimates falling within the two scatter bands associated with the uniaxial and torsional standard fatigue curves used to calibrate the MWCM itself. This means that the MWCM applied in terms of nominal stresses is capable of performing the fatigue assessment under both CA and VA fatigue loading by complying with the recommendations of the available standard codes, that is, by always resulting in estimates characterised, from a statistical point of view, by an adequate level of safety. As to the estimates obtained for those welded samples tested under VA fatigue loading (Fig. 5c), according to what recommended by the IIW \([3]\), fatigue lifetime was predicted by taking the critical value of the damage sum, \( D_{on} \), invariably equal to 0.5. Further, the estimates summarised in the chart of Fig. 5c were calculated adopting the classical “2k−1 correction” due to Haibach \([41]\). Eqs (22) and (23), with \( N_{fp} = 10^6 \) cycles to failure.

To conclude, it can be pointed out that the reached level of accuracy is definitely satisfactory because one cannot ask a predictive method to result in estimates that are, from a statistical point of view, less scattered than the experimental information used to calibrate the method itself.

### 6. Estimating lifetime in terms of hot-spot stresses

In order to correctly apply the MWCM in terms of hot-spot stresses to estimate multiaxial fatigue lifetime of weldments, the stress components relative to the critical plane are suggested as being determined by post-processing the geometrical stress components perpendicular (\( \sigma_{f,gs} \) in Fig. 6a) and parallel (\( \tau_{f,gs} \) in Fig. 6a) to the weld bead \([60]\). In particular, such stress components are recommended to be used to define the relevant stress state at the assumed critical section because in the presence of notches with opening angle larger than about \( 100^\circ \), fatigue strength is seen to depend mainly on the Mode I and III stress components \([16]\). On the contrary, the contribution of Mode II loading to the overall fatigue damage can be neglected with little loss of accuracy as a consequence of the fact that the resulting stress components are not singular \([6]\). According to the above considerations and owing to the fact that the magnitude of the geometrical stresses is seen to vary as the distribution of the linear-elastic stress fields in the vicinity of the hot-spots changes \([61]\), it is possible to hypothesise that the structural stresses perpendicular to the weld bead are somehow related to the Mode I stress components, whereas the

<table>
<thead>
<tr>
<th>Code</th>
<th>Material</th>
<th>Reference</th>
<th>( R )</th>
<th>( \Delta \Sigma / \Delta T )</th>
<th>( \delta (^\circ) )</th>
<th>F</th>
<th>Geometry</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>WSP1</td>
<td>StE 460[\textsuperscript{a}]</td>
<td>[14]</td>
<td>-1</td>
<td>( \infty )</td>
<td>-</td>
<td>-</td>
<td>Fig. 4a</td>
<td>Fig. 4g</td>
</tr>
<tr>
<td>WSP2</td>
<td>StE 460[\textsuperscript{a}]</td>
<td>[44]</td>
<td>-1</td>
<td>( \sqrt{3} )</td>
<td>0</td>
<td>1</td>
<td>Fig. 4a</td>
<td>Fig. 4g</td>
</tr>
<tr>
<td>WSP9</td>
<td>42CrMo4</td>
<td>[53]</td>
<td>-1</td>
<td>( \infty )</td>
<td>-</td>
<td>-</td>
<td>Fig. 4a</td>
<td>Fig. 4g</td>
</tr>
<tr>
<td>WSP10</td>
<td>6082-T6</td>
<td>[22]</td>
<td>-1</td>
<td>1</td>
<td>90</td>
<td>1</td>
<td>Fig. 4a</td>
<td>Fig. 4g</td>
</tr>
<tr>
<td>WSP10</td>
<td>6082-T6</td>
<td>[22]</td>
<td>-1</td>
<td>( \sqrt{3} )</td>
<td>90</td>
<td>1</td>
<td>Fig. 4a</td>
<td>Fig. 4g</td>
</tr>
</tbody>
</table>

\[ \text{a} \text{ Stress relieved.} \]
structural shear stresses parallel to the weld bead depend instead on the Mode III stress components [24,60].

As to the performed validation exercise, Fig. 6a schematically shows the strategy we followed to calculate the hot-spot stress states in the investigated welded geometries (Fig. 4a–f). In more detail, normal and shear structural stresses were extrapolated to the weld toe by using two superficial points positioned at a distance from the hot-spot equal to 0.4r and r, respectively [3], r being the reference thickness as defined in Fig. 4a–f. The stress values used to extrapolate the hot-spot quantities were determined at the above reference points through linear-elastic FE models whose mesh density was set according to Niemi’s recommendations [5].

With regard to the in-field usage of the MWCM applied in terms of hot-spot quantities to perform the fatigue assessment of stress relieved welded joints, superimposed static stresses were taken into account by directly correcting the modified Wöhler curves through enhancement factor \( f(\alpha_r) \), Eq. (25), where the above factor was calculated again according to definitions (27) and (28).

The experimental, \( N_f \) vs. estimated, \( N_{kp} \), number of cycles to failure diagrams reported in Fig. 6b–d were built by calibrating the constants in the MWCM’s governing equations through the reference values suggested by the IIW [3] and summarised in Table 4. The above error diagrams should make it evident that the use of the MWCM applied in terms of hot-spot stresses results in estimates falling, mainly on the conservative side, within the widest calibration scatter band.

To conclude, it can be highlighted that also in this case fatigue lifetime under VA fatigue loading (Fig. 6d) was estimated by taking the critical value of the damage sum, \( D_{cr} \), constant and equal to 0.5 [3] as well as by adopting, as recommended by Halbach [41], the classical “2\( k - 1 \)” correction, Eqs (22) and (23), with \( N_{kp} = 10^6 \) cycles to failure.

7. Estimating lifetime according to the reference radius concept

The most advanced design methodology which is recommended by the IIW [3] is that based on the use of a reference radius, \( r_{ref} \), equal to 1 mm, where such a method can be adopted solely to perform the fatigue assessment of welded joints having thickness of the main plate larger than (or equal to) 5 mm. In more detail, according to the above approach the profile of either the weld toe or root is rounded with a circular fillet having radius equal to 1 mm (Fig. 7a). From a scientific point of view, the above value for \( r_{ref} \) was first proposed by Radaj [4,62], who took full advantage of the micro-support theory due to Neuber to model the fatigue behaviour of sharp cracks: accordingly, the fatigue assessment performed in terms of the reference radius concept is seen not to be influenced by the actual values of the radii of either the weld toes or roots [62].

If lifetime under uniaxial fatigue loading is estimated according to such a well-established method, the IIW suggests using a reference stress range, \( \Delta \sigma_{cr} \), calculated according to the maximum principal stress criterion at \( N_f = 2 \times 10^6 \) cycles to failure, equal to 225 MPa for steel weldments and to 71 MPa for aluminium joints, the above reference ranges being determined for a probability of survival, \( P_s \), equal to 97.7% and estimated under a load ratio, \( \dot{R} \), equal to 0.5 (i.e., by simulating the effect of high tensile residual stresses). Further the CF uniaxial fatigue curve recommended by the IIW has the knee point, \( N_{kp} \), at 10^6 cycles to failure and negative inverse slope \( k \) is equal to 3 for \( N_f \leq N_{kp} \) and to 22 for \( N_f > N_{kp} \) [4,34], this schematisation applying to both steel and aluminium weldments.

Under CA torsional loading instead, the design curve, again determined according to the maximum principal stress hypothesis, suggested by Sonsino [34] as being used to perform the fatigue assessment according to the \( r_{ref} = 1 \) mm idea has reference shear stress range, \( \Delta \tau_{cr} \), at \( N_f = 2 \times 10^6 \) cycles to failure equal to 160 MPa and to 63 MPa for steel and aluminium welded joints, respectively. Further, the negative inverse slope, \( k_{\tau} \), is equal to 5 for \( N_f \leq N_{kp} \) and to 22 for \( N_f > N_{kp} \), where \( N_{kp} \) is taken equal to 10^6 cycles to failure.

Strictly speaking, the curves as defined above can safely be used to design against fatigue solely those as-welded structures that can be classified as “thick and stiff”. On the contrary, as long as “thin and flexible” welded structures are concerned, the negative inverse slope is seen to increase from 3 up to 5 under CA uniaxial fatigue loading and from 5 up to 7 under CA torsional loading [63]. As to the above schematisation, it has to be said that, even though in
situations of practical interest it is never straightforward to classify welded joints as either “thick and stiff” or “thin and flexible”, in the validation exercise discussed below the use of the $r_{ref} = 1\,\text{mm}$ idea will be extended to those situations involving both CA and VA multiaxial fatigue loading by taking into account also this important aspect: according to the above considerations, Table 5 summarises the fatigue curves used to calibrate the constants in the MWCM’s governing equations to make our criterion suitable for being applied, in conjunction with the $r_{ref} = 1\,\text{mm}$ approach [40], to the investigated experimental results.

Another important aspect which has to be considered in great detail here is the problem of correctly locating the point at which the critical stress has to be calculated in order to accurately perform the fatigue assessment according to the reference radius idea (point O in Fig. 7a). In particular, it has to be highlighted that, in the most general case, the peak of the stress state may be positioned, 

**Table 3**

Fatigue curves used to calibrate the MWCM to apply it in terms of nominal stresses.

<table>
<thead>
<tr>
<th>Code</th>
<th>Material</th>
<th>Reference</th>
<th>$k$</th>
<th>$\Delta\sigma^h$ (MPa)</th>
<th>$k_0$</th>
<th>$\Delta\sigma^h$ (MPa)</th>
<th>Geometry</th>
</tr>
</thead>
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<tr>
<td>WSP1</td>
<td>StE 460</td>
<td>[14]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>100</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP2</td>
<td>StE 460</td>
<td>[44]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>100</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP3</td>
<td>StE 460</td>
<td>[45]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>100</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP4</td>
<td>519</td>
<td>[46]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>100</td>
<td>Fig. 4b</td>
</tr>
<tr>
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<td>A519 – A36</td>
<td>[47]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>100</td>
<td>Fig. 4b</td>
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<tr>
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<td>52 steel</td>
<td>[48,49]</td>
<td>3</td>
<td>45</td>
<td>5</td>
<td>100</td>
<td>Fig. 4c</td>
</tr>
<tr>
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<td>BS43160</td>
<td>[50]</td>
<td>3</td>
<td>50</td>
<td>5</td>
<td>80</td>
<td>Fig. 4d</td>
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<td>BS43160 Gr. 50E</td>
<td>[51,52]</td>
<td>3</td>
<td>71</td>
<td>5</td>
<td>80</td>
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<td>[53]</td>
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<td>32</td>
<td>5</td>
<td>36</td>
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<td>22</td>
<td>5</td>
<td>28</td>
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<td>22</td>
<td>5</td>
<td>28</td>
<td>Fig. 4e</td>
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</table>

a Stress relieved.

b Nominal reference stress ranges ($P_S = 97.7\%$) extrapolated at $N_a = 2 \times 10^6$ cycles to failure.

![Fig. 6. Accuracy of the MWCM applied in terms of hot-spot stresses in estimating fatigue lifetime of steel and aluminium weldments subjected to CA (b and c) and VA (d) uniaxial/multiaxial fatigue loading ($ZMS = \text{zero mean stress}; N-ZMS = \text{non-zero mean stress}; AW = \text{as-welded}; SR = \text{stress relieved}; DF = \text{different frequencies}; F = f_R/f_T$).](image-url)
along the fictitious fillet, at different points as the degree of multiaxiality and non-proportionality of the applied load varies. However, in practise the above difference is seen to be so small to be negligible; if not, it is evident that the correct position of the critical location has to be found by looking for that point experiencing the largest extent of fatigue damage [40].

The uniaxial and torsional reference curves as defined above can efficiently be used to calibrate the constants in the MWCM’s governing equations to use our criterion solely to design welded joints working in the as-welded condition. On the contrary, in order to correctly take into account the mean stress effect in stress relieved welded connections, the stress range at $N_0$ cycles to failure of any modified Wöhler curve can directly be corrected through enhancement factor $f_{R_{cp}}$, Eq. (25), such a factor being again calculated according to definitions (27) and (28) [34].

The charts reported in Fig. 7b–d show the accuracy of the MWCM, used in conjunction with the reference radius concept, in estimating CA fatigue lifetime of “thin and flexible” as well as “thick and stiff” welded joints of both steel (Figs. 7b and c) and aluminium (Fig. 7d). As to the above diagrams, it is worth observing that the reference scatter bands are wider for “thin and flexible” joints due to the fact that in such circumstances the negative inverse slope is equal to 5 under uniaxial loading and to 7 under torsion [63]. Further, in the above charts the results generated by both Razmijoo [51,52] and Costa et al. [55] were not considered, because in those samples the thickness of the main tube was lower than 5 mm. The accuracy of the MWCM in estimating lifetime under VA fatigue loading is instead summarised in the error diagrams reported in Fig. 7b–e, the obtained estimates not only fall mainly within the calibration scatter bands, but they are also characterised by a level of conservatism that definitely complies with the recommendations of the available standard codes.

8. Estimating lifetime according to the Theory of Critical Distances

It is the author’s opinion that, even if its accuracy and reliability is not brought into question at all, the real difficulty in using the reference radius idea to design welded joints against fatigue in situations of practical interest is that determining the necessary stress states by rounding weld beads and roots with a fictitious radius is a complex and time-consuming numerical job, this holding true especially when complex tridimensional geometries are involved. In order to overcome the above problem without missing the advantages of linear-elastic local analyses, we have recently devised a novel design method which is based on the use of the MWCM applied along with the Theory of Critical Distances (TCD), the latter being formalised in terms of the so-called Point Method (PM) [10,11,24]. In more detail, such an approach can be used to estimate fatigue damage in welded connections by directly post processing the linear-elastic stress fields damaging the material in the vicinity of the assumed crack initiation sites, weld beads and roots being simply modelled as sharp notches. The most relevant peculiarity of our method is that, thanks to its specific features, it allows welded components to be assessed by simultaneously taking into account not only the degree of multiaxiality and non-proportionality of the linear-elastic stress field acting on the process zone, but also the detrimental effect of time-variable multiaxial stress gradients [64,65].

The in-field procedure suggested as being followed to perform the stress analysis according to our local approach is shown in Fig. 8. In particular, consider a welded joint damaged by a complex system of cyclic forces: as schematically displayed in Fig. 8, the stress state to be used to determine the necessary stress quantities relative the critical plane has to be determined, along the bisector, at a distance from the weld toe apex (or the weld root apex) equal to $M_{D_r}$, such a critical distance being equal to 0.5 mm and to 0.075 mm for steel and aluminium welded joints, respectively [64,65]. Further, it is worth pointing out here also that, according to the TCD’s philosophy, critical length $M_{D_r}$ is assumed to be a material property whose value does not depend on either the
As soon as the time-variable stress state at the critical point is known, the range of the maximum shear stress, $\Delta \tau$, and the range of the stress perpendicular to the critical plane, $\Delta \sigma_n$, can directly be determined by taking full advantage of either the CA or the VA definitions reviewed above. Finally, the calculated value for ratio $\rho_w$ allows the negative inverse slope, $k_s(\rho_w)$, and the reference shear stress range, $\Delta \tau_{\text{Ref}}(\rho_w)$, of the appropriate modified Wöhler curve to be estimated from the calibration functions reported below. In particular, for steel welded joints the negative inverse slope is suggested as being calculated as [64]:

$$k_s(\rho_w) = \frac{-2 \cdot \rho_w + 5}{\rho_w} \text{ for } \rho_w \leq 1$$  \hspace{1cm} (29)

$$k_s(\rho_w) = 3 \text{ for } \rho_w > 1$$  \hspace{1cm} (30)

whereas the reference shear stress ranges, $\Delta \tau_{\text{Ref}}(\rho_w)$, at $N_A = 5 \times 10^6$ cycles to failure takes on the following values [64]:

$$\Delta \tau_{\text{Ref}}(\rho_w) = -32 \cdot \rho_w + 96 \text{ [MPa]} \text{ for } \rho_w \leq 2$$  \hspace{1cm} (31)
Fatigue curves used to calibrate the MWCM to apply it along with the $r_{ref} = 1$ mm concept.

<table>
<thead>
<tr>
<th>Code</th>
<th>Material</th>
<th>Reference</th>
<th>$k$</th>
<th>$\Delta \sigma_i^a$ (MPa)</th>
<th>$k_0$</th>
<th>$\Delta \tau_i^a$ (MPa)</th>
<th>$\rho_{w, cor}$</th>
<th>Geometry</th>
</tr>
</thead>
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<tr>
<td>WSP1</td>
<td>StE 460$^a$</td>
<td>[14]</td>
<td>5</td>
<td>225</td>
<td>7</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP2</td>
<td>StE 460$^a$</td>
<td>[44]</td>
<td>5</td>
<td>225</td>
<td>7</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP3</td>
<td>StE 460</td>
<td>[45]</td>
<td>5</td>
<td>225</td>
<td>7</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP4</td>
<td>A519</td>
<td>[46]</td>
<td>5</td>
<td>225</td>
<td>7</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4a</td>
</tr>
<tr>
<td>WSP5</td>
<td>A319 – A36$^a$</td>
<td>[47]</td>
<td>5</td>
<td>225</td>
<td>7</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4b</td>
</tr>
<tr>
<td>WSP6</td>
<td>Fe 52 steel</td>
<td>[48,49]</td>
<td>3</td>
<td>225</td>
<td>5</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4c</td>
</tr>
<tr>
<td>WSP7</td>
<td>BS 4360</td>
<td>[50]</td>
<td>3</td>
<td>225</td>
<td>5</td>
<td>160</td>
<td>1.7</td>
<td>Fig. 4d</td>
</tr>
<tr>
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<td>42CrMo4$^a$</td>
<td>[53]</td>
<td>–</td>
<td>–</td>
<td>7</td>
<td>160</td>
<td>1.45</td>
<td>Fig. 4f</td>
</tr>
<tr>
<td>WSP10</td>
<td>6082-T6</td>
<td>[54]</td>
<td>5</td>
<td>71</td>
<td>7</td>
<td>63</td>
<td>1.45</td>
<td>Fig. 4a</td>
</tr>
</tbody>
</table>

$^a$ Stress relieved.

$^b$ Local reference stress ranges ($P_3 = 97.7\%$) extrapolated at $N_a = 2 \times 10^6$ cycles to failure.

$$ f(\tau) = \frac{2\tau_a}{\tau_m + \tau_a + 0.6(\tau_m - \tau_a)} \quad \text{for } (\tau_m - \tau_a) < 0 $$

(42)

As to the above correction, it is worth observing here that it can safely be used to design stress relieved welded joints made not only of steel (as recommended by Eurocode 3), but also of aluminium, since in the latter case the use of enhancement factor $f(\tau)$ results in corrections that are, in any case, more conservative than the ones obtained by adopting the enhancement factor values suggested by Sonsino as being used to specifically design stress relieved welded joints of aluminium [34].

Turning back to the determination of the stress state at the critical point (Fig. 8), it is worth highlighting here that, according to the TCD’s philosophy, such a stress tensor has to be determined by using a linear-elastic constitutive law to model the behaviour of both the parent and welded material. It is evident that the above hypothesis results in a great simplification of the design problem, since the total stress state at any instant of the load history can be determined by superposing the effect of every applied force, whose contributions being computed separately. Further, the stress state at a distance from the assumed crack initiation point equal to $M-D_D$ is recommended to be determined by always considering the contribution of the three fundamental modes (i.e., Mode I, II and III stress components) [24,64,65], even though, strictly speaking, fatigue damage in conventional welded joints can efficiently be evaluated by taking into account solely the contributions due to the singular modes (i.e., Mode I and III loading). Mode II stress components being no longer singular in the presence of stress raiser opening angles larger than about 100° [6]. This modus operandi results in a very similar method to the one proposed by considering a priori between singular and non-singular contributions.

In order to show the overall accuracy of our local approach, initially the error diagrams of Fig. 9a and b report the predictions made by considering a variety of steel welded details subjected to both CA (Fig. 9a) and VA (Fig. 9b) uniaxial nominal loading, the considered geometries as well as the investigated CA and VA loading paths being summarised, together with the corresponding bibliographical sources, in Refs [42,64]. The above charts should make it evident that our local method is capable of correctly evaluating the degree of multiaxiality of the stress field acting on the fatigue process zone, whose profile depends, in such circumstances, solely on the specific geometrical features of the assessed welded geometry. A similar level of accuracy is obtained also when the MWCM is used, along with the PM, to estimate the experimental results generated by testing steel weldments and summarised in Tables 1 and 2 (Fig. 9c and d), that is, when the degree of multiaxiality and non-proportionality of the local stress field depends not only on the geometrical features of the assessed welded...
connection, but also on the specific characteristics of the investigated CA (Fig. 9c) and VA (Fig. 9b) load history.

The chart of Fig. 10a summarises instead the accuracy and reliability of our local method in estimating fatigue lifetime of aluminium welded joints subjected to CA uniaxial nominal loading, the geometries of the considered welded samples being described in great detail in Ref. [65]. The above chart suggests that as long as joints in the as-welded condition are concerned, the use of our method results in predictions falling mainly within the calibration scatter bands. On the contrary the estimates associated with stress relieved welded joints are, as expected, slightly conservative. A similar level of conservatism was obtained also when our approach was used to estimate the experimental data summarised in Tables 1 and 2 and generated by testing, under both CA (Fig. 10b) and VA (Fig. 10c) fatigue loading, welded samples of aluminium [22].

It is worth pointing out here that that also in this case the number of cycles of failure under VA fatigue loading (Figs. 9b, d and 10c) was predicted not only by taking \(D_{cr} = 0.5\) [3], but also as by adopting the classical “2k–1 correction”, Eqs (22) and (23), with \(N_{kp} = 10^8\) cycles to failure.

Turning back to aluminium weldments, to understand the choice of suggesting the use of calibration constants which result in estimates characterised, from a statistical point of view, by an evident conservative trend, it is worth observing here that, according to the state of the art, the resources which have been invested so far to understand the response of aluminium welded joints to fatigue loadings are definitely very little compared to those invested to investigate the fatigue behaviour of welded steel. Such a lack of knowledge becomes absolutely evident when the available standard codes and recommendations are compared to each other: for instance, the fatigue design curves supplied by Eurocode 9 [2] are different from those recommended by the IIW [3] not only in terms of reference strength at 2 million cycles to failure, but also in terms of negative inverse slope [66]. The urgency of developing an efficient strategy to design aluminium weldments against multiaxial fatigue is proven by the fact that, as far as the writer is aware, only a few papers addressing the above problem and reporting original experimental results have been published in International Scientific Journals so far. Moreover, only one series of tests carried out under non-proportional loading is available in the technical literature. Finally, examination of the state of the art shows that, surprisingly, only few attempts have been made so far to devise reliable methodologies capable of efficiently estimating lifetime when aluminium weldments are subjected to variable amplitude multiaxial fatigue loading.

Another open question which should be answered urgently is the way of efficiently taking into account the presence of non-zero mean stresses, this holding true not only in stress relieved but also in as-welded joints: in fact, contrary to what happens in steel welded details, aluminium weldments can be highly sensitive to the presence of superimposed static stresses also when they work in the as-welded condition [67].

The above considerations should make it clear that, in light of such a level of uncertainty, as formalising our approach [24,65] we decided to calibrate the MWCM’s governing equations through uniaxial and torsional reference curves allowing an adequate
margin of safety to always be reached when addressing problems of practical interest.

It is possible to conclude by observing that, owing to the intrinsic flexibility of our local approach, structural engineers can follow the same strategy as the one described in Refs [24, 64, 65] by adopting, according to their in-field experience, different reference curves to calibrate the MWCM as well as to calculate M-DV in order to more efficiently perform the fatigue assessment of both steel and aluminium weldments.

9. Conclusions

(1). The most important peculiarity of the definitions adopted in the present study and suitable for calculating the stress quantities relative to the critical plane under both CA and VA uniaxial/multiaxial fatigue loading is that they can easily be implemented allowing the design problem to efficiently be addressed numerically.

(2). Independently of the adopted standard stress analysis (i.e., either nominal quantities, hot-spot stresses, or the reference radius idea), the MWCM is seen to be successful in estimating fatigue lifetime of both steel and aluminium weldments subjected to CA as well as VA uniaxial/multiaxial fatigue loading: accordingly, it is a powerful candidate to be considered for being included amongst those methods recommended by the available standard codes.

(3). Even if the TCD based methods are not included in any standard codes or recommendations yet, the MWCM applied in terms of the PM has proven to be highly accurate in estimating fatigue lifetime of steel and aluminium welded joints subjected to either CA or VA fatigue loading: this result is very promising, especially owing to the fact that our method can be applied by directly post-processing simple linear-elastic FE models, i.e., without the need for defining any nominal quantities.

References

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